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ON WEAKLY-PRÉSIMPLIFIABLE GROUP RINGS

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Abstract. A commutative ring R with unity is called weakly-présimplifiable (resp., présimplifiable) if for a, $b \in R$ with $a = ba$, then either $a = 0$ or b is a regular element (that is, b is not a zero-divisor) in R (resp., $a = 0$ or b is a unit in R). Let R be a commutative ring with unity and G be a nontrivial abelian group. In this paper, we give some characterizations for the group ring $R[G]$ to be weakly présimplifiable. Furthermore, we give a complete description of (weakly) présimplifiable circulant matrix ring.

1. Introduction

Throughout, all rings are considered to be commutative with unity and all groups are abelian and nontrivial unless otherwise indicated. The symbols $J(R)$, $Z(R)$, $U(R)$, $reg(R)$, and $nil(R)$ denote the Jacobson radical, the set of all zero-divisors, the set of all units of R, the set of regular elements (that is, nonzero elements in R that are not zero-divisors) of R and the nil radical of R, respectively. Moreover, for a ring R, $R[[x]]$ denotes the ring of formal power series in the variable x, and $R[x_1; \ldots; x_n; x_1^{-1}; \ldots; x_n^{-1}]$ denotes the ring of Laurent power series in the variables $x_1; \ldots; x_n$.

Let R be a commutative ring with unity and let G be a nontrivial abelian group. Then, $R[G]$ denotes the group ring of G over R. Any element in the ring $R[G]$ is a finite formal sum $\alpha = \sum_{g\in G} a_g g,\ \ a_g \in R$. The support of α in G is $\text{supp}(\alpha) = \{g \in G | a_g \neq 0\}$. Clearly, R can be regarded as a subring of $R[G]$. It is well known that

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if H and K are abelian groups, then $R[H \oplus K] = (R[H])[K]$ [17, p.134, Excercise 4]. For more details of the structure of the group rings we suggest [12], [18] and [17]. Note that if the order of every non-identity element in a group G is a power of the fixed prime p, then G is called a p-primary or a p-group.

A commutative ring R with non-zero unity is called *weakly-présimplifiable* (resp., *présimplifiable*) if for each two elements $a, b \in R$ such that $a = ab$, either $a = 0$ or b is a regular (resp., unit) in R. Domainlike rings (that is, all zero-divisors are nilpotent) and local rings and présimplifiable rings are examples of weakly-présimplifiable rings. We will present some known facts about weakly-présimplifiable rings throughout the paper. For more results on (weakly) présimplifiable ring, the reader is referred to [1], [3], and [8].

Let R be a ring and G a group. Let ρ be any ring theoretic property. It is a classical problem in algebra to find necessary and sufficient conditions on R and G so that the group ring $R[G]$ has the property ρ. Recently, D. D. Anderson et. al. in [3] studied a class of a présimplifiable ring of a commutative group ring, and they proved that, for a commutative ring R and an abelian torsion group G, the group ring $R[G]$ is présimplifiable, if and only if R is présimplifiable and G is p-primary with $p \in J(R)$.

In the light of that, motivated by aforementioned papers, we restrict ourselves here on the case when ρ is weakly-présimplifiable. We shall present necessary and sufficient conditions on G and R so that $R[G]$ is weakly-présimplifiable. For a nonzero torsion group G, we prove that $R[G]$ is weakly-présimplifiable if and only if R is weakly-présimplifiable of $char(R) = p^m$ where p is a prime number and $m \in \mathbb{N}$ and G is p-primary. Whenever G is torsion-free, $R[G]$ is weakly-présimplifiable if and only if R is weakly-présimplifiable. Finally, if G is mixed and $char(R) = p^m$, $R[G]$ is weakly-présimplifiable if and only if R is weakly-présimplifiable and the torsion subgroup of G is p-primary.

Furthermore, we give a complete description of (weakly) présimplifiable circulant matrix ring $CIRC_n(R)$. We prove that, $CIRC_n(R)$ is (resp.,weakly) présimplifiable where n is a power of a prime integer if and only if R is (resp., weakly) présimplifiable.

The results of this work on commutative group rings and circulant matrix ring well have very important applications in the algebraic coding theory, specially in studying cyclic codes and quasi cyclic codes. More generally in investigating many classes of linear codes over finite local rings [19].

2. Preliminaries

This part is consecrated on the elementary definitions and constructions needed throughout the paper.

Definition 2.1. *Let R be a commutative ring with unity. Then,*

(i) *R is said to be Présimplifiable if for $a, b \in R$ with $a = ab$, then either $a = 0$ or $b \in U(R)$.*

(ii) *R is said to be weakly-présimplifiable if for $a, b \in R$ with $a = ab$, then either $a = 0$ or $b \in reg(R)$.*

.

Examples of weakly-présimplifiable rings include présimplifiable rings, integral domains and local rings. A (weakly) présimplifiable ring R must be indecomposable (that is, R has no nontrivial idempotents). Thus, a direct product of (weakly) présimplifiable rings is never (weakly) présimplifiable. Also, it is known that $\mathbb{Z}_n$, the ring of integers modulo n, is (weakly) présimplifiable if and only if $n = p^m$ where p is a prime number and $m \in \mathbb{N}$, see [2], [3], and [7].

Lemma 2.2. [2, Theorem 6] *For a commutative ring R with unity. The following conditions are equivalent*

(i) *R is weakly présimplifiable.*

(ii) *$Z(R) \subseteq 1 - reg(R) \, (= 1 + reg(R))$.*

(iii) *For $a \in R$, either a or $a - 1$ is regular.*

(iv) *For $0 \neq r \in R$, $sRr = Rr$, implies s is regular.*

(v) *For (prime) ideals $P, Q \subseteq Z(R), P + Q \neq R$.*

(vi) *For $a, b \in Z(R)$, $< a, b > \neq R$.*

Theorem 2.3. [16, Theorem 3.2] *Let R be a ring with unity, $S = R[x, x^{-1}]$ and $f(x)$ an element of S. If $r_S(f(x)S) \neq 0$, then*

$$r_S(f(x)S) \cap R \neq 0.$$

Where $r_S(f(x)S) = \{s \in S | sf(x)S = \{0\}\}$ is the right annihilator of $f(x)S$ in S.

Lemma 2.4. *Let R be a commutative ring. If $g(x)$ is a zero-divisor in $R[x, x^{-1}]$, then there exists a nonzero element $c \in R$ such that $g(x)c = 0$.*

Proof. Let $g(x)$ be a zero-divisor in $S = R[x, x^{-1}]$. Then, $r_S(g(x)) \neq 0$ and $r_S(g(x)) \subseteq r_S(g(x)S)$. Hence, $r_S(g(x)S) \neq 0$ and Theorem 2.3 implies that $r_S(g(x)S) \cap R \neq 0$. Therefore, there exists $c \in R$ and $c \neq 0$ such that $g(x)c = 0$. □

Theorem 2.5. *Let R be a weakly-présimplifiable ring. Then,*

(i) *Every subring of R is weakly-présimplifiable.*
(ii) *R is indecomposable.*
(iii) *The characteristic of R is either zero or p^m for some prime number p and a positive integer m.*
(iv) *For any set of indeterminates $x_1; x_2; \ldots; x_n$ the polynomial ring $R[x_1; x_2; \ldots; x_n]$ is weakly-présimplifiable.*
(v) *For any set of indeterminates $x_1; x_2; \ldots; x_n$ the Laurent polynomial ring $R[x_1; x_2; \ldots x_n; x_1^{-1}; x_2^{-1}; \ldots; x_n^{-1}]$ is weakly-présimplifiable.*

Proof. (i) Assume R is a weakly-présimplifiable ring. Let S be a subring of R. Let $a, b \in S$ and $b \neq 0$ such that $ab = a$. Since $a, b \in S \subseteq R$ and R is weakly-présimplifiable, b is regular in R. Hence, b is regular in S. Therefore, S is weakly-présimplifiable.

(ii) Assume R is a weakly-présimplifiable ring. If R is not idecomposable, then it contains a nontrivial idempotent e (that is, $e^2 = e$, $e \neq 0$ and $e \neq 1$). So, $e(1 - e) = 0$. Hence, $e; e - 1$ are zero-divisors in R. It follows from Lemma 2.2(iii) that R is not a weakly-présimplifiable ring, which is a contradiction.

(iii) Let R be a weakly-présimplifiable ring. Assume that $char(R) \neq 0$. Namely, $char(R) = n$ where n is a positive integer. Clearly, R contains a subring isomorphic to Z_n. By using part(i), Z_n becomes weakly-présimplifiable. Now part (ii) asserts that Z_n is indecomposable. So, there is a prime number p and a positive integer m such that $n = p^m$. Thus, $char(R) = p^m$.

(iv) See [2, p.728, Theorem 18].

(v) Consider the ring $R[x; x^{-1}]$. Suppose that the Laurent polynomial ring $R[x; x^{-1}]$ is not weakly-présimplifiable. Lemma 2.2 (iii) implies that there exists $f(x) \in R[x, x^{-1}]$ such that $f(x)$ and $1 + f(x)$ are both zero-divisors. Let a be the constant term in $f(x)$. If $a = 0$, then Lemma 2.4 implies that there exists a nonzero element $c \in R$ such that $c(1 + f(x)) = 0$. So, being $1 + f(x)$ is a unit, $c = 0$, a contradiction. If $a \neq 0$, then by Lemma 2.4, being $f(x)$ and $1 + f(x)$ are both zero-divisors, there exists nonzero $c; c' \in R$ such that $cf(x) = 0$; and $c'(1 + f(x)) = 0$. Therefore, $ca = 0$, and $c'(1 + a) = 0$. Hence a, and $1 + a$ are zero-divisors in R. This contradicts the fact that R is weakly-présimplifiable. Thus, $R[x, x^{-1}]$ is a weakly-présimplifiable ring. Next, since $R[x_1; \dots x_n; x_1^{-1}; \dots; x_n^{-1}] = R[x_1; \dots x_{n-1}; x_1^{-1}, \dots; x_{n-1}^{-1}])[x_n; x_n^{-1}]$. and by induction argument, it follows that $R[x_1; \dots x_n; x_1^{-1}; \dots; x_n^{-1}]$ is a weakly-présimplifiable ring.

□

3. Circulant matrices

Let R be a commutative ring with unity and let n be a positive integer. If $v = (a_1; ...; a_n)$ is an n-tuple in R^n; then the $n \times n$ circulant matrix over the ring R, denoted by $CIRC_n(v)$ is a matrix in the form

$$CIRC_n(v) = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{bmatrix}$$

whose first row is v and the other rows are the cycle permutations of v.

Let $CIRC_n(R) = \{CIRC_n(v) \mid v \in R^n\}$. Then it is easy to show that $CIRC_n(R)$ is a commutative ring with unity under usual addition and multiplication of matrices over R. Now, as proved by D. D. Anderson et al. [4, Lemma 4.9.], if $A \in CIRC_n(R)$, then $A \in U(CIRC_n(R))$ (resp., $A \in reg(CIRC_n(R))$) if and only if $det(A) \in U(R)$ (resp., $det(A) \in reg(R)$). Also, Ghanem et al. [13, Theorem 1.1.] proved the following Lemma:

Lemma 3.1. [13, Theorem 1.1.] *Let R be a commutative ring with unity, $n = p^m$; where p is a prime integer, and let $v = (a_1, a_2, \cdots a_n) \in R^n$. Then,*

(i) $det(CIRC_n(v)) = (\sum_{i=1}^n \; a_i^n) + pt,\ t \in R$ *if p is odd;*
(ii) $det(CIRC_n(v)) = (\sum_{i=1}^n \; (-1)^i a_i^n) + pt,\ t \in R$ *if* $p = 2$.

This section is devoted to investigate the behavior of the ring of circulant matrices $CIRC_n(R)$ with respect to the (weakly) présimplifiable rings. It turns out that this is an important step towards solving our main question in the next section: When does the group ring $R[G]$ become weakly-présimplifiable?

Theorem 3.2. *Let R be a commutative ring with unity of $char(R) = p^m$ where p is a prime number and $m \in \mathbb{N}$. Then, $CIRC_{p^m}(R)$ is weakly présimplifiable if and only if R is weakly présimplifiable.*

Proof. Assume that R is weakly-présimplifiable and p is an odd prime. Let $A \neq 0$, $A \in CIRC_{p^m}(R)$ such that A is a zero-divisor. Lemma 2.2 indicates that it suffices to show that the matrix $A + I$ is a regular element in $CIRC_{p^m}(R)$. Set $A = CIRC(a_1, a_2, \cdots, a_n)$ where $n = p^m$. So,

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{bmatrix}$$

,

where $a_1, a_2, \cdots, a_n \in R$. Then,

$$A+I = \begin{bmatrix} a_1+1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1+1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1+1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1+1 \end{bmatrix}.$$

By Lemma 3.1 we have

$$det(A) = (\sum_{i=1}^{n} a_i^n) + pt_1. \tag{3.1}$$

for some $t_1 \in R$, and

$$det(A+I) = (\sum_{i=2}^{n} a_i^n) + (a_1+1)^n + pt_2. \tag{3.2}$$

for some $t_2 \in R$. Since $(a_1+1)^n = a_1^n + 1 + pt_3$ for some $t_3 \in R$, $det(A+I) = (\sum_{i=1}^{n} a_i^n) + 1 + pt_4$ for some $t_4 \in R$. Therefore,

$$det(A+I) - det(A) = 1 + pt_5 \tag{3.3}$$

for some $t_5 \in R$. Since $char(R) = p^m$, p is nilpotent in R. Thus, $1 + pt_5$ is a unit in R. being A is a zero-divisor, $det(A) = 0$ or $det(A)$ is a zero-divisor in R. If $det(A) = 0$, we have $det(A+I) = 1 + pt_5$ is a unit. Applying [4, Lemma 4.9] it follows that $A+I$ is regular, as desired. Suppose that $det(A)$ is a zero-divisor. Since $det(A+I) - det(A) = 1 + pt_5$, $1 + pt_5 \in < det(A+I) - det(A) >$. As $1 + pt_5$ is a unit in R, $< det(A+I) - det(A) >= R$. But R is weakly présimplifiable and $det(A)$ is a zero-divisor. Lemma 2.2(v) assures that $det(A+I)$ is not a zero-divisor. Further, if $det(A+I) = 0$, then $< det(A) >=< det(A+I) - det(A) >= R$. It implies that $det(A)$ and A are units, a contradiction. Thus, $det(A+I) \in reg(R)$ which implies that $A+I$ is regular by [4, Lemma 4.9]. Thus, if A is a zero-divisor then $A+I$ is regular. Therefore, $CIRC_n(R)$ is weakly présimplifiable. The same argument can be applied to the case $p = 2$. The converse follows directly since A is equivalent to a subring of $CIRC_n(R)$.

□

Proposition 3.3. Let D be an integral domain of characteristic zero and let $n = p^t$, where p is a prime number and $t \in \mathbb{N}$. Then, $CIRC_n(D)$ is weakly-présimplifiable if and only if p is a non-unit in D.

Proof. $\Rightarrow$: Assume that $CIRC_n(D)$ is weakly-présimplifiable. Then $CIRC_n(D)$ is indecomposable, if p is a unit in D, then n is unit and we have a nontrivial idempotent in $CIRC_n(D)$;

$$A = n^{-1} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

This contradicts the fact $CIRC_n(D)$ is indecomposable. Hence p must be non unit in D.

$\Leftarrow$: Assume that p is not a unit in D and $CIRC_n(D)$ is not weakly présimplifiable. Then, there exists a zero-divisor $A = CIRC(d_1, \ldots, d_n) \in CIRC_n(D)$ such that $I+A$ is also a zero-divisor in $CIRC_n(D)$. Now, we have $I + A = CIRC(1+d_1, d_2, \ldots, d_n)$. By [15, Corollary 1] we have $det(A)$ and $det(I + A)$ are zero-divisors in D. But since D is an integral domain, $det(A) = 0$ and $det(I + A) = 0$. By using the same method of the proof of Theorem 3.2 and Lemma 3.1, we conclude that $0 = det(A) = (\sum_{j=1}^{n} d_j{}^n) + pt_1$ and $0 = det(I + A) = 1 + (\sum_{j=1}^{n} d_j{}^n) + pt_2$ for some t_1, $t_2 \in D$. Therefore, being $\sum_{j=1}^{n} d_j{}^n = -pt_1$, we have $1 - pt_1 + pt_2 = 1 + pt_3$ for some $t_3 \in D$. This shows that p is a unit in D, a contradiction.

□

Theorem 3.4. *Let R be a commutative ring with unity of $char(R) = p^m$. If $n = p^m$, then $CIRC_n(R)$ is présimplifiable if and only if R is présimplifiable.*

Proof. Assume that R is présimplifiable. Let $A \in CIRC_n(R)$ such that A is a zero-divisor. It suffices to show that $A+I$ is a unit. By equations 3.1, 3.2 and 3.3 we find that

$$det(A + I) = det(A) + pt + 1$$

for some $t \in R$. Since A is a zero-divisor in R, $det(A)$ is a zero-divisor in R. And then $det(A) \in J(R)$ because R is présimplifiable. Since $char(R) = p^m$, p is niplotent and then $pt+1$ is a unit in R. Therefore, $det(A)+pt+1$ is unit in R. Thus, $det(A+I)$ is unit and it implies that $A+I$ is unit. So, $CIRC_n(R)$ is présimplifiable. The converse is clear. For the case $p = 2$, the same argument can be applied.

□

4. Group rings

Lemma 4.1. *Let R be a commutative ring with unity and H a nontrivial finite subgroup of a group G. If $|H| = m > 1$ such that $m \in U(R)$, then*

$$e = m^{-1}(\sum_{h\in H} h)$$

is a nontrivial idempotent in the group ring $R[G]$.

Proof. [17, Lemma 3.6.6, p.153].

□

Lemma 4.2. *Let G be an abelian group and R a commutative ring with unity. If $R[G]$ is weakly-présimplifiable, then $R[H]$ is weakly-présimplifiable for each subgroup H of G.*

Proof. Assume that $R[G]$ is weakly-présimplifiable and H be a subgroup of G. Then $R[H]$ is a subring of $R[G]$. Theorem 2.5(i) yields that $R[H]$ is weakly-présimplifiable.

□

Corollary 4.3. Let R be a commutative ring with unity and G be an abelian group. If $R[G]$ is weakly-présimplifiable, then R is weakly présimplifiable.

Proof. Put $H = \{e\}$, where e is the identity element in G, in Lemma 4.2. □

Lemma 4.4. [18, Lemma 1.4] *Let $a \in R[H] \subseteq R[G]$, where H is a subgroup of a group G. Then, a is a left (right) zero-divisor in $R[H]$ if and only if it is a left (right) zero-divisor in $R[G]$.*

Proposition 4.5. Let R be a commutative ring and G a nontrivial abelian group. If $R[H]$ is weakly-présimplifiable for each finitely generated subgroup H of G, then $R[G]$ is weakly présimplifiable.

Proof. Assume that $a, b \in R[G]$ and $ab = a$ such that $a \neq 0$. Let $H = \langle \text{supp}(b) \cup \text{supp}(a) \rangle$. Then, H is a finitely generated subgroup of G. So, $R[H]$ is a weakly-présimplifiable ring by our hypothesis. Since $a, b \in R[H]$ and $ab = a$ such that $a \neq 0$, b is regular in $R[H]$. Lemma 4.4 implies that b is regular in $R[G]$. Thus, $R[G]$ is a weakly-présimplifiable ring.

□

Theorem 4.6. *Let R be a commutative ring with unity of $char(R) = p^m$, $m \in \mathbb{N}$ where p is a prime number, and G be a cyclic p-primary group. Then, $R[G]$ is weakly présimplifiable if and only if R is weakly présimplifiable.*

Proof. $\Leftarrow$: Assume that G is a cyclic p-primary group. Let $n = |G| = p^t$, where t is a positive integer. Then, $R[G]$ is isomorphic to $CIRC_n(R)$ by [15, Theorem 1]. Hence $R[G]$ is weakly présimplifiable by Theorem 3.2.
$\Rightarrow$: Follows directly from Lemma 4.3.

□

Proposition 4.7. Let R be a commutative ring with unity and of $char(R) = p^m$, $m \in \mathbb{N}$ where p is a prime number, and G be a nonzero torsion abelian group. Then, $R[G]$ is weakly présimplifiable if and only if R is weakly présimplifiable and G is p-primary.

Proof. If $R[G]$ is weakly-présimplifiable, then Corollary 4.3 implies that R is weakly présimplifiable.
Suppose to the contrary that G is not p-primary. Let q be a prime number such that $q \neq p$ and G has an element g such that q divides the order of g. Let lq be the order of g where l is a positive integer. Cauchy's Theorem assures the existence of an element h in the subgroup $< g >$ such that $|h| = q$. Since p, q are different primes, $gcd(p, q) = 1$. Since $char(R) = p^m$, p is nilpotent in R. Therefore, q is a unit in R. Applying Lemma 4.1 yields that $e = q^{-1}(1 + h + \cdots + h^{q-1}) \in R[G]$ is a nontrivial idempotent, which

contradicts the fact that $R[G]$ weakly-présimplifiable. Therefore, G is a p-primary group. Conversely, assume that R is weakly présimplifiable and G is p-primary. By Proposition 4.5, without loss of generality, we can assume that G is finitely generated. Then, $G \cong C_{p^{m_1}} \times \cdots \times C_{p^{m_k}}$ for some positive integers $k, m_1, .., m_k$, where $C_{p^{m_i}}$ is the cyclic group of order p^{m_i}. Now, since $R[G] \cong R[C_{p^{m_1}} \times \cdots \times C_{p^{m_{k-1}}}][C_{p^{m_k}}]$, by using induction argument on k and using Theorem 4.6, it follows that $R[G]$ is weakly présimplifiable. □

Note that, if G is a finitely generated torsion-free abelian group, then $G \cong C_\infty^r$ where C_∞ is an infinite cyclic group and $r \in \mathbb{N}$. Moreover, $R[{C_\infty}^r]$ is isomorphic to the ring of Laurent power series $R[x_1, \ldots, x_r; x_1^{-1}; \ldots; x_r^{-1}]$.

Proposition 4.8. Let R be a commutative ring and G be a torsion-free abelian group. Then, $R[G]$ is weakly-présimplifiable if and only if R is weakly-présimplifiable

Proof. Assume that $R[G]$ is weakly-présimplifiable. From Corollary 4.3 it follows that R is weakly-présimplifiable. Conversely, by Proposition 4.5, without loss of generality, we can assume that G is finitely generated. So $G \cong C_\infty^r$ where $r \in \mathbb{N}$. Then, $R[G] \cong R[x_1; \ldots; x_t;\ x_1^{-1}, \ldots; x_t^{-1}]$. Theorem 2.5(v) yields that $R[G]$ is weakly-présimplifiable. □

Note that, if G is any group then we denote by $tor(G)$ the set of all periodic elements in G, that is, $tor(G) = \{g \in G \mid |g| < \infty\}$.

Proposition 4.9. Let R be a commutative ring with $char(R) = p^m$ where m is a positive integer and p is a prime, G a nontrivial abelian group. Then, $R[G]$ is weakly-présimplifiable if and only if R is weakly-présimplifiable and $tor(G)$ is either trivial or a p-primary group.

Proof. Assume R is weakly-présimplifiable such that $char(R) = p^m$ and $tor(G)$ is p-primary. By Proposition 4.5, and without loss of generality, we can assume that G is finitely generated. Then, we have two cases: case I: $tor(G)$ is trivial. It follows that G is torsion-free. Proposition 4.8 yields $R[G]$ is weakly-présimplifiable. case II: $tor(G)$ is non-trivial. Since G is finitely generated, we have that $G \cong {C_\infty}^r \times C_{p^{m_1}} \times \cdots \times C_{p^{m_k}}$ for some positive integers $r, k, m_1, .., m_k$. By Proposition 4.8,

$R[C_\infty^r]$ is weakly-présimplifiable. But we have $C_{p^{m_1}} \times \cdots \times C_{p^{m_k}}$ is p-primary. By [17, p. 134, Excecise 4], $A[C_\infty{}^r][C_{p^{m_1}} \times \cdots \times C_{p^{m_k}}] = R[G]$ and then Proposition 4.7 yields that $R[G]$ is weakly-présimplifiable. For the converse, assume that $R[G]$ is weakly présimplifiable, since $R[tor(G)]$ is a subring of $R[G]$, by using Theorem 2.5, we get that R is weakly-présimplifiable. From Proposition 4.8 it follows that $tor(G)$ is p-primary.

□

Proposition 4.10. Let R be a commutative ring with unity of $char(R) = 0$, and G an abelian group. If $R[G]$ is weakly-présimplifiable, then the order of each nontrivial finite subgroup (in particular, the order of each nontrivial periodic element) of G is non-unit in R.

Proof. Suppose to the contrary that H is a nontrivial finite subgroup of G and has order m and m is a unit in R. Lemma 4.1 implies that $R[G]$ has a nontrivial idempotent. Thus, $R[G]$ is not weakly-présimplifiable by Theorem 2.5(iii). Hence a contradiction.

□

Proposition 4.11. Let D be an integral domain of characteristic zero, and G be a mixed group such that the torsion subgroup of G is p-primary and p is non-unit in D where p is a prime number. Then, $D[G]$ is weakly-présimplifiable.

Proof. Let G_p be the torsion p-primary subgroup of G and p be a non-unit in D. So by Proposition 4.5 we can assume that G is finitely generated. Hence, $G \cong C_\infty^r \times G_p$. By [17, p. 134, Excecise 4], $D[G] = D[C_\infty{}^r][G_p]$. Since C_∞^r is a torsion free abelian group, $D' = D[C_\infty{}^r]$ is an integral domain. Now we have $D[G] = D'[G_p]$ isomorphic to the ring of all circulat matrices of size n by n over D' where $n = |G|$ [15, Theorem 1]. Therefore, $D[G]$ is weakly-présimplifiable by Theorem 3.3.

□

Proposition 4.12. Let F be a field with $char(F) = 0$ and G be a group. The following are equivalent:

(1) $F[G]$ is présimplifiable,

(2) $F[G]$ is weakly-présimplifiable, and

(3) G is torsion-free.

Proof. (1) $\Rightarrow$ (2) Since every présimplifiable is, clearly weakly présimplifiable. (2) $\Rightarrow$ (3) Assume G isn't torsion-free. Then there is an element in G of finite order m. Since $char(F) = 0$, m is a unit in F. Therefore, $F[G]$ has a nontrivial idempotent by Lemma 4.1. A contradiction, so G is torsion-free.
(3) $\Rightarrow$ (1) If G is torsion-free, then $F[G]$ is integral domain. Hence $F[G]$ is présimplifiable. □

Theorem 4.13. *Let R be a commutative ring, S a nonzero commutative cancellative torsion free monoid. Then, $R[S]$ is weakly présimplifiable if and only if R is weakly présimplifiable.*

Proof. Assume that R is weakly présimplifiable. Let $x \in R[S]$ be a zero-divisor, and suppose that $x + 1$ is a zero-divisor. If $1 \notin supp(x)$ and $x = x_1 s_1 + ... + x_m s_m$ where $x_1; ...; x_m \in R$ and $s_1; ...; s_m \in S$, then $x + 1 = 1 + x_1 s_1 + ... + x_m s_m$. Now by [14, Theorem 8.6], there is a non-zero element $a \in R$ such that $a(x+1) = 0$, which implies that $a = 0$, a contradiction.

If $1 \in supp(x)$ and $x = x_0 + x_1 s_1 + ... + x_m s_m$, then $x + 1 = (1 + x_0) + x_1 s_1 + ... + x_m s_m$. By using [14, Theorem 8.6], there are nonzero elements $a, a' \in R$ such that $a(x_0 + 1) = 0$ and $a' x_0 = 0$, which implies that $x_0, x_0 + 1$ are zero-divisors in R. Therefore R is not weakly présimplifiable, a contradiction. □

We conclude our work by suggesting that we will try to investigate presimplifiable semigroup rings $R[S]$, whrere R is a commutative ring with unity and S is a semi group.

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