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FRACTIONAL OSTROWSKI TYPE INEQUALITIES VIA $(s, r)-$CONVEX FUNCTION

ALI HASSAN [(1)] AND ASIF RAZA KHAN[(2)]

Abstract. We are introducing very first time a generalized class named it the class of $(s, r)-$convex functions in mixed kind. This generalized class contains many subclasses including class of $s-$convex functions in 1^{st} and 2^{nd} kind, $P-$convex functions, quasi convex functions and the class of ordinary convex functions. Also, we would like to state the generalization of the classical Ostrowski inequality via fractional integrals, which is obtained for functions whose first derivative in absolute values is $(s, r)-$ convex function in mixed kind. Moreover we establish some Ostrowski type inequalities via fractional integrals and their particular cases for the class of functions whose absolute values at certain powers of derivatives are $(s, r)-$ convex functions in mixed kind by using different techniques including Hölder's inequality and power mean inequality. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means would also be given.

1. Introduction

From literature, we recall and introduce some definitions for various convex (concave) functions.

Definition 1.1. [3] A function $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex (concave) function, if

$$\eta\,(tx + (1-t)y) \leq (\geq) t\eta(x) + (1-t)\eta(y),$$

$\forall x, y \in I, t \in [0, 1]$.

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We recall here definition of $P-$convex(concave) function from [14].

Definition 1.2. Let $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is a $P-$convex(concave) function, if η is a non-negative and $\forall x, y \in I$ and $t \in [0,1]$, we have

$$\eta\left(tx + (1-t)y\right) \leq (\geq) \eta(x) + \eta(y).$$

Here we also have definition of quasi$-$convex(concave) function (for detailed discussion see [16].

Definition 1.3. A function $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is known as quasi$-$convex(concave), if

$$\eta(tx + (1-t)y) \leq (\geq) \max\{\eta(x), \eta(y)\}$$

for all $x, y \in I$, $t \in [0,1]$.

Now we present definition of $s-$convex functions in the first kind as follows which are extracted from [22]:

Definition 1.4. Let $s \in [0,1]$. A function $\eta : I \subset [0,\infty) \to [0,\infty)$ is said to be $s-$convex (concave) function in the 1^{st} kind, if

$$\eta\left(tx + (1-t)y\right) \leq (\geq) t^s \eta(x) + (1-t^s)\eta(y),$$

$\forall x, y \in I, t \in [0,1]$.

Remark 1. *Note that in this definition we also included $s = 0$. Further if we put $s = 0$, we get quasi$-$convexity (see Definition 1.3).*

For second kind convexity we recall definition from [22].

Definition 1.5. Let $s \in [0,1]$. A function $\eta : I \subset [0,\infty) \to [0,\infty)$ is said to be $s-$convex (concave) function in the 2^{nd} kind, if

$$\eta\left(tx + (1-t)y\right) \leq (\geq) t^s \eta(x) + (1-t)^s \eta(y),$$

$\forall x, y \in I, t \in [0,1]$.

Remark 2. *In the similar manner, we have slightly improved definition of second kind convexity by including $s = 0$. Further if we put $s = 0$, we easily get $P-$convexity (see Definition 1.2).*

Definition 1.6. [5] Let $h_1, h_2 : [0,1] \to (0,\infty)$ and $m \in (0,1]$. The function $\eta : I \subset [0,\infty) \to \mathbb{R}$ is said to be $(m, h_1, h_2)-$convex(concave) function if

$$\eta\left(tx + m(1-t)y\right) \leq (\geq) h_1(t)\eta(x) + m h_2(t)\eta(y), \tag{1.1}$$

$\forall x, y \in I, t \in [0,1]$.

Now we introduce a new class of function which would be called class of $(s,r)-$ convex (concave) functions in the mixed kind:

Definition 1.7. Let $(s,r) \in [0,1]^2$. A function $\eta : I \subset [0,\infty) \to [0,\infty)$ is said to be $(s,r)-$convex (concave) function in mixed kind, if

$$\eta\left(tx + (1-t)y\right) \leq (\geq) t^{rs}\eta(x) + (1-t^r)^s \eta(y), \tag{1.2}$$

$\forall x, y \in I, t \in [0,1]$.

Remark 3. *In Definition* 1.7, *we have the following cases.*

1. *If we choose $s = 1$ in* (1.2), *we get $r-$convex (concave) in 1^{st} kind function.*
2. *If we choose $s = 1$, and $r = 0$, in* (1.2) *we get quasi$-$convex (concave) function.*
3. *If we choose $r = 1$, in* (1.2), *we get $s-$convex (concave) in 2^{nd} kind function.*
4. *If we choose $r = 1$, and $s = 0$ in* (1.2), *we get $P-$convex (concave) function.*
5. *If we choose $s = r = 1$ in* (1.2), *gives us ordinary convex (concave) function.*

In almost every field of science, inequalities play an important role. Although it is very vast discipline but our focus is mainly on Ostrowski type inequalities. In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives. This inequality is well known in the literature as Ostrowski inequality.

Theorem 1.1. [23] *Let $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be differentiable on (ρ_a, ρ_b) with the property that $|\varphi'(t)| \leq M$ for all $t \in (\rho_a, \rho_b)$. Then*

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} \varphi(t)dt\right| \leq (\rho_b - \rho_a)M\left[\frac{1}{4} + \left(\frac{x - \frac{\rho_a+\rho_b}{2}}{\rho_b - \rho_a}\right)^2\right], \tag{1.3}$$

for all $x \in (\rho_a, \rho_b)$.

Ostrowski inequality has applications in numerical integration, probability and optimization theory, statistics, information and integral operator theory. Until now, a large number of research papers and books have been written on generalizations of Ostrowski inequalities and their numerous applications in [7]-[13] and [17]-[21].

Definition 1.8. [25] The Riemann-Liouville integral operator of order $\zeta > 0$ with $\rho_a \geq 0$ is defined as

$$J^{\zeta}_{\rho_a}\varphi(x) = \frac{1}{\Gamma(\zeta)}\int_{\rho_a}^{x}(x-t)^{\zeta-1}\varphi(t)dt,$$

$$J^{0}_{\rho_a}\varphi(x) = \varphi(x). \tag{1.4}$$

In case of $\zeta = 1$, the fractional integral reduces to the classical integral.

Definition 1.9. [25] The Riemann-Liouville integrals $I^{\zeta}_{\rho_a^+}\varphi$ and $I^{\zeta}_{\rho_b^-}\varphi$ of $\varphi \in L_1([\rho_a, \rho_b])$ having order $\zeta > 0$ with $\rho_a \geq 0, \rho_a < \rho_b$ are defined by

$$I^{\zeta}_{\rho_a^+}\varphi(x) = \frac{1}{\Gamma(\zeta)}\int_{\rho_a}^{x}(x-t)^{\zeta-1}\varphi(t)dt,\ x > \rho_a$$

and

$$I^{\zeta}_{\rho_b^-}\varphi(x) = \frac{1}{\Gamma(\zeta)}\int_{x}^{\rho_b}(t-x)^{\zeta-1}\varphi(t)dt,\ x < \rho_b,$$

respectively. Here $\Gamma(\zeta) = \int_0^\infty e^{-u}u^{\zeta-1}du$ is the Gamma function and $I^{0}_{\rho_a^+}\varphi(x) = I^{0}_{\rho_b^-}\varphi(x) = \varphi(x)$.

Theorem 1.2. [25] *Let $\varphi : I \to \mathbb{R}$ be differentiable mapping on I^0, with $\rho_a, \rho_b \in I, \rho_a < \rho_b$ $\varphi' \in L_1[\rho_a, \rho_b]$ and for $\zeta > 1$, Montgomery identity for fractional integrals holds:*

$$\varphi(x) = \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta}J^{\zeta}_{\rho_a}\varphi(\rho_b) - J^{\zeta-1}_{\rho_a}(P_1(x,\rho_b)\varphi(\rho_b)) + J^{\zeta}_{\rho_a}(P_1(x,\rho_b)\varphi'(\rho_b)),$$

where $P_1(x,t)$ is the fractional Peano Kernel defined by:

$$P_1(x,t) = \begin{cases} \dfrac{t-\rho_a}{\rho_b-\rho_a}(\rho_b - x)^{1-\zeta}\Gamma(\zeta), & \text{if } t \in [\rho_a, x], \\[2ex] \dfrac{t-\rho_b}{\rho_b-\rho_a}(\rho_b - x)^{1-\zeta}\Gamma(\zeta), & \text{if } t \in (x, \rho_b]. \end{cases}$$

Let $[\rho_a, \rho_b] \subseteq (0, +\infty)$, we may define special means as follows:

(a) The arithmetic mean

$$A = A(\rho_a, \rho_b) := \frac{\rho_a + \rho_b}{2};$$

(b) The geometric mean

$$G = G(\rho_a, \rho_b) := \sqrt{\rho_a \rho_b};$$

(c) The harmonic mean

$$H = H(\rho_a, \rho_b) := \frac{2}{\frac{1}{\rho_a} + \frac{1}{\rho_b}};$$

(d) The logarithmic mean

$$L = L(\rho_a, \rho_b) := \begin{cases} \rho_a & \text{if } \rho_a = \rho_b \\ \dfrac{\rho_b - \rho_a}{\ln \rho_b - \ln \rho_a}, & \text{if } \rho_a \neq \rho_b \end{cases};$$

(e) The identric mean

$$I = I(\rho_a, \rho_b) := \begin{cases} \rho_a & \text{if } \rho_a = \rho_b \\ \dfrac{1}{e}\left(\dfrac{{\rho_b}^{\rho_b}}{{\rho_a}^{\rho_a}}\right)^{\frac{1}{\rho_b - \rho_a}}, & \text{if } \rho_a \neq \rho_b. \end{cases};$$

(f) The $p-$logarithmic mean

$$L_p = L_p(\rho_a, \rho_b) := \begin{cases} \rho_a & \text{if } \rho_a = \rho_b \\ \left[\dfrac{\rho_b^{p+1} - {\rho_a}^{p+1}}{(p+1)(\rho_a - \rho_b)}\right]^{\frac{1}{p}}, & \text{if } \rho_a \neq \rho_b. \end{cases};$$

where $p \in \mathbb{R} \setminus \{0, -1\}$.

In order to prove our main theorems, we need the following lemma that has been obtained in [26].

Lemma 1.1. *Let $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be a differentiable mapping on (ρ_a, ρ_b) with $a < b$. If $\varphi' \in L_1([\rho_a, \rho_b])$, then $x \in (\rho_a, \rho_b)$ the identity for fractional integrals holds:*

$$\left(\frac{(x-\rho_a)^\zeta + (\rho_b - x)^\zeta}{\rho_b - \rho_a}\right)\varphi(x) - \frac{\Gamma(\zeta+1)}{\rho_b - \rho_a}\left[I_{x^-}^\zeta \varphi(\rho_a) + I_{x^+}^\zeta \varphi(\rho_b)\right]$$
$$= \frac{(x-\rho_a)^{\zeta+1}}{\rho_b - \rho_a}\int_0^1 t^\zeta \varphi'(tx + (1-t)\rho_a)dt - \frac{(\rho_b - x)^{\zeta+1}}{\rho_b - \rho_a}\int_0^1 t^\zeta \varphi'(tx + (1-t)\rho_b)dt.$$

Throughout this paper, we denote

$$I(\varphi, x, \rho_a, \rho_b, \zeta) = \left(\frac{(x-\rho_a)^{\zeta} + (\rho_b - x)^{\zeta}}{\rho_b - \rho_a}\right)\varphi(x) - \frac{\Gamma(\zeta+1)}{\rho_b - \rho_a}\left[I^{\zeta}_{x^-}\varphi(\rho_a) + I^{\zeta}_{x^+}\varphi(\rho_b)\right],$$

$${}_{\zeta}\kappa^{\rho_b}_{\rho_a}(x) = \frac{(x-\rho_a)^{\zeta+1} + (\rho_b - x)^{\zeta+1}}{\rho_b - \rho_a}.$$

We also make use of Euler's beta function, which is for $x, y > 0$ defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The main aim of our study is to generalize the ostrowski inequality (1.3) for $(s,r)-$convex in mixed kind, which is given in Section 2. Moreover we establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $(s,r)-$convex functions in mixed kind by using different techniques including Hölder's inequality [28] and power mean inequality [27]. Also we give the special cases of our results and applications of midpoint inequalities in special means. In the last section gives us conclusion with some remarks and future ideas to generalize the results.

2. Generalization of Ostrowski inequality via fractional Integrals

Theorem 2.1. *Suppose all the assumptions of Lemma* 1.1 *hold. Additionally, assume that $|\varphi'|$ is $(s,r)-$convex function on $[\rho_a, \rho_b]$ and $|\varphi'(x)| \le M (M > 0)$, then*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \le M\left(\frac{1}{\zeta + rs + 1} + \frac{B\left(\frac{\zeta+1}{r}, s+1\right)}{r}\right) {}_{\zeta}\kappa^{\rho_b}_{\rho_a}(x).$$

$\forall x \in (\rho_a, \rho_b)$.

Proof. From the Lemma 1.1 we have

$$\begin{aligned} |I(\varphi, x, \rho_a, \rho_b, \zeta)| \le \; & \frac{(x-\rho_a)^{\zeta+1}}{\rho_b - \rho_a}\int_0^1 t^{\zeta}\left|\varphi'(tx + (1-t)\rho_a)\right| dt \\ & + \frac{(\rho_b - x)^{\zeta+1}}{\rho_b - \rho_a}\int_0^1 t^{\zeta}\left|\varphi'(tx + (1-t)\rho_b)\right| dt. \end{aligned} \tag{2.1}$$

Since $|\varphi'|$ is $(s,r)-$convex on $[\rho_a, \rho_b]$ and $|\varphi'(x)| \le M$, we have

$$\int_0^1 t^{\zeta}\left|\varphi'(tx + (1-t)\rho_a)\right| dt \le M\int_0^1 t^{\zeta}\left(t^{rs} + (1-t^r)^s\right) dt \tag{2.2}$$

and similarly

$$\int_0^1 t^{\zeta} |\varphi'(tx + (1-t)\rho_b)| \, dt \leq M \int_0^1 t^{\zeta} \left(t^{rs} + (1-t^r)^s\right) dt. \tag{2.3}$$

By using inequalities (2.2) and (2.3) in (2.1), we get

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq M \left(\frac{1}{\zeta + rs + 1} + \frac{B\left(\frac{\zeta+1}{r}, s+1\right)}{r} \right) {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

□

Corollary 2.1. *In Theorem* 2.1, *one can see the following.*

(1) If one takes $\alpha = \beta = s$ *and* $\gamma = \delta = 1$, *where* $s \in (0,1]$ *in inequality* (2.1), *then one has the Ostrowski inequality for* $s-$*convex functions in* 1^{st} *kind via fractional integrals:*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq M \left(\frac{1}{\zeta + s + 1} + \frac{B\left(\frac{\zeta+1}{s}, 2\right)}{s} \right) {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

(2) If one takes $\alpha = \delta = s$, *and* $\beta = \gamma = 1$, *where* $s \in (0,1]$ *in inequality* (2.1), *then one has the Ostrowski inequality for* $s-$*convex functions in* 2^{nd} *kind via fractional integrals:*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq M \left(\frac{1}{\zeta + s + 1} + B(\zeta+1, s+1) \right) {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

(3) If one takes $\alpha = \delta = s$, *and* $\zeta = \beta = \gamma = 1$, *where* $s \in (0,1]$ *in inequality* (2.1), *then one has the inequality* (2.1) *of Theorem* 2 *in* [1].

(4) If one takes $\alpha = \delta = s$, *and* $\beta = \gamma = 1$, *where* $s \in (0,1]$ *in inequality* (2.1), *then one has the inequality* (2.6) *of Theorem* 7 *in* [26].

(5) If one takes $\alpha = \delta = 0$, *and* $\beta = \gamma = 1$, *in inequality* (2.1), *then one has the Ostrowski inequality for* $P-$*convex functions via fractional integrals:*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq M \left(\frac{1}{\zeta + 1} + B(\zeta+1, 1) \right) {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

(6) If one takes $\alpha = \beta = \gamma = \delta = 1$, *in inequality* (2.1), *then one has the Ostrowski inequality for convex functions via fractional integrals:*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq M \left(\frac{1}{\zeta + 2} + B(\zeta+1, 2) \right) {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

(7) If one takes $\zeta = \alpha = \beta = \gamma = \delta = 1$, *in inequality* (2.1), *then one has the Ostrowski inequality* (1.3) *for convex function.*

Theorem 2.2. *Suppose all the assumptions of Lemma* 1.1 *hold. Additionally, assume that* $|\varphi'|^q$ *is* $(s,r)-$*convex function on* $[\rho_a,\rho_b], q\geq 1$ *and* $|\varphi'(x)|\leq M$*, then*

$$|I(\varphi,x,\rho_a,\rho_b,\zeta)|\leq \frac{M}{(\zeta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\zeta+rs+1}+\frac{B\left(\frac{\zeta+1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}\ {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x), \tag{2.4}$$

$\forall x\in(\rho_a,\rho_b)$.

Proof. From the Lemma 1.1 and using power mean inequality [27], we have

$$\begin{aligned}|I(\varphi,x,\rho_a,\rho_b,\zeta)|\leq{}& \frac{(x-\rho_a)^{\zeta+1}}{\rho_b-\rho_a}\left(\int_0^1 t^{\zeta}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\zeta}\left|\varphi'\left(tx+(1-t)\rho_a\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+\frac{(\rho_b-x)^{\zeta+1}}{\rho_b-\rho_a}\left(\int_0^1 t^{\zeta}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\zeta}\left|\varphi'\left(tx+(1-t)\rho_b\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned} \tag{2.5}$$

Since $|\varphi'|^q$ is $(s,r)-$convex on $[\rho_a,\rho_b]$. and $|\varphi'(x)|\leq M$, we get

$$\int_0^1 t^{\zeta}\left|\varphi'\left(tx+(1-t)\rho_a\right)\right|^q dt\leq M^q\int_0^1 t^{\zeta}\left(t^{rs}+(1-t^r)^s\right)dt \tag{2.6}$$

and

$$\int_0^1 t^{\zeta}\left|\varphi'\left(tx+(1-t)\rho_b\right)\right|^q dt\leq M^q\int_0^1 t^{\zeta}\left(t^{rs}+(1-t^r)^s\right)dt. \tag{2.7}$$

Using the inequalities (2.5) – (2.7), we get

$$|I(\varphi,x,\rho_a,\rho_b,\zeta)|\leq \frac{M}{(\zeta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\zeta+rs+1}+\frac{B\left(\frac{\zeta+1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}\ {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

□

Corollary 2.2. *In Theorem* 2.2, *one can see the following.*

1. *If one takes* $s=1$ *and* $r\in(0,1]$ *in inequality* (2.4), *then one has the Ostrowski inequality for* $r-$*convex functions in* 1^{st} *kind via fractional integrals:*

$$|I(\varphi,x,\rho_a,\rho_b,\zeta)|\leq \frac{M}{(\zeta+1)^{1-\frac{1}{q}}}\left(\frac{1}{\zeta+r+1}+\frac{B\left(\frac{\zeta+1}{r},2\right)}{r}\right)^{\frac{1}{q}}\ {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

(2) *If one takes $r = 1$ and $s \in (0, 1]$ in inequality (2.4), then one has the Ostrowski inequality for $s-$convex functions in 2^{nd} kind via fractional integrals:*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq \frac{M}{(\zeta+1)^{1-\frac{1}{q}}} \left(\frac{1}{\zeta+s+1} + B(\zeta+1, s+1)\right)^{\frac{1}{q}} {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

(3) *If one takes $r = 1$ and $s \in (0, 1]$ in inequality (2.4), then one has the inequality (2.8) of Theorem 9 in [26].*

(4) *If one takes $\zeta = r = 1$ and $s \in (0, 1]$ in inequality (2.4), then one has the inequality (2.3) of Theorem 4 in [1].*

(5) *If one takes $r = 1$ and $s = 0$ in inequality (2.4), then one has the Ostrowski inequality for $P-$convex functions via fractional integrals:*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq \frac{M}{(\zeta+1)^{1-\frac{1}{q}}} \left(\frac{1}{\zeta+1} + B(\zeta+1, 1)\right)^{\frac{1}{q}} {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

(6) *If one takes $r = s = 1$, in inequality (2.4), then one has the Ostrowski inequality for convex functions via fractional integrals:*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq \frac{M}{(\zeta+1)^{1-\frac{1}{q}}} \left(\frac{1}{\zeta+2} + B(\zeta+1, 2)\right)^{\frac{1}{q}} {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

Theorem 2.3. *Suppose all the assumptions of Lemma 1.1 hold. Additionally, assume that $|\varphi'|^q$ is $(s, r)-$convex function on $[\rho_a, \rho_b], q > 1$ and $|\varphi'(x)| \leq M (M > 0)$, then*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq \frac{M}{(\zeta p+1)^{\frac{1}{p}}} \left(\frac{1}{rs+1} + \frac{B\left(\frac{1}{r}, s+1\right)}{r}\right)^{\frac{1}{q}} {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x), \tag{2.8}$$

$\forall x \in (\rho_a, \rho_b)$, where $p^{-1} + q^{-1} = 1$.

Proof. From the Lemma 1.1 and using Hölder's inequality [28], we have

$$\begin{aligned} |I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq& \frac{(x-\rho_a)^{\zeta+1}}{\rho_b-\rho_a} \left(\int_0^1 t^{\zeta p} dt\right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(tx+(1-t)\rho_a)|^q dt\right)^{\frac{1}{q}} \\ &+ \frac{(\rho_b-x)^{\zeta+1}}{\rho_b-\rho_a} \left(\int_0^1 t^{\zeta p} dt\right)^{\frac{1}{p}} \left(\int_0^1 |\varphi'(tx+(1-t)\rho_b)|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{2.9}$$

Since $|\varphi'|^q$ is $(s, r)-$convex and $|\varphi'(x)| \leq M$, we have

$$\int_0^1 |\varphi'(tx+(1-t)\rho_a)|^q dt \leq M^q \int_0^1 t^{rs} + (1-t^r)^s dt \tag{2.10}$$

and

$$\int_0^1 |\varphi'(tx+(1-t)\rho_b)|^q dt \leq M^q \int_0^1 t^{rs} + (1-t^r)^s dt. \tag{2.11}$$

Using inequalities (2.9) – (2.11), we get

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq \frac{M}{(\zeta p+1)^{\frac{1}{p}}} \left(\frac{1}{rs+1} + \frac{B\left(\frac{1}{r}, s+1\right)}{r}\right)^{\frac{1}{q}} \ {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

□

Corollary 2.3. *In Theorem* 2.3, *one can see the following.*

(1) If one takes $s = 1$ *and* $r \in (0, 1]$ *in inequality* (2.8), *then one has the Ostrowski inequality for* $r-$*convex functions in* 1^{st} *kind via fractional integrals:*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq \frac{M}{(\zeta p+1)^{\frac{1}{p}}} \left(\frac{1}{r+1} + \frac{B\left(\frac{1}{r}, 2\right)}{r}\right)^{\frac{1}{q}} \ {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

(2) If one takes $r = 1$ *and* $s \in (0, 1]$ *in inequality* (2.8), *then one has the Ostrowski inequality for* $s-$*convex functions in* 2^{nd} *kind via fractional integrals:*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq \frac{M}{(\zeta p+1)^{\frac{1}{p}}} \left(\frac{1}{s+1} + B(1, s+1)\right)^{\frac{1}{q}} \ {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

(3) If one takes $r = 1$ *and* $s \in (0, 1]$ *in inequality* (2.8), *then one has the inequality* (2.7) *of Theorem* 8 *in* [26].

(4) If one takes $r = \zeta = 1$ *and* $s \in (0, 1]$ *in inequality* (2.8), *then one has the inequality* (2.2) *of Theorem* 3 *in* [1].

(5) If one takes $r = 1$ *and* $s = 0$ *in inequality* (2.8), *then one has the Ostrowski inequality for* $P-$*convex functions via fractional integrals:*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq \frac{M}{(\zeta p+1)^{\frac{1}{p}}} \left(1 + B(1, 1)\right)^{\frac{1}{q}} \ {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

(6) If one takes $s = r = 1$, *in inequality* (2.8), *then one has the Ostrowski inequality for convex functions via fractional integrals:*

$$|I(\varphi, x, \rho_a, \rho_b, \zeta)| \leq \frac{M}{(\zeta p+1)^{\frac{1}{p}}} \left(\frac{1}{2} + B(1, 2)\right)^{\frac{1}{q}} \ {}_{\zeta}\kappa_{\rho_a}^{\rho_b}(x).$$

Theorem 2.4. *Let* $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ *be differentiable on* (ρ_a, ρ_b), $\varphi' : [\rho_a, \rho_b] \to \mathbb{R}$ *be integrable on* $[\rho_a, \rho_b]$ *and* $\eta : I \subset \mathbb{R} \to \mathbb{R}$, *be a* $(s, r)-$*convex(concave) function in* *mixed sense, then we have the inequalities*

$$\begin{aligned}
&\eta\left[\varphi(x)-\frac{\Gamma(\zeta)}{\rho_b-\rho_a}(\rho_b-x)^{1-\zeta}J^{\zeta}_{\rho_a}\varphi(\rho_b)+J^{\zeta-1}_{\rho_a}(P_1(x,\rho_b)\varphi(\rho_b))\right]\\
&\leq(\geq)\frac{(\rho_b-x)^{1-\zeta}}{(\rho_b-\rho_a)^{rs}}\left[(x-\rho_a)^{rs-1}\int_{\rho_a}^{x}\eta\left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt\right.\\
&\left.+\frac{((\rho_b-\rho_a)^r-(x-\rho_a)^r)^s}{\rho_b-x}\int_{x}^{\rho_b}\eta\left[\frac{(t-\rho_b)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt\right], \qquad (2.12)
\end{aligned}$$

$\forall x\in[\rho_a,\rho_b]$.

Proof. Utilizing the Theorem 1.2, we get

$$\begin{aligned}
&\varphi(x)-\frac{\Gamma(\zeta)}{\rho_b-\rho_a}(\rho_b-x)^{1-\zeta}J^{\zeta}_{\rho_a}\varphi(\rho_b)+J^{\zeta-1}_{\rho_a}(P_1(x,\rho_b)\varphi(\rho_b))\\
&=J^{\zeta}_{\rho_a}(P_1(x,\rho_b)\varphi'(\rho_b))\\
&=\frac{1}{\Gamma(\zeta)}\int_{\rho_a}^{\rho_b}P_1(x,t)\frac{\varphi'(t)}{(\rho_b-t)^{1-\zeta}}dt\\
&=\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)\left[\frac{(\rho_b-x)^{1-\zeta}}{x-\rho_a}\int_{\rho_a}^{x}\frac{\{t-\rho_a\}\varphi'(t)}{(\rho_b-t)^{1-\zeta}}dt\right]\\
&+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)\right)\left[\frac{(\rho_b-x)^{1-\zeta}}{\rho_b-x}\int_{x}^{\rho_b}\frac{\{t-\rho_b\}\varphi'(t)}{(\rho_b-t)^{1-\zeta}}dt\right],
\end{aligned}$$

$\forall x\in[\rho_a,\rho_b]$. Next by using the $(s,r)-$convex(concave) function in mixed sense of $\eta:I\subset[0,\infty)\to\mathbb{R}$, we get

$$\begin{aligned}
&\eta\left[\varphi(x)-\frac{\Gamma(\zeta)}{\rho_b-\rho_a}(\rho_b-x)^{1-\zeta}J^{\zeta}_{\rho_a}\varphi(\rho_b)+J^{\zeta-1}_{\rho_a}(P_1(x,\rho_b)\varphi(\rho_b))\right]\\
&\leq(\geq)\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\alpha\gamma}\eta\left[\frac{(\rho_b-x)^{1-\zeta}}{x-\rho_a}\int_{\rho_a}^{x}\frac{\{t-\rho_a\}\varphi'(t)}{(\rho_b-t)^{1-\zeta}}dt\right]\\
&+\left(1-\left(\frac{x-\rho_a}{\rho_b-\rho_a}\right)^{\beta\gamma}\right)^{\delta}\eta\left[\frac{(\rho_b-x)^{1-\zeta}}{\rho_b-x}\int_{x}^{\rho_b}\frac{\{t-\rho_b\}\varphi'(t)}{(\rho_b-t)^{1-\zeta}}dt\right],
\end{aligned}$$

$\forall x\in[\rho_a,\rho_b]$. Applying Jensen's integral inequality [8], We get the Inequality (2.12). □

Corollary 2.4. *In Theorem* 2.4, *one can see the following.*

(1) *If one takes* $s = 1$ *and* $r \in (0,1]$ *in* (2.12), *then one has the Ostrowski inequality for* $r-$*convex(concave) functions in* 1^{st} *kind:*

$$\eta\left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta} J^{\zeta}_{\rho_a}\varphi(\rho_b) + J^{\zeta-1}_{\rho_a}(P_1(x,\rho_b)\varphi(\rho_b))\right]$$
$$\leq (\geq)\frac{(\rho_b - x)^{1-\zeta}}{(\rho_b - \rho_a)^r}\left[(x-\rho_a)^{r-1}\int_{\rho_a}^{x}\eta\left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b - t)^{1-\zeta}}\right]dt\right.$$
$$\left. + \frac{(\rho_b-\rho_a)^r - (x-\rho_a)^r}{(\rho_b - x)}\int_{x}^{\rho_b}\eta\left[\frac{(t-\rho_b)\varphi'(t)}{(\rho_b - t)^{1-\zeta}}\right]dt\right].$$

(2) *If one takes* $s = 1$ *and* $r = 0$ *in* (2.12), *we get quasi$-$convex(concave) function.*

$$\eta\left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta} J^{\zeta}_{\rho_a}\varphi(\rho_b) + J^{\zeta-1}_{\rho_a}(P_1(x,\rho_b)\varphi(\rho_b))\right]$$
$$\leq (\geq)\frac{(\rho_b - x)^{1-\zeta}}{(x - \rho_a)}\left[\int_{\rho_a}^{x}\eta\left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b - t)^{1-\zeta}}\right]dt\right].$$

(3) *If one takes* $r = 1$ *and* $s \in [0,1)$ (2.12), *then one has the Fractional Ostrowski type inequality for* $s-$*convex(concave) functions in* 2^{nd} *kind:*

$$\eta\left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta} J^{\zeta}_{\rho_a}\varphi(\rho_b) + J^{\zeta-1}_{\rho_a}(P_1(x,\rho_b)\varphi(\rho_b))\right]$$
$$\leq (\geq)\frac{(\rho_b - x)^{1-\zeta}}{(\rho_b - \rho_a)^s}\left[(x-\rho_a)^{s-1}\int_{\rho_a}^{x}\eta\left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b - t)^{1-\zeta}}\right]dt\right.$$
$$\left. + (\rho_b - x)^{s-1}\int_{x}^{\rho_b}\eta\left[\frac{(t-\rho_b)\varphi'(t)}{(\rho_b - t)^{1-\zeta}}\right]dt\right].$$

(4) *If one takes* $r = 1$ *and* $s = 0$ *in* (2.12), *then one has the Fractional Ostrowski type inequality for* $P-$*convex (concave) functions:*

$$\eta\left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta} J^{\zeta}_{\rho_a}\varphi(\rho_b) + J^{\zeta-1}_{\rho_a}(P_1(x,\rho_b)\varphi(\rho_b))\right]$$
$$\leq (\geq)(\rho_b - x)^{1-\zeta}\left[\frac{1}{x-\rho_a}\int_{\rho_a}^{x}\eta\left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b - t)^{1-\zeta}}\right]dt\right.$$
$$\left. + \frac{1}{\rho_b - x}\int_{x}^{\rho_b}\eta\left[\frac{(t-\rho_b)\varphi'(t)}{(\rho_b - t)^{1-\zeta}}\right]dt\right].$$

(5) *If one takes* $s = r = 1$ *in* (2.12), *then one has the Fractional Ostrowski type inequality for convex(concave) functions:*

$$\eta\left[\varphi(x) - \frac{\Gamma(\zeta)}{\rho_b - \rho_a}(\rho_b - x)^{1-\zeta} J^{\zeta}_{\rho_a}\varphi(\rho_b) + J^{\zeta-1}_{\rho_a}(P_1(x,\rho_b)\varphi(\rho_b))\right]$$
$$\leq (\geq)\frac{(\rho_b - x)^{1-\zeta}}{\rho_b - \rho_a}\left[\int_{\rho_a}^{x}\eta\left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b - t)^{1-\zeta}}\right]dt + \int_{x}^{\rho_b}\eta\left[\frac{(t-\rho_b)\varphi'(t)}{(\rho_b - t)^{1-\zeta}}\right]dt\right].$$

3. Applications of Midpoint Inequalties

If we replace φ by $-\varphi$ and $x = \frac{\rho_a+\rho_b}{2}$ in Theorem 2.4, we get

Theorem 3.1. *Let $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be differentiable on (ρ_a, ρ_b), $\varphi' : [\rho_a, \rho_b] \to \mathbb{R}$ be integrable on $[\rho_a, \rho_b]$ and $\eta : I \subset \mathbb{R} \to \mathbb{R}$, be a $(s, r)-$convex(concave) function in mixed sense, then*

$$\eta\left[\frac{\Gamma(\zeta)\left(\frac{\rho_b-\rho_a}{2}\right)^{1-\zeta}}{\rho_b-\rho_a} J^{\zeta}_{\rho_a}\varphi(\rho_b) - f\left(\frac{\rho_a+\rho_b}{2}\right) - J^{\zeta-1}_{\rho_a}\left(P_1\left(\frac{\rho_a+\rho_b}{2}, b\right)\varphi(\rho_b)\right)\right]$$
$$\le (\ge)\frac{2^{\zeta-1}}{(\rho_b-\rho_a)^{\zeta}}\left[\frac{1}{2^{sr-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_a}\eta\left[\frac{(t-\rho_a)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt\right.$$
$$\left.+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\rho_b}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{(t-\rho_b)\varphi'(t)}{(\rho_b-t)^{1-\zeta}}\right]dt\right]. \tag{3.1}$$

Remark 4. *In Theorem 3.1, if we put $\zeta = 1$ in (3.1). we get*

$$\eta\left(\frac{1}{\rho_b-\rho_a}\int_{\rho_a}^{\rho_b}\varphi(t)dt - \varphi\left(\frac{\rho_a+\rho_b}{2}\right)\right)$$
$$\le (\ge)\frac{1}{\rho_b-\rho_a}\left[\frac{1}{2^{sr-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta[(\rho_a-t)\varphi'(t)]dt\right.$$
$$\left.+\frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta[(\rho_b-t)\varphi'(t)]dt\right].$$

Remark 5. *Assume that $\eta : I \subset [0, \infty) \to \mathbb{R}$ be an $(s, r)-$convex(concave) function in mixed kind:*

(1) If we take $\zeta = 1, \varphi(t) = \frac{1}{t}$ in inequality (3.1) where $t \in [\rho_a, \rho_b] \subset (0, \infty)$, then we have

$$(\rho_b-\rho_a)\eta\left[\frac{A(\rho_a,\rho_b)-L(\rho_a,\rho_b)}{A(\rho_a,\rho_b)L(\rho_a,\rho_b)}\right]$$
$$\le (\ge)\frac{1}{2^{sr-1}}\int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}}\eta\left[\frac{t-\rho_a}{t^2}\right]dt + \frac{(2^r-1)^s}{2^{rs-1}}\int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b}\eta\left[\frac{t-\rho_b}{t^2}\right]dt.$$

(2) *If we take* $\zeta = 1, \varphi(t) = -\ln t$ *in inequality* (3.1)*, where* $t \in [\rho_a, \rho_b] \subset (0, \infty)$*, then we have*

$$(\rho_b - \rho_a)\eta \left[\ln\left(\frac{A(\rho_a, \rho_b)}{I(\rho_a, \rho_b)}\right)\right]$$
$$\leq (\geq) \frac{1}{2^{sr-1}} \int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}} \eta\left[\frac{t-\rho_a}{t}\right] dt + \frac{(2^r - 1)^s}{2^{rs-1}} \int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b} \eta\left[\frac{t-\rho_b}{t}\right] dt.$$

(3) *If we take* $\zeta = 1, \varphi(t) = t^p, p \in \mathbb{R} \setminus \{0, -1\}$ *in inequality* (3.1)*, where* $t \in [\rho_a, \rho_b] \subset (0, \infty)$*, then we have*

$$(\rho_b - \rho_a)\eta \left[L_p^p(\rho_a, \rho_b) + A^p(\rho_a, \rho_b)\right]$$
$$\leq (\geq) \frac{1}{2^{sr-1}} \int_{\rho_a}^{\frac{\rho_a+\rho_b}{2}} \eta\left[\frac{p(\rho_a - t)}{t^{1-p}}\right] dt + \frac{(2^r - 1)^s}{2^{rs-1}} \int_{\frac{\rho_a+\rho_b}{2}}^{\rho_b} \eta\left[\frac{p(\rho_b - t)}{t^{1-p}}\right] dt.$$

Remark 6. *In Theorem* 2.2*, one can see the following.*

(1) *Let* $x = \frac{\rho_a+\rho_b}{2}, \zeta = 1, 0 < \rho_a < \rho_b$, $q \geq 1$ *and* $\varphi : \mathbb{R} \to \mathbb{R}^+$, $\varphi(x) = x^n$ *in* (2.4)*. Then*

$$\left|A(\rho_a, \rho_b) - L_n^n(\rho_a, \rho_b)\right| \leq \frac{M(\rho_b - \rho_a)}{(2)^{2-\frac{1}{q}}} \left(\frac{1}{sr+2} + \frac{B\left(\frac{2}{r}, s+1\right)}{r}\right)^{\frac{1}{q}}.$$

(2) *Let* $x = \frac{\rho_a+\rho_b}{2}, \zeta = 1, 0 < \rho_a < \rho_b$, $q \geq 1$ *and* $\varphi : (0, 1] \to \mathbb{R}$, $\varphi(x) = -\ln x$ *in* (2.4)*. Then*

$$\left|\ln I(\rho_a, \rho_b) - \ln A(\rho_a, \rho_b)\right| \leq \frac{M(\rho_b - \rho_a)}{(2)^{2-\frac{1}{q}}} \left(\frac{1}{sr+2} + \frac{B\left(\frac{2}{r}, s+1\right)}{r}\right)^{\frac{1}{q}}.$$

Remark 7. *In Theorem* 2.3*, one can see the following.*

(1) *Let* $x = \frac{\rho_a+\rho_b}{2}, \zeta = 1, 0 < \rho_a < \rho_b$, $q \geq 1$ *and* $\varphi : \mathbb{R} \to \mathbb{R}^+$, $\varphi(x) = x^n$ *in* (2.8)*. Then*

$$\left|A(\rho_a, \rho_b) - L_n^n(\rho_a, \rho_b)\right| \leq \frac{M(\rho_b - \rho_a)}{2(p+1)^{\frac{1}{p}}} \left(\frac{1}{sr+1} + \frac{B\left(\frac{1}{r}, s+1\right)}{r}\right)^{\frac{1}{q}}.$$

(2) *Let* $x = \frac{\rho_a+\rho_b}{2}, \zeta = 1, 0 < \rho_a < \rho_b$, $q \geq 1$ *and* $\varphi : (0, 1] \to \mathbb{R}$, $\varphi(x) = -\ln x$ *in* (2.8)*. Then*

$$\left|\ln I(\rho_a, \rho_b) - \ln A(\rho_a, \rho_b)\right| \leq \frac{M(\rho_b - \rho_a)}{2(p+1)^{\frac{1}{p}}} \left(\frac{1}{sr+1} + \frac{B\left(\frac{1}{r}, s+1\right)}{r}\right)^{\frac{1}{q}}.$$

4. Conclusion and Remarks

4.1. **Conclusion.** Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $(s,r)-$convex functions in mixed kind, this class of functions contains many important classes including class of $s-$convex functions in 1^{st} and 2^{nd} kind [4], $P-$convex functions, quasi convex functions and the class of convex functions. We have stated our first main result in section 2, the generalization of Ostrowski inequality [23] via fractional integral and others results obtained by using different techniques including Hölder's inequality [28] and power mean inequality [27]. Also, various established results would be captured as special cases. Moreover, some applications in terms of special means would also be given.

4.2. **Remarks and Future Ideas.**

(1) One may do similar work to generalize all results stated in this article by applying weights.
(2) One may also do similar work by using various different classes of convex functions including Godunova-Levin $s-$convex function in 1^{st} and 2^{nd} kind and Godunova-Levin $(s,r)-$convex function in mixed kind and $h-$convex functions.
(3) One may try to state all results stated in this article for fractional integral with respect to another function.
(4) One may also state all results stated in this article for higher dimensions.
(5) One may also generalize all results using time scale domain.
(6) One may also generalize all results using Fractional sets, intutionistic Fractional sets and single valued Neutrosophic sets.

References

[1] M. Alomari, M. Darus, S.S. Dragomir and P. Cerone, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, *Appl. Math. Lett.*, **23**(2010), 1071–1076.

[2] A. Arshad and A. R. Khan, Hermite−Hadamard−Fejer Type Integral Inequality for $s-p-$Convex Functions of Several Kinds, *TJMM.*, **11 (2)** (2019), 25–40.

[3] E. F. Beckenbach, Convex functions, *Bull. Amer. Math. Soc.*, **54**(1948), 439–460.

[4] M. J. Vivas Cortez, J. E. Hernández, Ostrowski and Jensen-type inequalities via $(s,m)-$convex functions in the second sense, *Bol. Soc. Mat. Mex.*, **26**(2020), 287–302.

[5] M. J. Vivas Cortez, M. C. García, Ostrowski Type Inequalities for Functions Whose Derivative are $(m,h1,h2)$Convex, *Appl. Math. Inf. Sci.*, **11(1)**(2017), 79–86.

[6] M. J. Vivas Cortez, M. C. García, J. E. H. Hernández, Ostrowski-Type Inequalities for Functions Whose Derivative Modulus is Relatively $(m,h1,h2)-$Convex, *Appl. Math. Inf. Sci.* **13** (2019), 369–378.

[7] S. S. Dragomir, A Companion of Ostrowski's Inequality for Functions of Bounded Variation and Applications, *Int. J. Nonlinear Anal. Appl.*, **5** (2014), 89–97.

[8] S. S. Dragomir, A Functional Generalization of Ostrowski Inequality via Montgomery identity, *Acta Math. Univ. Comenianae, LXXXIV.*, **1** (2015), 63–78.

[9] S. S. Dragomir, On the Ostrowski's Integral Inequality for Mappings with Bounded Variation and Applications, *Math. Inequal. Appl.*, **4** (1) (2001), 59–66.

[10] S. S. Dragomir, Refinements of the Generalised Trapozoid and Ostrowski Inequalities for Functions of Bounded Variation, *Arch. Math.*, **91(5)** (2008), 450–460.

[11] S. S. Dragomir and N. S. Barnett, An Ostrowski Type Inequality for Mappings whose Second Derivatives are Bounded and Applications, *J. Indian Math. Soc. (N.S.)*, **66 (4)** (1999), 237–245.

[12] S. S. Dragomir, P. Cerone, N. S. Barnett and J. Roumeliotis, An Inequality of the Ostrowski Type for Double Integrals and Applications for Cubature Formulae, *Tamsui Oxf. J. Math. Sci.*, **16 (1)** (2000), 1–16.

[13] S. S. Dragomir, P. Cerone and J. Roumeliotis, A new Generalization of Ostrowski Integral Inequality for Mappings whose Derivatives are Bounded and Applications in Numerical Integration and for Special Means, *Appl. Math. Lett.*, **13** (2000), 19–25.

[14] S. S.Dragomir, J. Pečarić and L. Persson, Some inequalities of Hadamard type, *Soochow J. Math.*, **21 (3)**, (1995), 335–341.

[15] A. Ekinci, Klasik Eşitsizlikler Yoluyla Konveks Fonksiyonlar için Integral Eşitsizlikler, Ph.D. Thesis, Thesis ID: 361162 in tez2.yok.gov.tr Atatürk University, 2014.

[16] E. K. Godunova, V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions, *Numerical Mathematics and Mathematical Physics, (Russian)*, **166** (1985), 138–142.

[17] N. Irshad, A. R. Khan and A. Nazir, Extension of Ostrowki Type Inequality via Moment Generating Function, *Adv. Inequal. Appl.*, **2** (2020), 1–15.

[18] N. Irshad, A. R. Khan and M. A. Shaikh, Generalization of Weighted Ostrowski Inequality with Applications in Numerical Integration, *Adv. Ineq. Appl.*, **7** (2019), 1–14.

[19] N. Irshad, A. R. Khan, and Muhammad Awais Shaikh, Generalized Weighted Ostrowski-Gruss Type Inequality with Applications, *Global J. Pure Appl. Math.*, **15(5)** (2019), 675–692.

[20] N. Irshad and A. R. Khan, On Weighted Ostrowski Gruss Inequality with Applications, *TJMM*, **10 (1)** (2018), 15–22.

[21] N. Irshad and A. R. Khan, Generalization of Ostrowski Inequality for Differentiable functions and its applications to numerical quadrature rules, *J. Math. Anal*, **8 (1)** (2017), 79–102.

[22] M. A. Noor and M. U. Awan, Some integral inequalities for two kinds of convexities via fractional integrals, *TJMM*, **5** (2013), 129 – 136.

[23] A. M. Ostrowski, Uber die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert, *Comment. Math. Helv.*, **10** (1938), 226–227.

[24] D. P Shi, B.Y Xi, F. Qi, Hermite-Hadamard Type Inequalities for (m,h1,h2)-Convex Functions Via Riemann Liouville Fractional Integrals, *Turkish Journal of Analysis and Number Theory*, **2** (2014), 23-28.

[25] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives, *Theory and Applications Gordon and Breach New York*, **1** (1993).

[26] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are $s-$convex in the second sense via fractional integrals. *Comput Math Appl*, **63** (2012), 1147–1154.

[27] Z. G. Xiao, and A. H. Zhang, Mixed power mean inequalities, *Research Communication on Inequalities*, **8 (1)** (2002), 15–17.

[28] X. Yang, A note on Hölder inequality, *Appl. Math. Comput.*, **134** (2003), 319–322.

(1) Department of Mathematics, Shah Abdul Latif University Khairpur- 66020, Pakistan

Email address: alihassan.iiui.math@gmail.com

(2) Department of Mathematics, University of Karachi, University Road, Karachi-75270, Pakistan

Email address: asifrk@uok.edu.pk