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ALMOST GENERALIZED QUADRATIC FUNCTIONS OF THREE VARIABLES IN LIPSCHITZ SPACES

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Abstract. The notion of stability of functional equations was posed by Ulam, and then, Hyers gave the first significant partial solution. This type of stability has been established and developed by an increasing number of mathematicians in various spaces. In Lipschitz spaces, the notion of stability was introduced by Tabor and Czerwik. This notion has been considered less attention over the recent years in Lipschitz spaces. In this paper, we consider this type of stability and we prove the stability of the generalized quadratic functional equations of three variables in Lipschitz spaces. We generalize the stability of a quadratic functional equation from a special case to a general case and improve its approximation announced by Czerwik et al. in [4].

1. Introduction

The study of stability problems for functional equations concerning the stability of group homomorphisms was raised by a question of Ulam [22] and affirmatively was answered for Banach spaces by Hyers [9].

A stability problem for the following quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

was solved by Skof for functions $f:E_1\longrightarrow E_2$, where E_1 is a normed space and E_2 a Banach space (see [19]).

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The general solution and the stability of the following 2-variable quadratic functional equation

$$f(x+z, y+w) + f(x-z, y-w) = 2f(x, y) + 2f(z, w) \tag{1.2}$$

was established in complete normed spaces [1]. Ravi et al. [18] discussed the general solution and the stability of the 3-variable quadratic functional equation

$$f(x+y, z+w, u+v) + f(x-y, z-w, u-v) = 2f(x, z, u) + 2f(y, w, v). \tag{1.3}$$

In [5, 4], Czerwik et al. proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (1.1) in normed and Lipschitz spaces, respectively. The stability type problems for some functional equations were also studied by Tabor [20, 21] in Lipschitz spaces. The author of the present paper proved the stability of the cubic and quartic functional equations in Lipschitz spaces (see [7], [11]-[16]).

Any solution of (1.1) in the space of real numbers is of the form $h(x) = ax^2$ for all $x \in \mathbb{R}$, where $a \in \mathbb{R}$. Any solution of (1.2) is termed as a quadratic mapping. If $X = Y = \mathbb{R}$, the quadratic form $g(x, y) = ax^2 + bxy + cy^2$ is a solution of (1.2) for all $x, y \in \mathbb{R}$, where $a, b, c \in \mathbb{R}$. When $b = c = 0$, we obtain the quadratic form $h(x) = ax^2$ satisfying (1.1). So, any solution of (1.1) is a solution of (1.2) when viewed as a function of two variables, but not vice versa. Any solution of (1.3) in the space of real numbers is of the form

$$k(x, y, z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz \tag{1.4}$$

for all $x, y, z \in \mathbb{R}$, where $a_1, ..., a_6 \in \mathbb{R}$. It is remarkable that any solution of either (1.1) or (1.2) is a solution of (1.3), but not vice versa.

The stability of the quadratic functional equation (1.1) was proved by Czerwik et al. in Lipschitz spaces [4]. We remark that the solution of (1.1) is of the form $h(x) = ax^2$. Our motivation is to find the stability of a quadratic form of the general form (1.4). Note that the general form (1.4) contains any quadratic form of the special case $h(x) = ax^2$. For finding the stability of a quadratic form of the general form (1.4), we introduce the notion of the 3-quadratic functions and then we verify stability of these functions in Lipschitz spaces. Since a function f satisfies equation (1.1) if and only if f satisfies equation (1.5) (see [8]), our results recover the results announced by Czerwik et al. in [4]. The advantage of our discussion is improving the approximation that Czerwik et al. obtained in [4]. This improvement is possible by considering a suitable set valued function which we will denote it by **d**.

Rassias [17] and Najati [10] proved the Hyers–Ulam stability problem for the following quartic functional equation

$$f(2x+y)+f(2x-y)=4(f(x+y)+f(x-y))+24f(x)-6f(y)$$

for mappings from a linear space into a Banach space. Recently, EL-Fassi [8] investigated the general solution and the stability of the generalized qaudratic equation

$$f(x+y)+f(x-y)=f(x+2y)+f(x-2y)-6f(y) \tag{1.5}$$

in Lipschitz spaces. Moreover, the stability of generalized multi-quadratic mappings was proved in Lipschitz spaces [6].

We introduce the generalized quadratic functional equation of three variables as follows:

$$\begin{aligned} &f(x+y,z+w,u+v)+f(x-y,z-w,u-v)\\ &=f(x+2y,z+2w,u+2v)+2f(x-2y,z-2w,u-2v)-6f(y,w,v). \end{aligned} \tag{1.6}$$

We say that f is 3-quadratic if f satisfies (1.3) and f is generalized 3-quadratic if f satisfies (1.6). In this paper, we verify the stability of the generalized quadratic functional equation of three variables in Lipschitz spaces.

We start by introducing some quite standard notation. Let $G\times G\times G$ be the Cartesian product of an abelian group G with itself and denote by G^3. We denote By $B(G^3,S(V))$ the subset of all functions $f:G^3\longrightarrow V$ such that $\mathtt{Im}f\subset A$ for some $A\in S(V)$, where V is a vector space, $S(V)$ is a subset of $P(V)$, and $P(V)$ is the power set of V. The family $B(G^3,S(V))$ is a vector space and contains all constant functions.

Definition 1.1. The family $B(G^3,S(V))$ admits a left invariant mean (briefly LIM), if the subset $S(V)$ is linearly invariant, in the sense that $A+B\in S(V)$ for all $A,B\in S(V)$ and $x+\alpha A\in S(V)$ for all $x\in V$, $\alpha\in\mathbb{R}$, $A\in S(V)$ (see[2]), and there exists a linear operator $\Lambda: B(G^3,S(V))\longrightarrow V$ such that

(i) if $\mathtt{Im}f \subset A$ for some $A \in S(V)$, then $\Lambda[f] \in A$,

(ii) if $f \in B(G^3, S(V))$ and $(a,b,c) \in G^3$, then $\Lambda[f^{a,b,c}] = \Lambda[f]$,

where $f^{a,b,c}(x,y,z) = f(x+a, y+b, z+c)$.

Let $\mathbf{d} : G^3 \times G^3 \longrightarrow S(V)$ be a set-valued function such that

$$\mathbf{d}((x+a, y+b, z+c), (u+a, v+b, w+c)) = \mathbf{d}((x,y,z),(u,v,w))$$

for all $(a,b,c),(x,y,z),(u,v,w) \in G^3$ (cf. [3, 20]).

Definition 1.2. We say that a function $f : G^3 \longrightarrow V$ is $\mathbf{d}$-Lipschitz if

$$f(x,y,z) - f(u,v,w) \in \mathbf{d}((x,y,z),(u,v,w))$$

for all $(x,y,z),(u,v,w) \in G^3$.

In particular, when (G^3, d) is a metric group and V a normed space, we define the function $MC_f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ to be a module of continuity of the function $f : G^3 \longrightarrow V$ if for all $\delta > 0$ and all $(x,y,z),(u,v,w) \in G^3$ the condition $d((x,y,z),(u,v,w)) \leq \delta$ implies $||f(x,y,z) - f(u,v,w)|| \leq MC_f(\delta)$.

Definition 1.3. A function $f : G^3 \longrightarrow V$ is called Lipschitz function if there exists a constant $L > 0$ such that

$$||f(x,y,z) - f(u,v,w)|| \leq Ld((x,y,z),(u,v,w)) \tag{1.7}$$

for every $(x,y,z),(u,v,w) \in G^3$.

We consider $Lip(G^3, V)$ to be the Lipschitz space consisting of all bounded Lipschitz functions with the norm

$$||f||_{Lip} := ||f||_s + \mathtt{P}(f),$$

where $||.||_s$ is the supremum norm and

$$\mathtt{P}(f) = \sup\left\{\frac{||f(x,y,z) - f(u,v,w)||}{d((x,y,z),(u,v,w))} : (x,y,z),(u,v,w) \in G^3, (x,y,z) \neq (u,v,w)\right\}.$$

The function $||\cdot||_{Lip}$ is in fact a norm. Since $||.||_s$ is a norm, it is enough to show that the function $\mathtt{P}(\cdot)$ is a norm. Let $f \in Lip(G^3, V)$ and $\alpha \in \mathbb{R}$. Then it is clear that

(1) $\mathtt{P}(f) = 0$ if and only if $f = 0$.

(2) $\mathtt{P}(\alpha f) = |\alpha|\mathtt{P}(f)$.

(3) Let $f, g \in Lip(G^3, V)$. For every $(x,y,z),(u,v,w) \in G^3, (x,y,z) \neq (u,v,w)$ we have

$$\begin{aligned} ||(f+g)(x,y,z) - (f+g)(u,v,w)|| & \\ &= ||(f(x,y,z) - f(u,v,w)) + (g(x,y,z) - g(u,v,w))|| \\ &\leq ||(f(x,y,z) - f(u,v,w))|| + ||(g(x,y,z) - g(u,v,w))||. \end{aligned} \tag{1.8}$$

Define

$$A := \Big\{ \frac{||f(x,y,z) - f(u,v,w)||}{d((x,y,z),(u,v,w))} : (x,y,z),(u,v,w) \in G^3, (x,y,z) \neq (u,v,w) \Big\},$$

$$B := \Big\{ \frac{||g(x,y,z) - g(u,v,w)||}{d((x,y,z),(u,v,w))} : (x,y,z),(u,v,w) \in G^3, (x,y,z) \neq (u,v,w) \Big\},$$

Since $f, g \in Lip(G^3, V)$, the sets A and B are upper bounded sets. So, using (1.8), we get

$$\begin{aligned} \mathtt{P}(f+g) &\leq \sup(A+B) \\ &= \sup(A) + \sup(B) \\ &= \mathtt{P}(f) + \mathtt{P}(g). \end{aligned}$$

Lipschitz spaces have a rich algebra structure and can be worked on for future research. Some of open problems in this area are given in chapter 7 of [23].

2. Stability in Lipschitz spaces

For a given function $f : G^3 \longrightarrow V$ we define its generalized 3-variable quadratic difference as follows:

$$\begin{aligned} Q_f(x,y,z,u,v,w) &:= f(x+2u, y+2v, z+2w) + f(x-2u, y-2v, z-2w) \\ &\quad - f(x+u, y+v, z+w) - f(x-u, y-v, z-w) - 6f(u,v,w) \end{aligned}$$

for all $(x,y,z),(u,v,w) \in G^3$. Note that f is generalized 3-quadratic if and only if $Q_f = 0$.

Lemma 2.1. *Let G be an Abelian group. If $f : G^3 \longrightarrow V$ is a function and $Q_f(r,t,s,\cdot,\cdot,\cdot) : G^3 \longrightarrow V$ is $\boldsymbol{d}$-Lipschitz for all $(r,t,s) \in G^3$, then*

$$\mathit{Im}(\alpha Q_f(\cdot,\cdot,\cdot,x,y,z) - \alpha Q_f(\cdot,\cdot,\cdot,u,v,w)) \subseteq \alpha \boldsymbol{d}((x,y,z),(u,v,w))$$

for all $(x,y,z),(u,v,w)\in G^3$, *and* $\alpha\in\mathbb{R}$.

Proof. Fix $(r,t,s)\in G^3$. By assumption we have

$$\alpha Q_f(r,t,s,x,y,z)-\alpha Q_f(r,t,s,u,v,w)\in\alpha\mathbf{d}((x,y,z),(u,v,w)) \tag{2.1}$$

for all $(x,y,z),(u,v,w)\in G^3$, and $\alpha\in\mathbb{R}$. From this we deduce that

$$\mathtt{Im}(\alpha Q_f(\cdot,\cdot,\cdot,x,y,z)-\alpha Q_f(\cdot,\cdot,\cdot,u,v,w))\subseteq\alpha\mathbf{d}((x,y,z),(u,v,w)).$$

for all $(x,y,z),(u,v,w)\in G^3$, and $\alpha\in\mathbb{R}$. □

We generalize the stability of a quadratic functional equation from a special case to a general case.

Theorem 2.1. *Let G be an Abelian group and V a vector space. Assume that the family $B(G^3,S(V))$ admits LIM. If $f:G^3\longrightarrow V$ is a function and $Q_f(r,t,s,\cdot,\cdot,\cdot):G^3\longrightarrow V$ is $\boldsymbol{d}$-Lipschitz for all $(r,t,s)\in G^3$, then there exists a generalized 3-quadratic function S such that $f-S$ is $\frac{1}{6}\boldsymbol{d}$-Lipschitz.*

Proof. Since the family $B(G^3,S(V))$ admits LIM, there exists a linear operator $\Lambda:B(G^3,S(V))\longrightarrow V$ such that

(i) $\Lambda[H]\in A$ for some $A\in S(V)$,
(ii) if for $(u,v,w)\in G^3$, $H^{u,v,w}:G^3\longrightarrow V$ is defined by $H^{u,v,w}(r,t,s):=H(r+u,t+v,s+w)$ for every $(r,t,s)\in G^3$, then $H^{u,v,w}\in B(G^3,S(V))$ and $\Lambda[H]=\Lambda[H^{u,v,w}]$.

Consider the function $H_{a,b,c}:G^3\longrightarrow V$ by

$$\begin{aligned}H_{a,b,c}(x,y,z):=&\frac{1}{6}f(x+2a,y+2b,z+2c)+\frac{1}{6}f(x-2a,y-2b,z-2c)\\&-\frac{1}{6}f(x+a,y+b,z+c)-\frac{1}{6}f(x-a,y-b,z-c)\end{aligned}$$

for all $(a, b, c) \in G^3$. We have

$$\begin{aligned} H_{a,b,c}(x, y, z) = & \frac{1}{6}[f(x + 2a, y + 2b, z + 2c) + f(x - 2a, y - 2b, z - 2c) \\ & - f(x + a, y + b, z + c) - f(x - a, y - b, z - c) - 6f(a, b, c)] \\ & - \frac{1}{6}[f(x, y, z) + f(x, y, z) - f(x, y, z) - f(x, y, z) - 6f(0, 0, 0)] \\ & + f(a, b, c) - f(0, 0, 0) \\ & = \frac{1}{6}Q_f(x, y, z, a, b, c) - \frac{1}{6}Q_f(x, y, z, 0, 0, 0) + f(a, b, c) - f(0, 0, 0). \end{aligned}$$

for all $(x, y, z), (a, b, c) \in G^3$. This implies $H_{a,b,c} \in B(G^3, S(V))$. Then, $\mathtt{Im}\ H_{a,b,c} \subset A$, where $A := \frac{1}{6}\mathbf{d}((a, b, c), (0, 0, 0)) + f(a, b, c) - f(0, 0, 0) \in S(V)$.

We know that $B(G^3, S(V))$ contains constant functions. By using property (i) of Λ it is easy to verify that if $f : G^3 \longrightarrow V$ is constant, i.e., $f(x, y, z) = k$ for $(x, y, z) \in G^3$, where $k \in V$, then $\Lambda[f] = k$. Define the function $S : G^3 \longrightarrow V$ by $S(x, y, z) := \Lambda[H_{x,y,z}]$ for $(x, y, z) \in G^3$.

We now show that $f - S$ is $\frac{1}{6}\mathbf{d}$-Lipschitz. For every $(x, y, z) \in G^3$ define the constant function $k_{x,y,z} : G^3 \longrightarrow V$ by $k_{x,y,z}(u, v, w) := f(x, y, z)$ for all $(u, v, w) \in G^3$. Then,

$$\begin{aligned} & (f(x, y, z) - S(x, y, z)) - (f(u, v, w) - S(u, v, w)) \\ & \qquad = (\Lambda[k_{x,y,z}] - \Lambda[H_{x,y,z}]) - (\Lambda[k_{u,v,w}] - \Lambda[H_{u,v,w}]) \\ & \qquad = \Lambda[k_{x,y,z} - H_{x,y,z}] - \Lambda[k_{u,v,w} - H_{u,v,w}] \\ & \qquad = \Lambda[\frac{1}{6}Q_f(\cdot, \cdot, \cdot, x, y, z) - \frac{1}{6}Q_f(\cdot, \cdot, \cdot, u, v, w)] \end{aligned}$$

for all $(x, y, z), (u, v, w) \in G^3$.

Due to Lemma 2.1 and property (i) of Λ we have

$$\Lambda[\frac{1}{6}Q_f(\cdot, \cdot, \cdot, x, y, z) - \frac{1}{6}Q_f(\cdot, \cdot, \cdot, u, v, w)] \in \frac{1}{6}\mathbf{d}((x, y, z), (u, v, w))$$

for all $(x, y, z), (u, v, w) \in G^3$. So,

$$(f(x, y, z) - S(x, y, z)) - (f(u, v, w) - S(u, v, w)) \in \frac{1}{6}\mathbf{d}((x, y, z), (u, v, w))$$

for all $(x, y, z), (u, v, w) \in G^3$, i.e., $f - S$ is a $\frac{1}{6}\mathbf{d}$-Lipschitz function. We have

$$2S(x, y, z) + 2S(u, v, w) = 2\Lambda[H_{x,y,z}(r, t, s)] + 2\Lambda[H_{u,v,w}(r, t, s)].$$

By using property (ii) of Λ, we obtain

$$\Lambda[H_{x,y,z}] = \Lambda[H_{x,y,z}^{u,v,w}], \qquad \Lambda[H_{x,y,z}] = \Lambda[H_{x,y,z}^{-u,-v,-w}]$$
$$\Lambda[H_{u,v,w}] = \Lambda[H_{u,v,w}^{2x,2y,2z}], \ \Lambda[H_{u,v,w}] = \Lambda[H_{u,v,w}^{-2x,-2y,-2z}]$$

for $(u, v, w) \in G^3$ and hence

$$\begin{aligned} 2S(x,y,z) + 2S(u,v,w) &= 2\Lambda[H_{x,y,z}] + 2\Lambda[H_{u,v,w}] \\ &= \Lambda[H_{x,y,z}^{u,v,w}] + \Lambda[H_{x,y,z}^{-u,-v,-w}] + \Lambda[H_{u,v,w}^{2x,2y,2z}] \\ &+ \Lambda[H_{u,v,w}^{-2x,-2y,-2z}]. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\Lambda[H_{x,y,z}^{u,v,w}] + \Lambda[H_{x,y,z}^{-u,-v,-w}] + \Lambda[H_{u,v,w}^{2x,2y,2z}] + \Lambda[H_{u,v,w}^{-2x,-2y,-2z}] \\ &= \Lambda[\frac{1}{6}f(r+2x+u, t+2y+v, s+2z+w) + \frac{1}{6}f(r-2x+u, t-2y+v, s-2z+w) \\ &- \frac{1}{6}f(r+x+u, t+y+v, s+z+w) - \frac{1}{6}f(r-x+u, t-y+v, s-z+w)] \\ &+ \Lambda[\frac{1}{6}f(r+2x-u, t+2y-v, s+2z-w) + \frac{1}{6}f(r-2x-u, t-2y-v, s-2z-w) \\ &- \frac{1}{6}f(r+x-u, t+y-v, s+z-w) - \frac{1}{6}f(r-x-u, t-y-v, s-z-w)] \\ &+ \Lambda[\frac{1}{6}f(r+2u+2x, t+2v+2y, s+2w+2z) + \frac{1}{6}f(r-2u+2x, t-2v+2y, s-2w+2z) \\ &- \frac{1}{6}f(r+u+2x, t+v+2y, s+w+2z) - \frac{1}{6}f(r-u+2x, t-v+2y, s-w+2z)] \\ &+ \Lambda[\frac{1}{6}f(r+2u-2x, t+2v-2y, s+2w-2z) + \frac{1}{6}f(r-2u-2x, t-2v-2y, s-2w-2z) \\ &- \frac{1}{6}f(r+u-2x, t+v-2y, s+w-2z) - \frac{1}{6}f(r-u-2x, t-v-2y, s-w-2z)] \\ &= S(x+u, y+v, z+w) + S(x-u, y-v, z-w). \end{aligned}$$

This means that S is 3-variable quadratic. It remains to prove that S is generalized 3-variable quadratic and so $Q_S = 0$.

$$\begin{aligned}
&S(x+2u, y+2v, z+2w) + S(x-2u, y-2v, z-2w) - 6S(u,v,w)\\
&\quad = S(x+u+u, y+v+v, z+w+w) + S(x-u-u, y-v-v, z-w-w) - 6S(u,v,w)\\
&\quad = S(x+u+u, y+v+v, z+w+w) + S(x,y,z) - S(x,y,z)\\
&\qquad + S(x-u-u, y-v-v, z-w-w) + S(x,y,z) - S(x,y,z) - 6S(u,v,w)\\
&\quad = 2S(x+u, y+v, z+w) + 2S(u,v,w) - S(x,y,z)\\
&\qquad + 2S(x-u, y-v, z-w) + 2S(u,v,w) - S(x,y,z) - 6S(u,v,w)\\
&\quad = 2[S(x+u, y+v, z+w) + S(x-u, y-v, z-w)] - 2S(x,y,z) - 2S(u,v,w)\\
&\quad = 2[2S(x,y,z) + 2S(u,v,w)] - 2S(x,y,z) - 2S(u,v,w)\\
&\quad = 2S(x,y,z) + 2S(u,v,w)\\
&\quad = S(x+u, y+v, z+w) + S(x-u, y-v, z-w).
\end{aligned}$$

□

Corollary 2.1. *Under the hypotheses of Theorem 2.1, if* $\mathtt{Im}Q_f \subset A$ *for some* $A \in S(V)$*, then* $\mathtt{Im}(f-S) \subset \frac{1}{6}A$.

Proof. We know that

$$\mathtt{Im}(\frac{1}{6}Q_f(x,y,z,\cdot,\cdot,\cdot)) \subset \mathtt{Im}(\frac{1}{6}Q_f) \subset \frac{1}{6}A$$

and so $\frac{1}{6}Q_f(x,y,z,\cdot,\cdot,\cdot) \in B(G^3, S(V))$ for all $(x,y,z) \in G^3$. By property (i) of Λ, we have

$$f(x,y,z) - S(x,y,z) = \Lambda[\frac{1}{6}Q_f(\cdot,\cdot,\cdot,x,y,z)] \in \frac{1}{6}A$$

for all $(x,y,z) \in G^3$. Therefore, $\mathtt{Im}(f-S) \subset \frac{1}{6}A$. □

We now recover and refine [4, Theorem 1] as follows.

Corollary 2.2. *Let G be an Abelian group and V a vector space. Assume that the family $B(G, S(V))$ admits LIM. If $f : G \longrightarrow V$ is a function and $Q'_f(x,\cdot) = Q_f(x,0,0,\cdot,0,0) : G \longrightarrow V$ is* ***d****-Lipschitz for all $x \in G$, then there exists a quadratic function K such that $f-K$ is $\frac{1}{6}$***d****-Lipschitz.*

Proof. Define $f_1 : G^3 \longrightarrow V$ by $f_1(\cdot, y, z) := f(\cdot)$ for all $y, z \in G$. By assumption $Q_{f_1}(x, 0, 0, \cdot, 0, 0) : G \times \{0\} \times \{0\} \longrightarrow V$ is $\mathbf{d}$-Lipschitz for all $x \in G$. Using Theorem 2.1, one deduces there exists a generalized 3-quadratic function S such that $f_1 - S$ is $\frac{1}{6}\mathbf{d}$-Lipschitz. Consider $K(x) := S(x, 0, 0)$. So, K is a generalized quadratic function and hence K is a quadratic function (see [8]). Since $f_1(x, 0, 0) - S(x, 0, 0)$ is $\mathbf{d}$-Lipschitz for all $x \in G$ and $f_1(x, 0, 0) - S(x, 0, 0) = f(x) - K(x)$, $f - K$ is $\frac{1}{6}\mathbf{d}$-Lipschitz. $\square$

Remark 1. *The advantage of our discussion is improving the approximation that Czerwik et al. obtained in* [4]. *This improvement is possible by considering a suitable set valued function. Let G be a group with metric d and let V be a normed space. Let*

$$\boldsymbol{d}(x, y) := d(x, y)B(0, 1), \tag{2.2}$$

where $B(0, 1)$ is the closed ball with the center at 0 and the radius 1. Czerwik et al. proved in [4, Theorem 1], *under some hypotheses, there exists a quadratic function K such that $f - K$ is $\frac{1}{2}\boldsymbol{d}$-Lipschitz. We indicated in Corollary 2.2, under some hypotheses, there exists a quadratic function K such that $f - K$ is $\frac{1}{6}\boldsymbol{d}$-Lipschitz. Now, if we apply the suitable set valued function $\boldsymbol{d}$ defined in* (2.2), *then we conclude that our approximation is better, indeed*

$$\begin{aligned}(f(x) - K(x)) - (f(y) - K(y)) \in \frac{1}{6}\boldsymbol{d}(x, y) &= B(0, \frac{d(x, y)}{6}) \\ &\subseteq B(0, \frac{d(x, y)}{2}) = \frac{1}{2}\boldsymbol{d}(x, y)\end{aligned}$$

for all $x, y \in G$.

Definition 2.1. Suppose $(G^3, +)$ is an Abelian group. We say that a metric d on $(G^3, +)$ is invariant under translation if it satisfies the following condition

$$d((x + a, y + b, z + c), (u + a, v + b, w + c)) = d((x, y, z), (u, v, w))$$

for all $(a, b, c), (x, y, z), (u, v, z) \in G^3$.

Definition 2.2. A metric $\tilde{d}$ on $G^3 \times G^3$ is called a product metric if it is an invariant metric and the following condition holds

$$\begin{aligned}\tilde{d}((a,b,c,x,y,z),(a,b,c,u,v,w)) &= \tilde{d}((x,y,z,a,b,c),(u,v,w,a,b,c)) \\ &= d((x,y,z),(u,v,w))\end{aligned}$$

for all $(a,b,c),(x,y,z),(u,v,w) \in G^3$.

We use the following corollary to establish the next result.

Corollary 2.3. *Let $(G^3,+,d,\tilde{d})$ be a product metric, and V a normed space such that $B(G^3, CB(V))$ admits LIM, where $CB(V)$ is the family of all closed balls with center at zero. If $f : G^3 \longrightarrow V$ is a function, then there exists a generalized 3-quadratic function $S : G^3 \longrightarrow V$ such that $MC_{f-S} = \frac{1}{6}MC_{Q_f}$. Moreover, if $Q_f \in B(G^3 \times G^3, BC(V))$, then*

$$||f - S||_s \le \frac{1}{6}||Q_f||_s.$$

Proof. Let $\Theta : G^3 \times G^3 \longrightarrow \mathbb{R}^+$ be a positive real–valued function defined by

$$\Theta((x,y,z),(u,v,w)) := \inf_{d((x,y,z),(u,v,w)) \le \delta} MC_{Q_f}(\delta)$$

for all $(x,y,z),(u,v,w) \in G^3$. Define the set–valued function $\mathbf{d} : G^3 \times G^3 \longrightarrow S(V)$ by

$$\mathbf{d}((x,y,z),(u,v,w)) := \Theta((x,y,z),(u,v,w))B(0,1)$$

for all $(x,y,z),(u,v,w) \in G^3$, where $B(0,1)$ is the unit closed ball. Since MC_{Q_f} is the module of continuity of Q_f, then

$$\begin{aligned}||Q_f(r,s,t,x,y,z) - Q_f(r,s,t,u,v,w)|| &\le \inf_{\tilde{d}((r,s,t,x,y,z),(r,s,t,u,v,w)) \le \delta} MC_{Q_f}(\delta) \\ &= \inf_{d((x,y,z),(u,v,w)) \le \delta} MC_{Q_f}(\delta) \\ &= \Theta((x,y,z),(u,v,w))\end{aligned}$$

for all $(x,y,z),(u,v,w) \in G^3$ and hence $Q_f(r,s,t,\cdot,\cdot,\cdot)$ is a $\mathbf{d}$-Lipschitz function. On the other hand, Theorem 2.1 implies there exists a generalized 3-quadratic function $S : G^3 \longrightarrow V$ such that $f - S$ is a $\frac{1}{6}\mathbf{d}$-Lipschitz function. Thus,

$$(f(x,y,z) - S(x,y,z)) - (f(u,v,w) - S(u,v,w)) \in \frac{1}{6}\mathbf{d}((x,y,z),(u,v,w))$$

for all $(x,y,z),(u,v,w) \in G^3$. So,

$$||(f(x,y,z)-S(x,y,z))-(f(u,v,w)-S(u,v,w))|| \leq \frac{1}{6}\Theta((x,y,z),(u,v,w)) = \inf_{d((x,y,z),(u,v,w))\leq\delta} \frac{1}{6}MC_{Q_f}(\delta)$$

for all $(x,y,z),(u,v,w) \in G^3$. Consequently,

$$MC_{f-S} = \frac{1}{6}MC_{Q_f}.$$

Suppose that $Q_f \in B(G^3 \times G^3, CB(V))$. Then, clearly $\mathtt{Im}Q_f \subset ||Q_f||_s B(0,1)$. By applying Corollary 2.1, we get

$$\mathtt{Im}(f-S) \subset \frac{1}{6}||Q_f||_s B(0,1)$$

and hence $||f-S||_s \leq \frac{1}{6}||Q_f||_s$. □

We prove the stability of the generalized 3-quadratic functional equation in the Lipschitz norm. One can arrange a similar corollary like Corollary 2.2 for a quadratic function.

Corollary 2.4. *Let $(G^3,+,d,\tilde{d})$ be a product metric, and V a normed space such that $B(G^3, CB(V))$ admits LIM, where $CB(V)$ is the family of all closed balls with center at zero. If $f : G^3 \longrightarrow V$ is a function and $Q_f \in Lip(G^3 \times G^3, V)$, then there exists a generalized 3-quadratic function S such that*

$$||f-S||_{Lip} \leq \frac{1}{6}||Q_f||_{Lip}.$$

Proof. Define the function $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by

$$\Phi(t) := \mathtt{P}(Q_f)t$$

for all $t \in \mathbb{R}^+$. Since $Q_f \in Lip(G^3 \times G^3, V)$, we conclude that

$$||Q_f(r,t,s,a,b,c) - Q_f(x,y,z,u,v,w)|| \leq \mathtt{P}(Q_f)\tilde{d}((r,t,s,a,b,c),(x,y,z,u,v,w))$$

for all $(r,t,s),(a,b,c),(x,y,z),(u,v,w) \in G^3$. The definition of Φ implies

$$||Q_f(r,t,s,a,b,c) - Q_f(x,y,z,u,v,w)|| \leq \Phi(\tilde{d}((r,t,s,a,b,c),(x,y,z,u,v,w)))$$

for all $(r,t,s),(a,b,c),(x,y,z),(u,v,w) \in G^3$. So, Φ is the module of continuity of Q_f and Theorem 2.3 ensures there exists a generalized 3-quadratic function $S : G^3 \longrightarrow V$ such that $MC_{f-S} = \frac{1}{6}\Phi$. Hence,

$$\begin{aligned}||(f(x,y,z)-S(x,y,z))-(f(u,v,w)-S(u,v,w))|| &\leq MC_{f-S}(d((x,y,z),(u,v,w)))\\ &= \frac{1}{6}\Phi(d((x,y,z),(u,v,w)))\\ &= \frac{1}{6}\mathtt{P}(Q_f)d((x,y,z),(u,v,w))\end{aligned}$$

for all $(x,y,z),(u,v,w) \in G^3$. From the last inequality it follows that $f-S$ is a Lipschitz function and

$$\mathtt{P}(f-S) \leq \frac{1}{6}\mathtt{P}(Q_f). \tag{2.3}$$

Since $Q_f \in Lip(G^3 \times G^3, V)$, then Q_f is bounded and $\mathtt{Im}Q_f \subset B(0,M)$ for some $M>0$. This implies $Q_f \in B(G^3 \times G^3, CB(V))$ and Theorem 2.3 entails

$$||f-S||_s \leq \frac{1}{6}||Q_f||_s. \tag{2.4}$$

This means the function $f-S$ is bounded and hence, $f-S \in Lip(G^3, V)$. By using (2.3) and (2.4) we find that

$$\begin{aligned}||f-S||_{Lip} &= ||f-S||_s + \mathtt{P}(f-S)\\ &\leq \frac{1}{6}||Q_f||_s + \frac{1}{6}\mathtt{P}(Q_f) = \frac{1}{6}||Q_f||_{Lip}.\end{aligned}$$

□

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