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SOME PROPERTIES AND CRITERIA FOR SUB-CHAOTIC C_0-SEMIGROUPS

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ABSTRACT. In this paper, we get a closer view to sub-chaotic C_0-semigroups. We show that if a C_0-semigroup contains a subspace-chaotic operator, then it is sub-chaotic. We prove that there are sub-chaotic C_0-semigroups that contain no subspace-chaotic operator. We also prove that if φ is a bounded and holomorphic function on the unit disk, then the multiplication C_0-semigroup generated by φ can not be sub-chaotic. Moreover, we state some criteria for a C_0-semigroup to be sub-chaotic based on the properties of the operators that made the semigroup.

1. Introduction

A bounded and linear operator T on a Banach space X is called hypercyclic if $orb(T,x) = \{x, Tx, ..., T^n x\}$ is dense in X for some $x \in X$. The concept of hypercyclicity and some related concepts like chaoticity are interesting topics for researchers in dynamical systems and investigated by them in various mathematical structures. One of these structures is a semigroup. By a C_0-semigroup on a Banach space X, we mean a family $(T_t)_{t\geq 0}$ of bounded linear operators on X such that:

(i) $T_0 = I$,
(ii) for any $s \geq 0$ and any $t \geq 0$, $T_{s+t} = T_s T_t$,
(iii) for any $s \geq 0$ and any $x \in X$, $\lim_{t\to s} T_t x = T_s x$.

The orbit of an element $x \in X$ under C_0-semigroup $(T_t)_{t\geq 0}$ is defined by

$$orb((T_t)_{t\geq 0}, x) = \{T_t x : t \geq 0\}.$$

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We say a C_0-semigroup $(T_t)_{t\geq 0}$ is hypercyclic, if $orb((T_t)_{t\geq 0}, x)$ is dense in X for some $x \in X$. It is notable that if $(T_t)_{t\geq 0}$ is hypercyclic, then T_t is hypercyclic for any $t > 0$, see [9, Theorem 2.3].

If for every pair of sets U and V of open subsets of X, we have $T_t(U) \cap V \neq \phi$ for some $t \geq 0$, then $(T_t)_{t\geq 0}$ is called topologically transitive. When X is a separable space, hypercyclicity of C_0-semigroup $(T_t)_{t\geq 0}$ is equivalent to topologically transitivity of it, see [13, p. 186].

Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. If we consider

$$Ax := \lim_{t\to 0}\frac{1}{t}(T_t x - x),$$

then there is a dense subset of X that Ax exists on it and we denote this subset by $D(A)$. In this case, A with domain $D(A)$ is named the generator of $(T_t)_{t\geq 0}$, see [13].

Subrahmonian Moothathu established in [20] that for a C_0-semigroup $(T_t)_{t\geq 0}$ on a complex Banach space with generator A, if $\sigma(A) \neq \phi$, then $orb((T_t)_{t\geq 0}, x)$ is linearly independent for any hypercyclic vector x. Also, one can see more properties for C_0-semigroups with $\sigma(A) \neq \phi$ in [21].

A new category of C_0-semigroups that is named recurrent C_0-semigroups was introduced and investigated in [18]. Moreover, sufficient conditions for recurrence of C_0-semigrous and their direct sum can be found in [18]. Also, one can find the concept of γ-boundedness for C_0-semigroups and a characterization for generation of γ-bounded C_0-semigroups in [1].

A C_0-semigroup $(T_t)_{t\geq 0}$ on X is said a chaotic C_0-semigroup if it is transitive and the set of its periodic points is dense in X. Remember that if $T_t x = x$ for some $t > 0$, then x is said a periodic point for $(T_t)_{t\geq 0}$.

We can not find any hypercyclic C_0-semigroups on finite-dimensional spaces, see [13, Theorem 7.15]. Thus, chaotic C_0-semigroups can not exist on finite-dimensional spaces, too. Moreover, it was established in [8] that hypercyclic C_0-semigroups can be constructed in any complex Banach spaces that are separable and infinite-dimensional. But chaotic C_0-semigroups do not satisfy this matter.

Desh et al. investigated various semigroups for hypercyclicity and chaoticity like translation semigroups and semigroups that are generated by discrete shifts in [11].

They also, stated various sufficient conditions for hypercyclicity and chaoticity of semigroups. One can also see [10] and [15] for more information. Banasiak and Moszynski stated some other conditions for chaoticity of semigroups based on eigenvectors in [2] by refining some conditions in [11]. They also introduced the concept of subspace chaotic C_0-semigroups or briefly sub-chaotic C_0-semigroup as follows:

Definition 1.1. For a closed and non-trivial subspace M of X, we name a C_0-semigroup $(T_t)_{t\geq 0}$ is sub-chaotic with the space of chaoticity M if $(T_t|_M)_{t\geq 0}$ is topologically chaotic and M is invariant under $(T_t)_{t\geq 0}$, i.e., $T_t(M) \subseteq M$ for any $t > 0$, see [2].

In 2012, Madore and Martinez-Avendano described subspace-hypercyclic operators. For an operator T on X and for a closed subspace M of X, if there is a vector $x \in M$ such that $orb(T, x) \cap M$ is dense in M, then T is called M-hypercyclic, see [16]. With a routine procedure it can be seen that if $T(M) \subseteq M$ and $T|_M$ is a hypercyclic operator, then T is M-hypercyclic. An operator on a Banach space X is called subspace-chaotic with respect to closed subspace M or M-chaotic if it is M-transitive and has a dense set of periodic points in M, see [23].

The concept of subspace-diskcyclicity was introduced in [3] and semi chaotic operators on Banch spaces were introduced in [5]. It was proved in [5] that semi chaotic operators exist on every finite dimensional Banach spaces. One can also see [4] and [19] for more information about this matter.

For a closed and non-trivial subspace M of X, we say that $(T_t)_{t\geq 0}$ is M-hypercyclic if $orb((T_t)_{t\geq 0}, x) \cap M$ is dense in M for some $x \in X$, see [22]. If $(T_t)_{t\geq 0}$ is a C_0-semigroup such that M is invariant under it and $(T_t|_M)_{t\geq 0}$ is hypercyclic, then we can conclude that $(T_t)_{t\geq 0}$ is M-hypercyclic. Throught this paper, X indicates a complex and infinite-dimensional Banach space and we briefly called bounded linear operators on X, operators. Also, we denote by M a closed and non-trivial subspace of X.

In this paper, we want to discover some relations between subspace-chaotic operators and sub-chaotic C_0-semigroups. Moreover, we want to obtain sufficient conditions for a C_0-semigroup to be sub-chaotic.

In Setion 2, we show that if a C_0-semigroup contains a subspace-chaotic operator, then it is sub-chaotic. We establish that there are sub-chaotic C_0-semigroups that contain no subspace-chaotic operator. Also, we costruct a non-sub-chaotic multiplication C_0-semigroup. In Section 3, we state some various criteria for a C_0-semigroup to be sub-chaotic based on properties of operators that made the semigroup.

2. Subspace-chaotic Operators and Sub-chaotic C_0-Semigroups

As it was mentioned in the introduction, chaotic C_0-semigroups do not exist on finite-dimensional spaces. Now, we establish in the next theorem that for a finite-dimensional subspace M of a Banach space X and a C_0-semigroup $(T_t)_{t\geq 0}$ on this space, $(T_t)_{t\geq 0}$ can not be sub-chaotic with the space of chaoticity M.

Theorem 2.1. If $(T_t)_{t\geq 0}$ is sub-chaotic C_0-semigroup on X with the space of chaoticity M, then M is infinite-dimensional.

Proof. Let M be a finite-dimensional subspace for X such that $(T_t)_{t\geq 0}$ is sub-chaotic for the space of chaoticity M. Hence, $(T_t)_{t\geq 0}$ is invariant under M and $(T_t|_M)_{t\geq 0}$ is chaotic. So we can conclude that $(T_t|_M)_{t\geq 0}$ is a chaotic semigroup on finite-dimentional space M. This is a contradiction as mentioned before the theorem and so M is an infinite-dimensional space.

□

In the following lemma, we give a primarily relation between subspace-chaoticity of an operator and its restriction.

Lemma 2.1. *Consider that T is an M-chaotic operator on X with $T(M) \subseteq M$. Then $T|_M$ is a chaotic operator.*

Proof. Since T is M-chaotic, T is M-transitive. On the other hand, by [16, Theorem 3.5], T is M-hypercyclic. Let $x \in M$ be an M-hypercyclic vector for T. Since $T(M) \subseteq M$, x is an hypercyclic vector for $T|_M$. Also, by M-chaoticity of T, this operator has a dense set of periodic points in M. Hence, $T|_M$ has a dense set of periodic points in M and so $T|_M$ is chaotic. □

The next corollary shows that an M-chaotic operator in a semigroup can build a sub-chaotic semigroup.

Corollary 2.1. *For a C_0-semigroup $(T_t)_{t\geq 0}$, let M be a closed and non-trivial subspace of X and invariant under $(T_t)_{t\geq 0}$.*

(i) *If $T_{t_0}|_M$ is chaotic for some $t_0 > 0$, then $(T_t)_{t\geq 0}$ is sub-chaotic with the space of chaoticity M.*

(ii) *If there is some $t_0 > 0$ such that T_{t_0} is M-chaotic, then $(T_t)_{t\geq 0}$ is sub-chaotic with the space of chaoticity M.*

Proof. Part (i) is clear by definition of sub-chaotic semigroups. For proving part (ii), note to this matter that by Lemma 2.1, this condition implies that $T|_M$ is chaotic. □

By using chaotic semigroups, one can construct sub-chaotic semigroups as follows.

Example 2.1. *Presume that $(T_t)_{t\geq 0}$ is a chaotic C_0-semigroup. If we consider $S_t := T_t \oplus I$ and $M := X \oplus \{0\}$, then $(S_t|_{X\oplus\{0\}})_{t\geq 0} = (T_t \oplus I|_{X\oplus\{0\}})_{t\geq 0}$ is sub-chaotic with the space of chaoticity M. Since $(T_t \oplus I)_{t\geq 0}$ is not hypercyclic, $(T_t \oplus I)_{t\geq 0}$ is not chaotic.*

For instance, consider $X = C_0(\mathbb{R}^+)$ with supremum norm, where

$$C_0(\mathbb{R}^+) = \{f : f : \mathbb{R}^+ \to \mathbb{C}; \lim_{x\to\infty} f(x) = 0\}.$$

Let α be a positive and fixed integer. Then if we define

$$(T_t f)(x) = e^{\alpha t} f(x + t), \quad x \in \mathbb{R}^+,$$

then $(T_t)_{t\geq 0}$ is a chaotic C_0-semigroup, see [13, p. 188]. *Hence, $(T_t \oplus I)_{t\geq 0}$ is a sub-chaotic C_0-semigroup with the space of chaoticity $M := C_0(\mathbb{R}^+) \oplus \{0\}$.*

In the following, we make a non-sub-chaotic semigroup.

Example 2.2. *Consider that φ is bounded and holomorphic on the unit disk $\mathbb{D}$. If we define*

$$T^{\varphi}{}_t f = e^{t\varphi} f, \quad t \geq 0,$$

then $(T_t^{\varphi})_{t\geq 0}$ is a C_0-semigroup on H^2, see [13, p. 206] *and is called multiplication C_0-semigroup. Recall that the Hardy-Hilbert space consists of analytic functions which* *have power series representation with square-summable complex coefficients, see* [17, Definition 1.1.1]. *That means*

$$H^2 = \{f : f(z) = \Sigma_{n=0}^{\infty} a_n z^n \quad \textit{and} \quad \Sigma_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

In [12] *interesting theorems about multiplication C_0-semigroups can be found. We claim that multiplication C_0-semigroup can not be sub-chaotic. Suppose on the contrary that $(T_t^{\varphi})_{t\geq 0}$ is a sub-chaotic multiplication C_0-semigroup with the space of chaoticity M. Thus, $(T_t^{\varphi}|_M)_{t\geq 0}$ is hypercyclic and so as we mentioned in the introduction, $T_t^{\varphi}|_M$ is hypercyclic for any $t > 0$.*

This means that $T_{t_0}^{\varphi}$ is an analytic Toeplitz operator, that is M-hypercyclic. This is a contradiction since analytic Toeplitz operators can not be subspace-hypercyclic, see [16, p. 504].

We know that if $(T_t)_{t\geq 0}$ is a hypercyclic C_0-semigroup, then T_t is a hypercyclic operator for any $t \geq 0$. Now, it is natural to ask this question that does the sub-chaoticity of a C_0-semigroup $(T_t)_{t\geq 0}$ imply that for any $t > 0$, T_t is a subspace-chaotic operator? In the next corollary, we show that the answer is negative.

Corollary 2.2. *There are C_0-semigroups on a Banach space X such that they are sub-chaotic with the space of chaoticity M, but they do not contain any subspace-chaotic operators with respect to M.*

Proof. As it is constructed in [6], there is a chaotic C_0-semigroup $(S_t)_{t\geq 0}$ such that for any $t \geq 0$, the operator S_t is not chaotic. Hence, $S_t \oplus I$ can not be a subspace-chaotic operator with respect to $X \oplus \{0\}$. We claim that $(S_t \oplus I|_{X\oplus\{0\}})_{t\geq 0}$ is a sub-chaotic C_0-semigroup with space chaoticity $M := X \oplus \{0\}$.

For this, consider that $U \oplus \{0\}$ and $V \oplus \{0\}$ are optional open sets in M. Thus, U and V are open sets in the space X. Now, since $(S_t)_{t\geq 0}$ is chaotic, we can conclude that $S_{t_0}(U) \cap V$ is non-empty for some $t_0 \geq 0$. Therefore,

$$\begin{aligned}
&(S_{t_0} \oplus I)(U \oplus \{0\}) \cap (V \oplus \{0\}) \\
&= (S_{t_0}(U) \oplus \{0\}) \cap (V \oplus \{0\}) \\
&= (S_{t_0}(U) \cap V) \oplus \{0\} \neq \phi.
\end{aligned}$$

So, $(S_t \oplus I|_{X\oplus\{0\}})_{t\geq 0}$ is topologically transitive. Also,

$$per((S_t)_{t\geq 0}) = \{x \in X : \exists t > 0; S_t(x) = x\}$$

is dense in X. Let $x \in per((S_t)_{t\geq 0})$. Then we can find a $t_0 > 0$ so that $S_{t_0}(x) = x$. Therefore,

$$(S_{t_0} \oplus I)(x \oplus \{0\}) = S_{t_0}(x) \oplus I(0) = x \oplus \{0\}.$$

Hence, $x \oplus \{0\}$ is a periodic element for $(S_{t_0} \oplus I)_{t\geq 0}$. Thus,

$$per((S_t)_{t\geq 0}) \oplus \{0\} \subseteq per(S_t \oplus I)_{t\geq 0}.$$

Now, since $per((S_t)_{t\geq 0})$ is dense in X, then $per((S_t)_{t\geq 0}) \oplus \{0\}$ is dense in $X \oplus \{0\}$. Therefore, $(S_t \oplus I)_{t\geq 0}$ has a dense set of periodic points in $X \oplus \{0\}$ and hence, $(S_t \oplus I|_{X\oplus\{0\}})_{t\geq 0}$ is chaotic.

□

3. Some Critaria for Sub-chaoticity of C_0-Semigroups

In this section, we find some adequate conditions for sub-chaoticity of the C_0-semigroups. The idea of the first theorem is given from such a condition for hypercyclicity of operators in [16].

Theorem 3.1. Let $(A_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ be C_0-semigroups on a Banach space X. Let E be an operator on X with a closed range M and let M be invariant under $(T_t)_{t\geq 0}$. If $A_tE = ET_t$ for any $t > 0$ and $(T_t)_{t\geq 0}$ is a chaotic C_0-semigroup, then $(A_t)_{t\geq 0}$ is sub-chaotic with the space of chaoticity M.

Proof. First, we show that M is an invariant subspace for $(A_t)_{t\geq 0}$. Let $t > 0$ and let $y \in M$. Thus, there is $x \in X$ such that $Ex = y$ and so,

$$A_t(y) = A_t(Ex) = ET_t(x) \in M.$$

This means $A_t(M) \subseteq M$ for any $t > 0$. By hypothesis, $(T_t)_{t\geq 0}$ is chaotic. It follows that $per((T_t)_{t\geq 0})$ is dense in X. Let $x \in per((T_t)_{t\geq 0})$. Then $T_{t_0}(x) = x$ for some $t_0 > 0$. Hence,

$$A_{t_0}Ex = ET_{t_0}(x) = E(x).$$

This means that if $x \in per((T_t)_{t\geq 0})$, then $E(x) \in per((A_t)_{t\geq 0})$ and so,

$$E(per((T_t)_{t\geq 0})) \subseteq per((A_t)_{t\geq 0}).$$

Because $per((T_t)_{t\geq 0})$ is dense X so $E(per((T_t)_{t\geq 0}))$ is dense in M and hence, $per((A_t)_{t\geq 0}) \cap M$ is dense in M. This means the set of periodic points of $(A_t)_{t\geq 0}$ forms a dense set in M.

On the other hand, $(T_t)_{t\geq 0}$ is chaotic and hence it is hypercyclic. Let x be a hypercyclic vector for $(T_t)_{t\geq 0}$. So,

$$orb((A_t)_{t\geq 0}, Ex) = \{A_t(Ex) : t \geq 0\} = \{E(T_t x) : t \geq 0\}.$$

Hence,

$$M \supseteq \overline{orb((A_t)_{t\geq 0}, Ex)} = \overline{\{A_t(Ex) : t \geq 0\}} = \overline{E(orb((T_t)_{t\geq 0}, x)} = M.$$

Thus,

$$\overline{orb((A_t)_{t\geq 0}, Ex)} = M.$$

Now, since $Ex \in M$ and $A_t(M) \subseteq M$, we conclude that $(A_t|_M)_{t\geq 0}$ is hypercyclic. Therefore, $(A_t|_M)_{t\geq 0}$ is chaotic and so $(A_t)_{t\geq 0}$ is sub-chaotic with the space of chaoticity M.

□

By using the idea of hypercyclicity criterion for operators [14, Theorem 2], we present the following condition for semigroups.

Theorem 3.2. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup. Let M be a closed subspace of X that is invariant under $(T_t)_{t\geq 0}$. If

(i) $Z = \{x \in M : T_t x \to 0\}$ is dense in M,
(ii) $Y = \{y \in M : \exists (u_t) \text{ in } M, u_t \to 0 \text{ and } T_t u_t \to y\}$ is dense in M,
(iii) $Per((T_t)_{t\geq 0}) \cap M$ forms a dense subset in M,

then $(T_t)_{t\geq 0}$ is sub-chaotic for the space of chaoticity M.

Proof. First, we show that $(T_t|_M)_{t\geq 0}$ is topologically transitive. For proving this, take $U, V \subseteq M$ are relatively open sets. By (i), we can conclude that there exist $x_0 \in Z$

and $y_0 \in Y$ such that

$$x_0 \in U \cap Z \quad \text{and} \quad y_0 \in V \cap Y. \tag{3.1}$$

Hence, $T_t x_0 \to 0$ and there is $(u_t) \subseteq M$ such that $u_t \to 0$ and $T_t u_t \to y_0$ and so,

$$x_0 + u_t \to x_0 \quad \text{and} \quad T_t(x_0 + u_t) = T_t(x_0) + T_t(u_t) \to y_0. \tag{3.2}$$

By (3.1) and (3.2) there is sufficiently large t_0 such that for any $t \geq t_0$,

$$x_0 + u_t \in U \quad \text{and} \quad T_t(x_0 + u_t) \in V.$$

Because $x_0 + u_t \in M$ and $T_t(M) \subseteq M$ for any $t \geq t_0$ so $(T_t|_M)_{t\geq 0}$ is topologically transitive. By hypothesis, $(T_t)_{t\geq 0}$ has a dense set of periodic points in M, so has $(T_t|_M)_{t\geq 0}$. This implies $(T_t|_M)_{t\geq 0}$ is chaotic. Consequently, $(T_t|_M)_{t\geq 0}$ is sub-chaotic for the space of chaoticity M.

□

The idea of the following conditions is from [11] for the chaoticity of semigroups.

Theorem 3.3. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Let M be a closed and non-trivial subspace of X that is invariant under $(T_t)_{t\geq 0}$. If

(i) for any $x, y \in M$ and for any $\varepsilon > 0$, there exist $w \in M$ and $t > 0$ such that $||x - w|| < \varepsilon$ and $||y - T_t(w)|| < \varepsilon$,

(ii) $per((T_t)_{t\geq 0}) \cap M$ forms a dense subset of M,

then $(T_t)_{t\geq 0}$ is sub-chaotic for the space of chaoticity M.

Proof. We claim that $(T_t|_M)_{t\geq 0}$ is topologically transitive. Let U and V be a pair of relatively open sets in M. Suppose that $x \in U$ and $y \in V$. Thus, there is $\varepsilon > 0$ such that $B(x, \varepsilon) \cap M \subseteq U$ and $B(y, \varepsilon) \cap M \subseteq V$. By hypothesis, there is $w \in M$ and there is $t_0 > 0$ such that

$$||x - w|| < \varepsilon \quad \text{and} \quad ||y - T_{t_0}(w)|| < \varepsilon.$$

Therefore, $w \in U$ and $T_{t_0}(w) \in V$. Hence, there is $t_0 > 0$ such that $T_{t_0}(U) \cap V \neq \phi$. Since $w \in M$ and $T_{t_0}(M) \subseteq M$, one can conclude that $T_{t_0}|_M(U) \cap V \neq \phi$. Hence, $(T_t|_M)_{t\geq 0}$ is topologically transitive. By hypothesis, $(T_t)_{t\geq 0}$ has a dense set of periodic points in M. Thus, $(T_t|_M)_{t\geq 0}$ is chaotic and hence $(T_t)_{t\geq 0}$ is sub-chaotic for the space of chaoticity M.

$\square$

Corollary 3.1. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Let M be a closed and non-trivial subspace of X. If

(i) $T_t(M) \subseteq M$ for any $t > 0$,

(ii) for any $\varepsilon > 0$, there is a subset $D \subseteq M$ so that $\overline{D} = M$ and for any $x \in D$, there is a subset $D_x \subseteq M$ so that $\overline{D_x} = M$ and for any $y \in D_x$, there is $v \in M$ and $t > 0$ such that

$$||y - v|| < \varepsilon \quad \text{and} \quad ||x - T_t(v)|| < \varepsilon,$$

(iii) $per((T_t)_{t\geq 0}) \cap M$ forms a dense subset of M,

then $(T_t)_{t\geq 0}$ is sub-chaotic for the space of chaoticity M.

Proof. We show that the conditions of Theorem 3.3 hold. We first prove that part (i) of Theorem 3.3 can be concluded from part (ii) of this corollary. For this, let $x \in M$ and let $y \in M$. Suppose that $\varepsilon > 0$. By (ii), we have a dense subset D of M. By density of D, we can gain $z \in D$ such that

$$||y - z|| < \frac{\varepsilon}{2}. \tag{3.3}$$

It follows from (ii) that there is a dense subset D_z of M. By density of D_z, we can find $w \in D_z$ such that

$$||x - w|| < \frac{\varepsilon}{2}. \tag{3.4}$$

Also, by (ii), for this w we can find $v \in M$ and $t > 0$ such that

$$||w - v|| < \frac{\varepsilon}{2} \quad \text{and} \quad ||z - T_t(v)|| < \frac{\varepsilon}{2}. \tag{3.5}$$

Therefore, by (3.4) and (3.5), $||x - v|| < \varepsilon$ and by (3.3) and (3.5),

$$||y - T_t v|| \leq ||y - z|| + ||z - T_t(v)|| < \varepsilon,$$

which is part (i) of Theorem 3.3. Note that M is invariant under $(T_t)_{t\geq 0}$, by the assumption in part (i) and part (iii) of this corollary is the same as part (ii) of Theorem 3.3. So, Theorem 3.3 implies the result.

□

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