Jordan Journal of Mathematics and Statistics (JJMS), 15(4B), 2022, pp 1077 - 1092
DOI: https://doi.org/10.47013/15.4.19

PROPER HELIX OF ORDER 6 AND LC HELIX IN PSEUDO-EUCLIDEAN SPACE E_4^8

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Abstract. In this paper, we used the result that complex hyperbolic spaces $CH^2(-\frac{4c}{3})$ with holomorphic sectional curvature $\frac{-4c}{3}$ are isometrically embedded in E_4^8. By considering a circle in $CH^2(-\frac{4c}{3})$, we prove that the image of the circle by isometric embedding is a proper helix of order 6 in E_4^8. Moreover, we define a generalized LC helix on a submanifold of E_4^8. Also, we show that the image of a circle by isometric embedding from complex hyperbolic plane $CH^2(-\frac{4}{3})$ to pseudo-Euclidean space E_4^8 is a generalized LC helix on some submanifold of E_4^8.

1. Introduction

A curve $\gamma(s)$ parametrized by arc length parameter is said to be a circle in Kähler manifold M, if there exist a unit vector field Y orthonormal to tangent vector field X along $\gamma(s)$ and satisfies the relations

$$\nabla_X X = \kappa Y, \;\; \nabla_X Y = -\kappa X,$$

where κ is some positive constant known as curvature of the circle and ∇ is Riemannian connection on M. The complex torsion for circle is defined as $\tau = g(X, JY)$, where J is a complex structure and g is a Riemannian metric on M. A Kähler manifold M with constant holomorphic sectional curvature c is known as a complex space form $M(c)$. Two circles in $M(c)$ are congruent by holomorphic isometries to each other if and only if they have the same curvature and complex torsion [17]. In [3, 5, 6], the various properties of circles in complex projective space and complex hyperbolic

2010 *Mathematics Subject Classification.* 53C50, 53C40.

Key words and phrases. Curve of proper order, complex hyperbolic space, generalized LC helix, isometric embedding, parallel second fundamental form.

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Received: Oct. 21, 2021 Accepted: April 14, 2022 .

space were studied. O. J. Garay and A. Romeo [11] in 1990, studied the isometric embedding from a complex hyperbolic space to a pseudo-Euclidean space. The authors used the application of isometric embedding to study the real hypersurfaces in complex hyperbolic space [11].

A smooth curve $\gamma(s)$ with frame $\{V_1, ..., V_d\}$, of orthonormal fields in Riemannian manifold M^n is said to be a curve of proper order d, if frame fields satisfies the following Frenet formula

$$\nabla'_{V_1} V_i = -\kappa_{i-1} V_{i-1} + \kappa_i V_{i+1}, \qquad 1 \leq i \leq d,$$

where $V_0 = V_{d+1} = 0$, $\kappa_0 = 0$. Also ∇' is a Levi-Civita connection on M^n and κ_i are known as the i^{th} curvatures of $\gamma(s)$. If all the i^{th} curvatures of $\gamma(s)$ are constants, then curve is called a helix of proper order d in M. The definition for proper curve and proper helix of order d in pseudo-Euclidean space is slightly different from the definition of Riemannian manifold [18]. The holomorphic helices of proper order d were considered in [15, 4]. A holomorphic helix of proper order d in Kähler manifold is a helix of proper order d with constant torsion function. Also torsion for helix of proper order d in Kähler manifold is defined as

$$\tau_{ij} = \langle V_i, V_j \rangle; 1 \leq i, j \leq d.$$

A general helix in Euclidean space is a curve whose tangent vector makes a constant angle with some fixed direction. The necessary and sufficient conditions for a curve γ to become a general helix in Euclidean space is that the ratio of curvature and torsion of γ is constant. In 1802, M. A. Lancret stated this condition, the first proof for the condition was given by B. de Saint Venant in 1845. In 1997, M. Barros [9], proved the same condition by using a Killing vector field. A slant helix in Euclidean space is a curve whose principal normal makes a constant angle with some fixed direction. In 2004, S. Izumiya and N. Takeuchi [12], defined the slant helix in Euclidean space. A curve γ in E^3_1 is said to be a slant helix if there exist a constant direction vector say W whose inner product with principal normal N of the curve is constant i.e, $\langle W, N \rangle = c$ [8]. In [7, 14], authors studied the position vector and spherical indicatrix of slant helices.

In [13], generalized LC helix on hypersurfaces of Minkowski space E_1^{n+1} were studied. A non null regular curve γ lying on hypersurface M of Minkowski space E_1^{n+1} is said to be generalized LC helix if there exist a vector field X along the curve γ such that $\langle \gamma', X\rangle = c(constant)$ and $\nabla_{\gamma'} X = 0$, where ∇ is Levi-Civita connection on hypersurface. Any line parallel to vector field X is said to be an axis of the generalized LC helix. A regular curve in hypersurface of Minkowski space E_1^4 be a generalized LC helix iff ratio of curvatures ($\kappa_1 = \kappa, \kappa_2 = \tau$) are constant $i.e, \frac{\kappa_1}{\kappa_2} = c(constant)$ [13]. In [16], authors studied the helices of order 6 in Euclidean sphere by using minimal embedding from complex projective space to a $7-$ dimensional Euclidean sphere.

We organize our paper as follows: In section 2, we discuss some definitions and important results that are useful to prove our main Theorems. In section 3, we study the proper helix of order 6 in E_4^8 by using isometric embedding with parallel second fundamental form from complex hyperbolic space $CH_2(-\frac{4c}{3})$ to pseudo-Euclidean space E_4^8. After that, we discussed the behavior of a curve into a submanifold M_ν^d of manifold E_4^8 in which the curve is lying on.

2. Preliminaries and some results

Let C^{n+1} be a complex vector space equipped with Hermitian metric

$$\tilde{g}(z, w) = -\bar{z}_1 w_1 + \sum_{i=2}^{i=n+1} \bar{z}_i w_i,$$

where $z = (z_1, ...z_{n+1})$ and $w = (w_1, ...w_{n+1})$ are vectors of C^{n+1}. Then the hypersurface anti-di Sitter space of vector space C^{n+1} is defined as

$$H_1^{2n+1} = \{z \in C^{n+1}; \tilde{g}(z, z) = -1\}.$$

The tangent space at $z \in H_1^{2n+1}$, is given by

$$T_z H_1^{2n+1} = \{(z, w); w \in C^{n+1}\, and\, Re(\tilde{g}(z, w)) = 0\}.$$

If $H_z(T_z H_1^{2n+1})$ and $V_z(T_z H_1^{2n+1})$, representing the horizontal and vertical decomposition of tangent space $T_z H_1^{2n+1}$, then

$$H_z(T_z H_1^{2n+1}) = \{(z, w); w \in C^{n+1}\, and \;\; \tilde{g}(z, w) = 0\},$$

$$V_z(T_z H_1^{2n+1}) = \{(z, i\nu z); \nu \in R\}.$$

Consider Π is a canonical projection from hypersurface anti-di Sitter space to CH^n with S^1- fiber bundle. Then the tangent space at $\Pi(z) \in CH^n$ is identified as a horizontal subspace of $T_z H_1^{2n+1}$ and follows by

$$T_{\pi(z)}CH^n = \{d\Pi((z,w)); z \in H_1^{2n+1}\, and\, (z,w) \in H_z(T_z H_1^{2n+1})\}.$$

Let g is the restriction of Hermitian metric $\tilde{g}$ on C^{n+1} to $H_z(T_z H_1^{2n+1})$, then

$$g(u,v) = \frac{4}{c} Re(\tilde{g}(u,v)), \qquad u,v \in T_{\pi(z)}CH^n \simeq H_z(T_z H_1^{2n+1}),$$

is a positive definite metric on CH^n. For some positive constant c, $CH^n(-c)$ is known as a complex hyperbolic space with constant holomorphic sectional curvature $-c$ and complex structure J. Where complex structure J is induced from canonical complex structure on C^{n+1}.

Let $\bar{\nabla}$, $\tilde{\nabla}$ and ∇ are the Riemannian connections on C^{n+1}, H_1^{2n+1} and $CH^n(-c)$ respectively, then we get

Lemma 2.1. [2] *For all horizontal vector fields X, Y on H_1^{2n+1}, we have*

$$\begin{cases} \bar{\nabla}_X Y = \tilde{\nabla}_X Y + g(X,Y)N, \\ \tilde{\nabla}_X Y = \nabla_X Y - g(X,JY)JN, \end{cases} \tag{2.1}$$

where N is normal to real hypersurface $H_1^{2n+1} \subset C^{n+1}$ and $g(N,N) = -1$.

LC Helix in Pseudo-Euclidean space E_ν^n

Pseudo-Euclidean $n-$ dimensional space E_ν^n with index ν is a real vector space R^n equipped with the metric

$$g'(x,y) = -\sum_{i=1}^{i=\nu} x^i y^i + \sum_{j=\nu+1}^{j=n} x^j y^j,$$

where $x = (x^1, ...x^n)$ and $y = (y^1, ...y^n) \in R^n$. Let $\gamma : (a,b) \longrightarrow E_\nu^n;\ a,b \in R$ be a curve parametrized by arc length parameter s, then the curve is said to be, spacelike if $g'(\gamma'(s), \gamma'(s)) = 1$, timelike if $g'(\gamma'(s), \gamma'(s)) = -1$ and lightlike if $g'(\gamma'(s), \gamma'(s)) = 0$.

Definition 2.1. [18] A non - null curve $\gamma(s)$ in a pseudo-Euclidean space parametrized by arc length is called a proper curve of order d if there exists a frame $\{V_1, ..., V_d\}$ of orthonormal fields along $\gamma(s)$ and satisfy the Frenet formula

$$\nabla'_{V_1} V_i = -\lambda_{i-2}\lambda_{i-1}\kappa_{i-1}V_{i-1} + \kappa_i V_{i+1}, \qquad 1 \leq i \leq d,$$

where $V_0 = V_{d+1} = 0$, $\kappa_0 = 0$ and ∇' is Levi-Civita connection on pseudo-Euclidean space. Now

$$\kappa_i = \parallel \nabla'_{V_1} V_i + \lambda_{i-2}\lambda_{i-1}\kappa_{i-1}V_{i+1} \parallel, \qquad 1 \leq i < d,$$

$$\lambda_{i-1} = g'(V_i, V_i), \qquad 1 \leq i < d,$$

where κ_i are known as the i^{th} curvatures of $\gamma(s)$. If κ_i $(1 \leq i \leq d-1)$ are constants along $\gamma(s)$, then $\gamma(s)$ is said to be a proper helix of order d in pseudo-Euclidean space.

Definition 2.2. [1] A curve γ in pseudo-Riemannian manifold is said to be a circle if it satisfies the following differential equation $\nabla'_X \nabla'_X X + g'(\nabla'_X X, \nabla'_X X) g'(X, X) X = 0$, where X is a tangent vector field along γ .

Next, we generalize the definition of generalized LC helix in hypersurface of E_1^n [13] to any submanifold of pseudo-Euclidean space E_ν^n.

Definition 2.3. Let M^d be a sub-manifold of E_ν^n, then a non null curve $\gamma(s)$ lying on M^d is said to be a generalized LC helix, if there exist a vector field U such that $g'(T, U) = C$ (some constant) and $\dot{\nabla}_T U = 0$. Where T is tangent vector field of $\gamma(s)$ and $\dot{\nabla}$ is Levi-Civita connection on M^d . Any line parallel to U is said to be an axis of $\gamma(s)$.

Definition 2.4. Let $\gamma : I \subset R \longrightarrow M^d$ be a generalized LC helix, then harmonic curvatures of γ are defined by $H_j : I \subset R \longrightarrow R$,

$$H_1 = \lambda_0 \lambda_1 \frac{\kappa_1}{\kappa_2},$$

$$H_j = (\lambda_{j-1}\lambda_j \kappa_j H_{j-2} + \dot{\nabla}_{V_1} H_{j-1}) \frac{1}{\kappa_{j+1}}, \qquad 2 \leq j \leq d-2,$$

where $H_0 = 0$ and κ_1, κ_2, . . ., κ_{d-1} are curvature functions of γ.

Let $f : M \longrightarrow N$ be an isometric embedding from Riemannian manifold M into a Riemannian manifold N. Also let ∇ and $\tilde{\nabla}$ are Riemannian connections on M and N respectively. Then the Gauss Weingarten formulae are given as follows

$$\tilde{\nabla}_T U = \nabla_T U + \sigma(T, U),$$

$$\tilde{\nabla}_T \eta = -A_\eta T + D_T \eta,$$

where T, U are smooth sections of TM and η is a smooth section of NM. The symbols $\sigma(T, U)$ and $A_\eta X$ are known as second fundamental form and shape operator on M respectively. The connection $\hat{\nabla}$ of the second fundamental form σ on the space of tangent bundle and normal bundle is defined as [10]

$$(\hat{\nabla}_T \sigma)(U, V) = D_T(\sigma(U, V)) - \sigma(\nabla_T U, V) - \sigma(U, \nabla_T V),$$

where T, U and V are smooth sections of tangent bundle TM. The second fundamental form is said to be parallel if $\hat{\nabla}\sigma = 0$. An isometric embedding from complex hyperbolic space CH^n to pseudo-Euclidean space $E^{n^2+2n+1}_{n^2+1}$ was defined in [11] and provides the following remark.

Remark. The isometric embedding from CH^n to $E^{n^2+2n}_{n^2}$ has a parallel second fundamental form.

For $n = 2$, the above remark ensure the existence of an isometric embedding from the complex hyperbolic plane CH^2 to E^8_4 with parallel second fundamental form.

3. Main Results

Theorem 3.1. *Every circle of curvature $\kappa > \sqrt{\frac{4c}{3}}$ and torsion $0 < \tau < 1$ in complex hyperbolic plane of holomorphic sectional curvature $-\frac{4c}{3}$ ($CH^2(-\frac{4c}{3})$) are proper helix of order 6 in E^8_4 by an isometric embedding $i : CH^2(-\frac{4c}{3}) \longrightarrow E^8_4$ which has parallel second fundamental .*

Proof. Let $i : CH^2(-\frac{4c}{3}) \longrightarrow E^8_4$ be an isometric embedding with parallel second fundamental form. Now Gauss Weingarten formulae are

$$\begin{cases} \nabla'_T U = \nabla_T U + \sigma(T, U), \\ \nabla'_T \eta = -A_\eta T + D_T \eta, \end{cases} \tag{3.1}$$

where T, U are smooth sections of $TCH^2(-\frac{4c}{3})$ and η is smooth section of $NCH^2(-\frac{4c}{3})$. Also ∇ and ∇' are Riemannian connections on $CH^2(-\frac{4c}{3})$ and E^8_4 respectively. The scalar product of first normal space is defined as [11, 10]

$$\begin{aligned}\langle\sigma(T,U),\sigma(V,W)\rangle &= -\frac{c}{3}\{2\langle T,U\rangle\langle V,W\rangle+\langle T,V\rangle\langle U,W\rangle+\langle T,W\rangle\langle U,V\rangle\\ &+\langle T,JV\rangle\langle U,JW\rangle+\langle T,JW\rangle\langle U,JV\rangle\}.\end{aligned} \tag{3.2}$$

As $\langle A_{\sigma(V,W)}T,U\rangle = \langle\sigma(T,U),\sigma(V,W)\rangle$, therefore from equation (3.2), we obtain

$$A_{\sigma(V,W)}T = -\frac{c}{3}\{2\langle V,W\rangle T+\langle T,V\rangle W+\langle T,W\rangle V+\langle T,JV\rangle JW+\langle T,JW\rangle JV\}. \tag{3.3}$$

Consider γ be a circle of curvature $\kappa > \sqrt{\frac{4c}{3}}$ and torsion $0 < \tau < 1$ on $CH^2(-\frac{4c}{3})$. Then by the definition of circle

$$\nabla_T T = \kappa U, \qquad \textit{and} \qquad \nabla_T U = -\kappa T, \tag{3.4}$$

where T is tangent vector field of $\gamma(s)$ and U is some vector field orthonormal to T. Let $\{T,U,V,W,X,Y,Z,N\}$ be an orthonormal frame along $i\circ\gamma$. Then T,U,V and W will be smooth sections of tangent bundle $TCH^2(-\frac{4c}{3})$. Whereas the remaining vector fields X,Y,Z and N will be sections of normal bundle. The vector fields V and W are already calculated in [5]. The vector fields X and Y are easy to assume from (3.1) and whether they are independent or not can be checked by using equation (3.2). Since $\langle\sigma(T,U),\sigma(U,U)\rangle = 0$. Therefore, Z can be calculated by taking the combination of X and Y. Whereas the vector field N is orthogonal to the first normal space. Thus

$$V = \frac{JT+\tau U}{\sqrt{1-\tau^2}}, \qquad W = \frac{JU-\tau T}{\sqrt{1-\tau^2}}, \qquad X = \frac{\sigma(T,T)}{\sqrt{\frac{4c}{3}}},$$

$$Y = \frac{\sigma(T,U)}{\sqrt{\frac{c(1-\tau^2)}{3}}}, \qquad Z = \frac{\sigma(U,U)-(\frac{1+\tau^2}{2})\sigma(T,T)}{\sqrt{\frac{c(3-\tau^4-2\tau^2)}{3}}},$$

where T,U,V and W are spacelike vector fields whereas X,Y and Z are timelike vector fields. Also

$$\begin{aligned}\sigma(T,JU) &= a_1X+a_2Y+a_3Z\\ &= \tau\sigma(T,T)+2\tau(1-\tau^2)\sqrt{\frac{c}{3(3-\tau^4-2\tau^2)}}Z.\end{aligned} \tag{3.5}$$

Here $a_1 = \langle \sigma(T, JU), X\rangle = \tau$, $a_2 = \langle \sigma(T, JU), Y\rangle = 0$ and $a_3 = \langle \sigma(T, JU), Z\rangle = 2\tau(1-\tau^2)\sqrt{\frac{c}{3(3-\tau^4-2\tau^2)}}$. Now, for circle γ of curvature $\kappa > \sqrt{\frac{4c}{3}}$ and torsion $0 < \tau < 1$ on $CH^2(-\frac{4c}{3})$, from (3.4) and (3.1), $i \circ \gamma$ holds the relation

$$\begin{aligned} \nabla'_T T &= \nabla_T T + \sigma(T, T) \\ &= \kappa_1 \left(\frac{1}{\kappa_1}(\kappa U + \sigma(T, T)) \right) \\ &= \kappa_1 F_2. \end{aligned} \tag{3.6}$$

Here $\kappa_1 = \sqrt{\frac{3\kappa^2-4c}{3}}$ and $F_2 = \frac{1}{\kappa_1}(\kappa U + \sigma(T, T))$ is a spacelike unit vector field. To calculate the κ_1, we use the property that i is parallel embedding and equation (3.2). Now, the covariant derivative of F_2 with respect to T, will follow the following equation

$$\begin{aligned} \nabla'_T F_2 &= \frac{\kappa}{\kappa_1}\nabla'_T U + \kappa \nabla'_T \sigma(T, T) \\ &= \frac{1}{\kappa_1}\left(\kappa(\nabla_T U + \sigma(T, U)) - A_{\sigma(T,T)}T + D_T(\sigma(T, T))\right). \end{aligned}$$

As i is parallel, therefore $D_T(\sigma(T, T)) = \sigma(\nabla_T T, T) + \sigma(T, \nabla_T T)$. Thus above equation gives us

$$\nabla'_T F_2 = -\kappa_1 T + \kappa_2 F_3, \tag{3.7}$$

where $\kappa_2 = 3\kappa\sqrt{\frac{c(1-\tau^2)}{3\kappa^2-4c}}$ and $F_3 = \frac{\sigma(T,U)}{\sqrt{\frac{c(1-\tau^2)}{3}}} = Y$ is a timelike vector field. Now the covariant derivative of F_3 with respect to T, gives the relation

$$\nabla'_T F_3 = \kappa_2 F_2 + \kappa_3 F_4, \tag{3.8}$$

where $\kappa_3 = \sqrt{\frac{(6\kappa^2-2c)^2-27c\kappa^2\tau^2}{3(3\kappa^2-4c)}}$ and $F_4 = \sqrt{\frac{3(3\kappa^2-4c)}{(6\kappa^2-2c)^2-27c\kappa^2\tau^2}}\{\sqrt{3(1-\tau^2)}(\frac{-3\kappa^2}{3\kappa^2-4} + \frac{1}{3})U + \sqrt{\frac{c}{3}}\tau V + \kappa\sqrt{\frac{3-\tau^4-2\tau^2}{1-\tau^2}}Z - 2\sqrt{1-\tau^2}(\frac{3c\kappa}{3\kappa^2-4c} + \frac{\kappa}{2})X\}$ is a spacelike unit vector field. Similarly

$$\nabla'_T F_4 = -\kappa_3 F_3 + \kappa_4 F_5, \tag{3.9}$$

where $\kappa_4 = 3\kappa\tau\sqrt{\frac{c(3\kappa^2-4c)}{(6\kappa^2-2)^2-27\kappa^2\tau^2}}$ and $F_5 = W = \frac{JU-\tau T}{\sqrt{1-\tau^2}}$. Now

$$\nabla'_T F_5 = \kappa_4 F_4 + \kappa_5 F_6, \tag{3.10}$$

where $\kappa_5 = \sqrt{\kappa_4^2 + \kappa^2 - \frac{4c\tau^2(1-\tau^2)}{3(3-\tau^4-2\tau^2)}}$ and $F_6 = \frac{1}{\kappa_5}(-\kappa V - \kappa_4 F_4 + \frac{2\tau\sqrt{c(1-\tau^2)}}{\sqrt{3(3-\tau^4-2\tau^2}} Z)$. The covariant derivative of F_6 along the curve $i \circ \gamma$, will be

$$\nabla'_T F_6 = -\kappa_5 F_5. \tag{3.11}$$

Equations (3.6) - (3.11), showing that $\{T = F_1, F_2, F_3, F_4, F_5, F_6\}$ is an orthogonal frame satisfying the condition

$$\nabla'_T F_i = -\lambda_{i-2}\lambda_{i-1}\kappa_{i-1}F_{i-1} + \kappa_i F_{i+1}, \qquad 1 \leq i \leq 6,$$

where $F_0 = F_7 = 0$ and $\kappa_0 = 0$, along the curve $i \circ \gamma$ in E^8_4. Thus, the curve $i \circ \gamma$ is a proper curve of order 6 in E^8_4. Since curvature functions along $i \circ \gamma$ are constants. Thus $i \circ \gamma$ is a proper helix of order 6 in E^8_4.
In frame $\{F_1, F_2, F_3, F_4, F_5, F_6\}$, the vector fields F_3 and F_4 are timelike vector fields whereas the remaining vector fields are spacelike vector fields. □

Corollary 3.1. *If $i \circ \gamma$ be a curve with $\kappa_1(s) = \frac{2c}{\sqrt{\kappa_2^2 - 3c}}$ and $\kappa_2 > \sqrt{3c}$ in E^8_4, then $i \circ \gamma$ is a proper helix of order 4 in E^8_4.*

Proof. From equation (3.6) and (3.7), we have $\kappa = \sqrt{\frac{3\kappa_1^2+4c}{3}}$ and $\tau = \sqrt{\frac{c(3\kappa_1^2+4c)-\kappa_1^2\kappa_2^2}{c(3\kappa_1^2+4c}}$. After substituting these values of κ, τ in κ_3 and κ_4 of above theorem, we get $\kappa_3 = \sqrt{\kappa_1^2 + \kappa_2^2 + 5c}$ and $\kappa_4 = 0$. Thus $i \circ \gamma$ is a proper helix of order 4 in E^8_4. □

Theorem 3.2. *For a circle $\gamma(s)$ with $0 < \kappa < \sqrt{\frac{c}{13}}$ and $\frac{1}{3} < \tau < 1$ in complex hyperbolic plane $CH^2(-\frac{4c}{3})$. The image $i \circ \gamma$ is a proper helix of order 6 in E^8_4, where i is an isometric embedding from $CH^2(-\frac{4c}{3})$ to E^8_4 with parallel second fundamental form.*

Proof. Let $\gamma(s)$ be a circle with $0 < \kappa < \sqrt{\frac{c}{13}}$ and $\frac{1}{3} < \tau < 1$ in complex hyperbolic plane $CH^2(-\frac{4c}{3})$. If $i : CH^2(-\frac{4c}{3}) \longrightarrow E^8_4$ be an isometric embedding with parallel second fundamental form, then similar to Theorem 3.1, we define the orthonormal frame along $i \circ \gamma$. Now from (3.1), (3.2) and (3.4), we obtain

$$\nabla'_T T = \bar{\kappa}_1 E_2, \tag{3.12}$$

where $\bar{\kappa}_1 = \sqrt{\frac{4c-3\kappa^2}{3}}$ and $E_2 = \frac{1}{\bar{\kappa}_1}(\kappa U + \sigma(T, T))$. Now, we have

$$\nabla'_T E_2 = \bar{\kappa}_1 T + \bar{\kappa}_2 E_3, \tag{3.13}$$

where $\bar{\kappa}_2 = 3\kappa\sqrt{\frac{c(1-\tau^2)}{4c-3\kappa^2}}$ and $E_3 = \frac{\sigma(T,U)}{\sqrt{\frac{c(1-\tau^2)}{3}}} = Y$. Also

$$\nabla'_T E_3 = -\bar{\kappa}_2 E_2 + \bar{\kappa}_3 E_4, \tag{3.14}$$

where $\bar{\kappa}_3 = \sqrt{\frac{(6\kappa^2-2c)^2-27c\kappa^2\tau^2}{3(4c-3\kappa^2)}}$ and $E_4 = \sqrt{\frac{3(4c-3\kappa^2)}{(6\kappa^2-2c)^2-27c\kappa^2\tau^2}}\{\sqrt{3c(1-\tau^2)}(\frac{3\kappa^2}{4c-3\kappa^2}+\frac{1}{3})U$ $+\sqrt{\frac{c}{3}}\tau V + \kappa\sqrt{\frac{3-\tau^4-2\tau^2}{1-\tau^2}}Z + 2\sqrt{1-\tau^2}(\frac{3c\kappa}{4c-3\kappa^2}-\frac{\kappa}{2})X\}$. Similarly

$$\nabla'_T E_4 = \bar{\kappa}_3 E_3 + \bar{\kappa}_4 E_5, \tag{3.15}$$

where $\bar{\kappa}_4 = 3\kappa\tau\sqrt{\frac{c(4c-3\kappa^2)}{(6\kappa^2-2c)^2-27c\kappa^2\tau^2}}$ and $E_5 = W = \frac{JU-\tau T}{\sqrt{1-\tau^2}}$. Now

$$\nabla'_T E_5 = -\bar{\kappa}_4 E_4 + \bar{\kappa}_5 E_6, \tag{3.16}$$

where $\bar{\kappa}_5 = \sqrt{\bar{\kappa}_4^2 - \kappa^2 + \frac{4c\tau^2(1-\tau^2)}{3c(3-\tau^4-2\tau^2)}}$ and $E_6 = \frac{1}{\kappa_5}(-\kappa V + \kappa_4 E_4 + \frac{2\tau\sqrt{c(1-\tau^2)}}{\sqrt{3(3-\tau^4-2\tau^2}}Z)$. The covariant derivative of E_6 along $i \circ \gamma$, gives us

$$\nabla'_T E_6 = \bar{\kappa}_5 E_5, \tag{3.17}$$

From equation (3.12) - (3.17), we get the frame $\{T = E_1, E_2, E_3, E_4, E_5, E_6\}$ of orthonormal fields satisfying the condition

$$\nabla'_T E_i = -\lambda_{i-2}\lambda_{i-1}\kappa_{i-1}E_{i-1} + \kappa_i E_{i+1}, \qquad 1 \le i \le 6,$$

where $E_0 = E_7 = 0$ and $\kappa_0 = 0$, along the curve $i \circ \gamma$ in E_4^8. Thus, the curve $i \circ \gamma$ is a proper helix of order 6.

In frame $\{E_1, E_2, E_3, E_4, E_5, E_6\}$, the vector fields E_2, E_3 and E_6 are timelike vector fields, whereas remaining three vector fields are spacelike vector fields. □

Theorem 3.3. *Any circle with torsion $\tau = 1$ in $CH^2(-\frac{4c}{3})$ be remain a circle in E_4^8 by an isometric embedding $i : CH^2(-\frac{4c}{3}) \longrightarrow E_4^8$ which has parallel second fundamental form.*

Proof. Let γ be a circle in $CH^2(-\frac{4c}{3})$ with $\tau = 1$, then the unit tangent vector field T of γ and the vector field U orthogonal to T, holds the relation

$$\nabla_T T = \kappa U, \qquad \textit{and} \qquad \nabla_T U = -\kappa T,$$

where κ is some positive constant function. From equation (3.1), we have

$$\nabla'_T T = \kappa_1 V_2,$$

were κ_1 is either, $\kappa_1 = \sqrt{\frac{3\kappa^2-4c}{3}}$ if $\kappa > \sqrt{\frac{4c}{3}}$, or $\kappa_1 = \sqrt{\frac{4c-3\kappa^2}{3}}$ if $\kappa < \sqrt{\frac{4c}{3}}$. Thus, the covariant derivative of $V_2 = \frac{1}{\kappa_1}(\kappa U + \sigma(T,T))$, along $i \circ \gamma$ is

$$\nabla'_T V_2 = -\epsilon\kappa_1 T,$$

where $\epsilon = g'(V_2, V_2)$. Also $\nabla'_T\nabla'_T T + g'(\nabla'_T T, \nabla'_T T)g'(T,T)T = 0$, hence $i \circ \gamma$ is a circle in E^8_4. □

Theorem 3.4. *A circle of curvature $\kappa > \sqrt{\frac{4}{3}}$ and torsion $0 < \tau < 1$ in complex hyperbolic plane $CH^2(-\frac{4}{3})$ is a generalized LC helix in some submanifold of dimension 6 and index 2 (M^6_2) of manifold E^8_4 by an isometric embedding $i : CH^2(-\frac{4}{3}) \longrightarrow E^8_4$ with parallel second fundamental form.*

Proof. Let $i : CH^2(-\frac{4}{3}) \longrightarrow E^8_4$ be an isometric embedding with parallel second fundamental form and $\gamma(s)$ is a circle with $\kappa > \sqrt{\frac{4}{3}}$ and $0 < \tau < 1$ lying on $CH_2(-\frac{4}{3})$, then

$$\nabla_T T = \kappa U, \qquad \textit{and} \qquad \nabla_T U = -\kappa T,$$

where T is tangent vector field of $\gamma(s)$ and the vector field U is orthonormal to T. As we proved in Theorem 3.1, there exists a frame $\{T, F_2, F_3, F_4, F_5, F_6\}$ of orthonormal vector fields along $i \circ \gamma$ in E^8_4. Now corresponding to this frame we consider a submanifold of dimension 6 and index 2 (M^6_2) of the manifold E^8_4 in which $i \circ \gamma$ is lying and these orthonormal fields are smooth sections of $T_{i\circ\gamma}M^6_2$. Now to prove $i \circ \gamma$ is a generalized LC helix in M^6_2, we have to find a vector field U which is parallel along $i \circ \gamma$ and $g(U,T) = a = \textit{constant}$. The metric g and the Levi-Civita connection $\dot{\nabla}$ on M^6_2 are defined as $g = g'|_{M^6_2}$ and $\dot{\nabla} = \nabla'|_{M^6_2}$ respectively.
Let U be a vector field parallel along $i \circ \gamma$ and $g(U,T) = a$, then U can be expressed as a linear combination of orthonormal fields $\{T, F_2, F_3, F_4, F_5, F_6\}$ *i.e*,

$$U = aT + u_2F_2 + u_3F_3 + u_4F_4 + u_5F_5 + u_6F_6, \tag{3.18}$$

where $u_i = \lambda_{i-1} g(U, F_i)$, $2 \leq i \leq 6$ and $\lambda_{i-1} = g(F_i, F_i)$. Now

$$0 = Tg(T,U) = Tg'|_{M_2^6}(T,U) = \kappa_1 g(F_2,U),$$

$$0 = Tg(F_2,U) = -\kappa_1 g(T,U) + \kappa_2 g(F_3,U),$$

$$0 = Tg(F_3,U) = \kappa_2 g(F_2,U) + \kappa_3 g(F_4,U) = \kappa_3 g(F_4,U),$$

$$0 = Tg(F_4,U) = -\kappa_3 g(F_3,U) + \kappa_4 g(F_5,U),$$

$$0 = Tg(F_5,U) = -\kappa_5 g(F_6,U).$$

Substituting these values in equation (3.18), we obtain

$$U = aF_1 - a\frac{\kappa_1}{\kappa_2}F_3 + a\frac{\kappa_1}{\kappa_2}\frac{\kappa_3}{\kappa_4}F_5. \tag{3.19}$$

Since, $g(U,U) = a^2 - (a\frac{\kappa_1}{\kappa_2})^2 + (a\frac{\kappa_1}{\kappa_2}\frac{\kappa_3}{\kappa_4})^2$. Therefore using data from Theorem 3.1, we obtain

$$g(U,U) = a^2\left(\frac{27\kappa^2\tau^2(27\kappa^2 - (3\kappa^2-4)^2 - 2(6\kappa^2-2)^2) + (6\kappa^2-4)^4}{729\kappa^4\tau^2}\right). \tag{3.20}$$

Next we shell show that $g(U,U) \neq 0$. Suppose $g(U,U) = 0$, then

$$\tau^2 = \frac{(6\kappa^2-4)^4}{27\kappa^2(-27\kappa^2 + (3\kappa^2-4)^2 + 2(6\kappa^2-2)^2)}. \tag{3.21}$$

Since $0 < \tau < 1$, therefore the equation (3.21), will have a solution only if the right hand side of the equation is also lie between zero and one, whereas

$$(6\kappa^2-4)^4 - 27\kappa^2(-27\kappa^2 + (3\kappa^2-4)^2 + 2(6\kappa^2-2)^2) > 0,$$

which implies that

$$\frac{(6\kappa^2-4)^4}{27\kappa^2(-27\kappa^2 + (3\kappa^2-4)^2 + 2(6\kappa^2-2)^2)} > 1. \tag{3.22}$$

From (3.21) and (3.22), we can conclude that $g(U,U) \neq 0$. Thus U is non null vector field and an axis of generalized LC helix $i \circ \gamma$ in M_2^6. □

Theorem 3.5. *If $\gamma(s)$ be a circle with $0 < \kappa < \frac{1}{4\sqrt{3}}$ and $\frac{1}{3} < \tau < 1$ in $CH^2(-\frac{4}{3})$), then $\gamma(s)$ is a generalized LC helix in some submanifold M_3^6 of manifold E_4^8 by isometric embedding $i : CH^2(-\frac{4}{3}) \longrightarrow E_4^8$ with parallel second fundamental form.*

Proof. Let $i : CH^2(-\frac{4}{3}) \longrightarrow E^8_4$ be an isometric embedding with parallel second fundamental form and $\gamma(s)$ is a circle with $0 < \kappa < \frac{1}{\sqrt{13}}$, $\frac{1}{3} < \tau < 1$ lying on $CH_2(-\frac{4}{3})$. As we proved in Theorem 3.2, we can find an orthonormal frame $\{T, E_2, E_3,\ E_4, E_5, E_6\}$ along $i \circ \gamma$ in E^8_4. In frame $\{T, E_2, E_3,\ E_4, E_5, E_6\}$ vector fields E_2, E_3 and E_6 are timelike vector fields whereas the remaining vector fields are spacelike vector fields. Now corresponding to this frame we consider a submanifold of dimension 6 and index 3 (M^6_3) of the manifold E^8_4 in which $i \circ \gamma$ is lying and these orthonormal fields are smooth sections of $T_{i\circ\gamma}M^6_3$. To prove $i \circ \gamma$ be a generalized LC helix in M^6_3, we have to find a vector field S which is parallel along $i \circ \gamma$ and $g(S, T) = a(constant)$.

Let S be a vector field parallel along $i \circ \gamma$ and $g(S, T) = a$, then

$$S = aT + s_2E_2 + s_3E_3 + s_4E_4 + s_5E_5 + s_6E_6, \tag{3.23}$$

where $s_i = \lambda_{i-1}g(S, T)$, $2 \leq i \leq 6$ and $\lambda_{i-1} = g(E_i, E_i)$. As $\tilde{\nabla}_T S = 0$ and $g(S, T) = a$. If we use these two relations to solve equation (3.23), we get

$$g(E_2, S) = g(E_4, S) = g(E_6, S) = 0,$$

$$g(E_3, S) = -a\frac{\bar{\kappa}_1}{\bar{\kappa}_2}\ \ and\ \ g(E_5, S) = a\frac{\bar{\kappa}_1}{\bar{\kappa}_2}\frac{\bar{\kappa}_3}{\bar{\kappa}_4}.$$

Substituting these values in equation (3.23), we obtain

$$S = aT - a\frac{\bar{\kappa}_1}{\bar{\kappa}_2}E_3 + a\frac{\bar{\kappa}_1}{\bar{\kappa}_2}\frac{\bar{\kappa}_3}{\bar{\kappa}_4}E_5. \tag{3.24}$$

Since $g(S, S) = a^2 - (a\frac{\bar{\kappa}_1}{\bar{\kappa}_2})^2 + (a\frac{\bar{\kappa}_1}{\bar{\kappa}_2}\frac{\bar{\kappa}_3}{\bar{\kappa}_4})^2$, therefore substitution of $\bar{\kappa}_1$, $\bar{\kappa}_2$, $\bar{\kappa}_3$ and $\bar{\kappa}_4$ from Theorem 3.3, provides

$$g(S, S) = a^2\left(\frac{27\kappa^2\tau^2(27\kappa^2 - (3\kappa^2 - 4)^2 - 2(6\kappa^2 - 2)^2) + (6\kappa^2 - 4)^4}{729\kappa^4\tau^2}\right). \tag{3.25}$$

Next we shell show that $g(S, S) \neq 0$. If $g(S, S) = 0$, then

$$\tau^2 = \frac{(6\kappa^2 - 4)^4}{27\kappa^2(-27\kappa^2 + (3\kappa^2 - 4)^2 + 2(6\kappa^2 - 2)^2)}. \tag{3.26}$$

In our problem $\frac{1}{3} < \tau < 1$, whereas

$$\begin{aligned}(6\kappa^2-4)^4 - & 27\kappa^2(-27\kappa^2+(3\kappa^2-4)^2+2(6\kappa^2-2)^2) \\ &= 12896\kappa^8 - 3915\kappa^6 + 3537\kappa^4 - 840\kappa^2 + 16 \\ &= (3\kappa^2-1)(48\kappa^2-1)(3\kappa^2-4)^2 > 0 \ \forall \ 0 < \kappa < \frac{1}{4\sqrt{3}},\end{aligned}$$

which implies that

$$\frac{(6\kappa^2-4)^4}{27\kappa^2(-27\kappa^2+(3\kappa^2-4)^2+2(6\kappa^2-2)^2)} > 1. \tag{3.27}$$

From (3.26) and (3.27), we can conclude that $g(S, S) \neq 0$. Thus S is non null vector field and an axis of generalized LC helix $i \circ \gamma$ with $0 < \kappa < \frac{1}{4\sqrt{3}}$ and $\frac{1}{3} < \tau < 1$ in M_2^6. □

Conclusion

In this article, we use an isometric embedding, so we can conclude that there exists both bounded and unbounded proper helix of order 6 (generalized LC helix) in E_4^8. For example,

Let $\gamma(s)$ be a circle of curvature κ and torsion $\tau = \pm 1$ in $CH^2(-4)$, then from [3]

(i) For $\kappa > 2$, γ is a simple closed curve with prime period $\frac{2\pi}{\sqrt{\kappa^2-4}}$.

(2) For $\kappa \leq 2$, γ is a simple two sides unbounded open curve.

Since i is an isometric embedding, therefore from Theorem 3.1 ($c = 3$), we can conclude that

(1) For $\kappa = 3$ and $\tau = 1$, $i \circ \gamma$ is a circle of length $\frac{2\pi}{\sqrt{\kappa^2-4}}$.

(2) For $\kappa = 1$ and $\tau = 1$, $i \circ \gamma$ will be unbounded circle.

Similarly, if we consider $\gamma(s)$ be a circle of curvature κ and torsion $-1 < \tau < 1$ in $CH^2(-4)$, then for $\kappa \leq \kappa(\tau)$, $\gamma(s)$ is a simple two sides unbounded curve [3], where $\kappa(\tau)$ is a unique solution of

$$\kappa^2\tau^2 - \frac{4}{27}(\kappa^2-1)^3 = 0.$$

Now, using the fact that i is an isometric embedding in Theorem 3.1 (Theorem 3.2), we have

(3) For $\kappa \leq \frac{1}{6}\frac{\sqrt{6}\sqrt{\left(108+12\sqrt{69}\right)^{\frac{1}{3}}\left(\left(108+12\sqrt{69}\right)^{\frac{2}{3}}+12+6\left(108+12\sqrt{69}\right)^{\frac{1}{3}}\right)}}{\left(108+12\sqrt{69}\right)^{\frac{1}{3}}}$ and $\tau^2 = \frac{4}{27}$, $i \circ \gamma$ is a unbounded proper helix in E_4^8.

Acknowledgement

We would like to thank the editor and the referees for their valuable suggestions toward the improvement of the paper. Santosh Kumar would also like to thank UGC India for their financial support, Ref. No. 1068/ (CSIR - UGC NET JUNE 2019).

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