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QUASI BI-SLANT SUBMANIFOLDS OF NEARLY KAEHLER MANIFOLDS

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Abstract. We introduce the notion of quasi bi-slant submanifolds of nearly Kaehler manifolds and study some of thier properties. The necessary and sufficient conditions for the integrability of distributions, involved in the definition of quasi bi-slant submanifolds of nearly Kaehler manifolds, are obtained. We also investigate the necessary and sufficient conditions for quasi bi-slant submanifolds of nearly Kaehler manifolds to be totally geodesic, and we study the geometry of foliations. Finally, we construct some non-trivial examples of quasi bi-slant manifolds of nearly Kaehler manifolds.

1. Introduction

The theory of submanifolds has several important applications in Mathematics, Physics and Mechanics. In the last two decades, the applications of Kaehler manifolds are widely recognized (especially in Physics, for the target spaces of non-linear σ-models with super-symmetry [10].

Bejancu [2] introduced the notion of CR-submanifolds of Kaehler manifolds, which is the generalization of holomorphic submanifolds and totally real submanifolds. Another generalization of holomorphic and totally submanifolds of Kaehler manifolds were given by Chen [7], named as slant submanifolds. The properties of slant submanifolds of an almost Hermitian manifolds have been studied by many authors for instance [1, 8, 9]. A natural generalization of CR-submanifolds, slant submanifolds,

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holomorphic submanifolds and totally real submanifolds of almost Hermitian manifolds was defined by N. Papaghiuc [19], which is known as semi-slant submanifolds. A. Lotta [17, 18] defined and studied slant submanifolds in contact geometry. Cabrerizo et al. [4, 5] have studied slant and semi-slant submanifolds in contact geometry, respectively. In [6], Carriazo studied the properties of bi-slant submanifolds within the context of contact geometry. Several geometers have established many remarkable results of bi-slant submanifolds in the context of different structures for instance, [16, 20, 21, 22, 23, 25]. A nearly Kaehler structure on a complex manifold provides an interesting study with differential geometric point of view see [11, 13, 14]. Consequently, the study of submanifolds of a nearly Kaehler manifold vis-á-vis that of a Kaehler manifold assumes significance in general.

In 1981, Chen [8] defined the canonical de Rham cohomology class for closed CR-submanifolds in a Kaehler manifold, and it has been further studied by Deshmukh [12] within the framework of nearly Kaehler manifolds in 1982. In 2014, Sahin [24] studied the de Rham Cohomology class of hemi-slant submanifolds of Kaehler manifolds.

Motivated from above interesting and significant studies, we define quasi bi-slant submanifolds of nearly Kaehler manifolds. We organize our work as follows: In the second section, we mention the basic definitions and some known results of complex structures. In Section 3, we define quasi bi-slant submanifolds of nearly Kaehler manifolds. The necessary and sufficient conditions for quasi bi-slant submanifolds of Kaehler manifolds to be integrable are given in Section 4. The fifth section deals with the study of geometry of leaves of distributions, which are involved in the definition of quasi bi-slant submanifolds, and give the necessary and sufficient conditions for such submanifolds to be totally geodesic. In the last section, we provide non-trivial examples of quasi bi-slant submanifolds of Kaehler manifolds.

2. Preliminaries

Let M be an even dimensional differentiable manifold and J denotes a $(1,1)$ tensor field on M, such that

$$J^2 = -I,$$

where I is the identity operator. Then J is called an almost complex structure on M. The manifold M with an almost complex structure J is called an almost complex manifold [26].

Let g be a Riemannian metric on M, such that

$$g(JX, JY) = g(X, Y), \quad \forall\ X, Y \in \Gamma(TM), \tag{2.1}$$

where $\Gamma(TM)$ represents the Lie algebra of vector fields in M. Then g is called an almost Hermitian metric on M and the manifold M equipped with Hermitian metric g is called an almost Hermitian manifold. Additionally, if the almost complex structure J satisfies

$$(\overline{\nabla}_X J)Y + (\overline{\nabla}_Y J)X = 0 \tag{2.2}$$

for every $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ is the Levi-Civita connection on M, then M is said to be a nearly Kaehler manifold. Putting X for Y in (2.2), we find

$$(\overline{\nabla}_X J)X = 0. \tag{2.3}$$

Throughout this paper A and h denote the shape operator and second fundamental form of submanifold L into M, respectively.

If ∇ is the induced Riemannian connection on L, then the Gauss and Weingarten formulae of L into M are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.4}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.5}$$

for all vector $X, Y \in \Gamma(TL)$ and $V \in \Gamma(T^{\perp}L)$, where $\nabla^{\perp}$ denotes the connection on the normal bundle $(T^{\perp}L)$ of L. The shape operator and the second fundamental form are related by

$$g(A_V X, Y) = g(h(X, Y), V).$$

The mean curvature vector is defined by

$$H = \frac{1}{n} trace(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where n denotes the dimension of submanifold L and $\{e_1, e_2, e_3, \ldots, e_n\}$ is the local orthonormal basis of tangent space at each point of L.

A submanifold L of a nearly Kaehler manifold M is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H,$$

where H is the mean curvature vector. If $h(X, Y) = 0$ for every $X, Y \in \Gamma(TL)$, then L is said to be a minimal submanifold.

For any $X \in \Gamma(TL)$, we can write

$$JX = lX + mX, \tag{2.6}$$

where lX and mX are the tangential and normal components of JX on L, respectively. Similarly for any $V \in \Gamma(T^L)$, we have

$$JV = BV + CV, \tag{2.7}$$

where BV and CV are the tangential and normal components of JV on L, respectively.

The covariant derivative of complex structure J is defined as

$$(\overline{\nabla}_X J)Y = \overline{\nabla}_X JY - J\overline{\nabla}_X Y. \tag{2.8}$$

Now, for $X, Y \in \Gamma(TL)$,

$$(\overline{\nabla}_X J)Y = \mathfrak{P}_X Y + \mathfrak{Q}_X Y, \tag{2.9}$$

where $\mathfrak{P}_X Y$ and $\mathfrak{Q}_X Y$ denote the tangential and normal parts of $(\overline{\nabla}_X J)Y$, respectively.

From (2.4), (2.5), (2.6) and (2.7), we get

$$\mathfrak{P}_X Y = (\overline{\nabla}_X l)Y - A_{mY} - Bh(X, Y)$$

and

$$\mathfrak{Q}_X Y = (\overline{\nabla}_X m)Y + h(X, lY) - Ch(X, Y).$$

Similarly, for any $V \in T^{\perp}M$, $\mathfrak{P}_X V$ and $\mathfrak{Q}_X V$ denote, respectively, the tangential and normal parts of $(\overline{\nabla}_X J)V$, and

$$\mathfrak{P}_X V = (\overline{\nabla}_X B)V + lA_V X - A_{CV} X,$$

$$\mathfrak{Q}_X V = (\overline{\nabla}_X C)V + h(BV, X) + mA_V X.$$

Notice that on a Riemannian submanifold L of a nearly Kaehler manifold M, from equations (2.3) and (2.9), we have

$$\mathfrak{P}_X Y + \mathfrak{P}_Y X = 0,$$

$$\mathfrak{Q}_X Y + \mathfrak{Q}_Y X = 0.$$

The covariant derivative of projection morphisms in (2.6) and (2.7) are defined as:

$$(\overline{\nabla}_X l)Y = \nabla_X lY - l\nabla_X Y,$$

$$(\overline{\nabla}_X m)Y = \nabla^{\perp}_X mY - m\nabla_X Y,$$

$$(\overline{\nabla}_X B)V = \nabla_X BV - B\nabla^{\perp}_X V,$$

$$(\overline{\nabla}_X C)V = \nabla^{\perp}_X CV - C\nabla^{\perp}_X V$$

for any $X, Y \in \Gamma(TL)$ and $V \in \Gamma(T^{\perp}L)$.

Now we recall the following definitions, assuming that L is a submanifold of Hermitian manifold M.

Definition 2.1. *Let L be a Riemannian manifold isometrically immersed in an almost Hermitian manifold M. A submanifold L is said to be invariant (holomorphic or complex)* [3] *if $J(T_xL) \subseteq T_xL$, for every point $x \in L$.*

Definition 2.2. *A submanifold L is said to be anti-invariant (totally real)* [15] *if $J(T_xL) \subseteq T^{\perp}_xL$, for every point $x \in L$.*

Definition 2.3. *A submanifold L is said to be slant* [4]*, if for each non-zero vector X tangent to L at $x \in L$, the angle $\theta(X)$ between JX and T_xL is constant, i.e., it does not depend on the choice of the point $x \in L$ and $X \in T_xL$. In this case, θ is called the slant angle of the submanifold. The slant submanifold L is called proper slant submanifold if neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$.*

We note that for a slant submanifold L if $\theta = 0$, then it is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then it is an anti-invariant submanifold. This means that the slant submanifold is a generalization of invariant and anti-invariant submanifolds.

Definition 2.4. *A submanifold L is said to be semi-invariant ([2], [19]), if there exist two orthogonal complementary distributions D and $D^{\perp}$ on L such that*

$$TL = D \oplus D^{\perp},$$

where D is invariant and $D^{\perp}$ is anti-invariant.

Definition 2.5. *A submanifold L is said to be semi-slant [19], if there exist two orthogonal complementary distributions D and D_{θ} on L such that*

$$TL = D \oplus D_{\theta},$$

where D is invariant and D_{θ} is slant with slant angle θ. In this case, the angle θ is called semi-slant angle.

Definition 2.6. *A submanifold L is said to be hemi slant [16], if there exist two orthogonal complementary distributions D_{θ} and $D^{\perp}$ on L such that*

$$TL = D_{\theta} \oplus D^{\perp},$$

where D_{θ} is slant with slant angle θ and $D^{\perp}$ is anti-invariant. In this case, the angle θ is called hemi-slant angle.

Definition 2.7. *A submanifold L is said to be bi-slant, if there exist two orthogonal complementary distributions D_{θ_1} and D_{θ_2} on L such that*

$$TL = D_{\theta_1} \oplus D_{\theta_2},$$

where D_{θ_1} and D_{θ_2} are slants with slant angles θ_1 and θ_2, respectively.

3. Quasi Bi-Slant Submanifolds

In this section, we introduce quasi bi-slant submanifolds of almost Hermitian manifolds and obtain the necessary and sufficient conditions for the distributions involved in the definition of such manifolds to be integrable.

Definition 3.1. *A submanifold L of an almost Hermitian manifold M is called a quasi bi-slant submanifold if there exist the distributions D, D_1 and D_2 such that*

(1) TL admits the orthogonal direct decomposition as

$$TL = D \oplus D_1 \oplus D_2,$$

(2) $J(D) = D$, *i.e.,* D *is invariant,*

(3) $J(D_1) \perp D_2$,

(4) For any non-zero vector field $X \in (D_1)_p$, $p \in L$, *the angle* θ_1 *between* JX *and* $(D_1)_p$ *is constant and independent of the choice of point* p *and* X *in* $(D_1)_p$,

(5) For any non-zero vector field $Z \in (D_2)_q$, $q \in L$, *the angle* θ_2 *between* JZ *and* $(D_2)_q$ *is constant and independent of the choice of point* q *and* Z *in* $(D_2)_q$.

Here the angles θ_1 *and* θ_2 *are called slant angles of* L.

We easily observe that

(a) If $dimD \neq 0$, $dimD_1 = 0$ and $dimD_2 = 0$, then L is an invariant submanifold.

(b) If $dimD \neq 0$, $dimD_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $dimD_2 = 0$, then L is a proper semi-slant submanifold.

(c) If $dimD = 0$, $dimD_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $dimD_2 = 0$, then L is a slant submanifold with slant angle θ_1.

(d) If $dimD = 0$, $dimD_1 = 0$, $dimD_2 \neq 0$ and $0 < \theta_2 < \frac{\pi}{2}$, then L is a slant submanifold with slant angle θ_2.

(e) If $dimD = 0$, $dimD_1 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $dimD_2 = 0$, then L is an anti-invariant submanifold.

(f) If $dimD \neq 0$, $dimD_1 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $dimD_2 = 0$, then L is a semi-invariant submanifold.

(g) If $dimD = 0$, $dimD_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $dimD_2 \neq 0$ with $\theta_2 = \frac{\pi}{2}$, then L is a hemi-slant submanifold.

(h) If $dimD = 0$, $dimD_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, $dimD_2 \neq 0$ and $0 < \theta_2 < \frac{\pi}{2}$, then L is a bi-slant submanifold.

(i) If $dimD \neq 0$, $dimD_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, $dimD_2 \neq 0$ and $\theta_2 = \frac{\pi}{2}$, then L is a quasi hemi-slant submanifold.

(j) If $dimD \neq 0$ and $0 < \theta_1 = \theta_2 < \frac{\pi}{2}$, then L is a proper semi-slant submanifold.

(k) If $dimD \neq 0$, $dimD_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $dimD_2 \neq 0, 0 < \theta_2 < \frac{\pi}{2}$, then L is a proper quasi bi-slant submanifold.

This means that the notion of quasi bi-slant submanifold is a natural generalization of invariant, anti-invariant, slant, hemi-slant and semi-slant submanifolds.

Let L be a quasi bi-slant submanifold of an almost Hermitian manifold M. We denote the projections of $X \in \Gamma(TL)$ on the distributions D, D_1 and D_2 by P, Q and R, respectively. Then for any $X \in \Gamma(TL)$, we can write

$$X = PX + QX + RX. \tag{3.1}$$

Now, we put

$$JX = lX + mX, \tag{3.2}$$

where lX and mX are tangential and normal components of JX on L, respectively.

In view of (3.1) and (3.2), we infer

$$JX = JPX + JQX + JRX = lPX + mPX + lQX + mQX + lRX + mRX,$$

which assumes the form

$$JX = lPX + lQX + mQX + lRX + mRX, \tag{3.3}$$

since $J(D) = D$ and $mPX = 0$. This means that, for any $X \in \Gamma(TL)$, the tangential and normal components of JX can be expressed as

$$lX = lPX + lQX + lRX$$

and

$$mX = mQX + mRX.$$

Now, we have the following decomposition

$$J(TL) = D \oplus lD_1 \oplus mD_1 \oplus lD_2 \oplus mD_2.$$

Since $mD_1 \subset (T^{\perp}L)$ and $mD_2 \subset (T^{\perp}L)$, therefore

$$T^{\perp}L = mD_1 \oplus mD_2 \oplus \mu,$$

where μ is the orthogonal complement of $mD_1 \oplus mD_2$ in $T^{\perp}L$, and it is invariant with respect to J.

For any $Z \in T^{\perp}L$, we put

$$JZ = BZ + CZ, \tag{3.4}$$

where $BZ \in \Gamma(TL)$ and $CZ \in (T^{\perp}L)$. It is noticed that,

(a) $lD \subseteq D$,

(b) $mD = \{0\}$,

(c) $lD_i \subseteq D_i$ for $i = 1, 2$,

(d) $B(T^{\perp}L) = D_1 \oplus D_2$.

Lemma 3.1. *Let L be a quasi bi-slant submanifold of an almost Hermitian manifold M. Then the endomorphism l and the projection morphisms m, B and C on the tangent bundle of L satisfy the following identities:*

(i) $l^2 + Bm = -I$ *on* TL,

(ii) $ml + Cm = 0$ *on* TL,

(iii) $mB + C^2 = -I$ *on* $(T^{\perp}L)$,

(iv) $lB + BC = 0$ *on* $(T^{\perp}L)$,

where I is the identity operator.

Lemma 3.2. *Let L be a quasi bi-slant submanifold of an almost Hermitian manifold M. Then we have*

(i) $l^2X = -(\cos^2\theta_1)X$,

(ii) $g(lX, lY) = (\cos^2\theta_1)g(X, Y)$,

(iii) $g(mX, mY) = (\sin^2\theta_1)g(X, Y)$

for any $X, Y \in \Gamma(D_1)$, where θ_1 denotes the slant angle of D_1.

Lemma 3.3. *Let L be a quasi bi-slant submanifold of an almost Hermitian manifold. Then*

(i) $l^2Z = -(\cos^2\theta_2)Z$,

(ii) $g(lZ, lW) = (\cos^2\theta_2)g(Z, W)$,

(iii) $g(mZ, mW) = (\sin^2\theta_2)g(Z, W)$

for any $Z, W \in \Gamma(D_2)$, where θ_2 denotes the slant angle of D_2.

4. Integrability Of Distributions and Decomposition Theorems

In this section, we investigate the integrability conditions for the distributions involved in the definition of quasi bi-slant submanifolds.

Theorem 4.1. *Let L be a proper quasi bi-slant submanifold of a nearly Kaehler manifold M. Then the invariant distribution D is integrable if and only if*

$$(\nabla_Z lW - \nabla_W lZ, lX) = g(h(W, lZ) - h(Z, lW), mX) - g(2\mathfrak{P}_Z W, lX).$$

Proof. For any $Z, W \in \Gamma D$ and $X = QX + RX \in \Gamma(D_1 \oplus D_2)$, using (2.1), (2.4), (2.6), (2.8) and (2.9), we get

$$\begin{aligned} g([Z, W], X) &= g(\overline{\nabla}_Z JW, JX) - g((\overline{\nabla}_Z J)W, JX) \\ &\quad - g(\overline{\nabla}_W JZ, JX) + g((\overline{\nabla}_W J)Z, JX) \\ &= g(\overline{\nabla}_Z lW, JX) - g(\mathfrak{P}_Z W, JX) - g(\overline{\nabla}_W lZ, JX) + g(\mathfrak{P}_W Z, JX) \\ &= g(\nabla_Z lW - \nabla_W lZ, lX) + g(h(Z, lW) - h(W, lZ), mX) - g(2\mathfrak{P}_Z W, lX) \end{aligned}$$

Let us suppose that the distribution D is invariant. Then the above equation prove that $g(\nabla_Z lW - \nabla_W lZ, lX) + g(h(Z, lW) - h(W, lZ), mX) - g(2\mathfrak{P}_Z W, lX) = 0$. The converse part is obvious. □

Theorem 4.2. *Let L be a proper quasi bi-slant submanifold of a nearly Kaehler manifold M. Then the slant distribution D_1 is integrable if and only if*

$$\begin{aligned} g(2\mathfrak{P}_Z W + A_{mW} Z - A_{mZ} W, lX) &= g(A_{mlW} Z - A_{mlZ} W + \mathfrak{P}_Z lW - \mathfrak{P}_W lZ, X) \\ &\quad + g(\nabla^{\perp}_Z mW - \nabla^{\perp}_W mZ, mRX). \end{aligned}$$

Proof. For any $Z, W \in \Gamma(D_1)$ and $X = PX + RX \in \Gamma(D \oplus D_2)$, we have

$$\begin{aligned} g([Z, W], X) &= g(\overline{\nabla}_Z lW, JX) + g(\overline{\nabla}_Z mW, JX) - g((\overline{\nabla}_Z J)W, JX) \\ &\quad - g(\overline{\nabla}_W lZ, JX) - g(\overline{\nabla}_W mZ, JX) + g((\overline{\nabla}_W J)Z, JX) \\ &= g(\overline{\nabla}_Z mW, JX) - g(\overline{\nabla}_Z JlW, X) - g(\overline{\nabla}_W mZ, JX) + g(\overline{\nabla}_W JlZ, X) \\ &\quad - g(\mathfrak{P}_Z W, JX) + g(\mathfrak{P}_W Z, JX) + g((\overline{\nabla}_Z J)lW, X) - g((\overline{\nabla}_W J)lZ, X), \end{aligned}$$

where equations (2.1), (2.6), (2.8) and (2.9) have been used. Now, using equations (2.4), (2.5), (3.3) and Lemma 3.2, we have-

$$\begin{aligned} g([Z, W], X) &= -g(A_{mW} Z - A_{mZ} W, JX) + \cos^2\theta_1 g([Z, W], X) + g(A_{mlW} Z \\ &\quad - A_{mlZ} W, X) + g(\nabla^{\perp}_Z mW - \nabla^{\perp}_W mZ, JX) \\ &\quad - g(2\mathfrak{P}_Z W, JX) + g(\mathfrak{P}_Z lW - \mathfrak{P}_W lZ, X), \end{aligned}$$

which implies

$$\begin{aligned}\sin^2\theta_1 g([Z,W],X) &= g(A_{mlW}Z - A_{mlZ}W, X) + g(\nabla^{\perp}_Z mW - \nabla^{\perp}_W mZ, mRX)\\ &\quad - g(A_{mW}Z - A_{mZ}W, lX) - g(2\mathfrak{P}_Z W, lX)\\ &\quad + g(\mathfrak{P}_Z lW - \mathfrak{P}_W lZ, X)\\ &= g(A_{mlW}Z - A_{mlZ}W + \mathfrak{P}_Z lW - \mathfrak{P}_W lZ, X) + g(\nabla^{\perp}_Z mW\\ &\quad - \nabla^{\perp}_W mZ, mRX) - g(2\mathfrak{P}_Z W + A_{mW}Z - A_{mZ}W, lX).\end{aligned}$$

Thus, the above equation implies the proof. □

From Theorem 4.2, we can state the following sufficient conditions for the slant distribution D_1 to be integrable on L.

Corollary 4.1. *Let L be a proper quasi bi-slant submanifold of a nearly Kaehler manifold M. If $\nabla^{\perp}_Z mW - \nabla^{\perp}_W mZ \in mD_1 \oplus \mu$, $A_{mlW}Z - A_{mlZ}W + \mathfrak{P}_Z lW - \mathfrak{P}_W lZ \in D_1$ and $2\mathfrak{P}_Z W + A_{mW}Z - A_{mZ}W \in D_1$, for any $Z, W \in \Gamma(D_1)$, then the slant distribution D_1 is integrable.*

By following the similar argument to Theorem 4.2, we conclude the following:

Corollary 4.2. *Let L be a proper quasi bi-slant submanifold of a nearly Kaehler manifold M. Then the slant distribution D_2 is integrable if and only if*

$$\begin{aligned}g(2\mathfrak{P}_Z W + A_{mW}Z - A_{mZ}W, lX) &= g(A_{mlW}Z - A_{mlZ}W + \mathfrak{P}_Z lW - \mathfrak{P}_W lZ, X)\\ &\quad + g(\nabla^{\perp}_Z mW - \nabla^{\perp}_W mZ, mQX)\end{aligned}$$

for any, $Z, W \in \Gamma(D_2)$ and $X \in \Gamma(D \oplus D_1)$.

From Theorem 4.2, we have the following sufficient conditions for the slant distribution D_2 to be integrable.

Corollary 4.3. *Let L be a proper quasi bi-slant submanifold of a nearly Kaehler manifold M. Then the slant distribution D_2 is integrable, provided $\nabla^{\perp}_Z mW - \nabla^{\perp}_W mZ \in mD_2 \oplus \mu$, $A_{mlW}Z - A_{mlZ}W + \mathfrak{P}_Z lW - \mathfrak{P}_W lZ \in D_2$ and $2\mathfrak{P}_Z W + A_{mW}Z - A_{mZ}W \in D_2$, $\forall\ Z, W \in \Gamma(D_2)$ and $X \in \Gamma(D \oplus D_1)$.*

5. Necessary and sufficient conditions for quasi bi-slant submanifolds to be totally geodesics

We investigate the geometry of leaves of foliations determined by the distributions D, D_1 and D_2.

Theorem 5.1. *Let L be a proper quasi bi-slant submanifold of a nearly Kaehler manifold M. Then the invariant distribution D defines a totally geodesic foliation on L if and only if $g(\nabla_X lY - \mathfrak{P}_X Y, lZ) = -g(h(X, lY), mZ)$ and $g(\nabla_X lY - \mathfrak{P}_X Y, BU) = -g(h(X, lY), CU)$ for any $X, Y \in \Gamma(D), Z \in \Gamma(D_1 \oplus D_2)$ and $U \in \Gamma(T^\perp L)$.*

Proof. In view of equations (2.1), (2.6), (2.8), (2.9) and $mY = 0$, we have

$$\begin{aligned} g(\overline{\nabla}_X Y, Z) &= g(\overline{\nabla}_X lY, JZ) - g((\overline{\nabla}_X J)Y, JZ) \\ &= g(\nabla_X lY, lZ) + g(h(X, lY), mZ) - g(\mathfrak{P}_X Y, lZ) \\ &= g(\nabla_X lY - \mathfrak{P}_X Y, lZ) + g(h(X, lY), mZ) \end{aligned}$$

for any $X, Y \in \Gamma(D)$, $Z = QZ + RZ \in \Gamma(D_1 \oplus D_2)$. Now, for any $U \in \Gamma(T^\perp L)$ and $X, Y \in \Gamma(D)$, we have

$$\begin{aligned} g(\overline{\nabla}_X Y, U) &= g(\overline{\nabla}_X lY, JU) - g((\overline{\nabla}_X J)Y, JU) \\ &= g(\nabla_X lY, BU) + g(h(X, lY), CU) - g(\mathfrak{P}_X Y, BU) \\ &= g(\nabla_X lY - \mathfrak{P}_X Y, BU) + g(h(X, lY), CU). \end{aligned}$$

The last two equations infer the required result. □

Theorem 5.2. *Let L be a proper quasi bi-slant submanifold of a nearly Kaehler manifold M. Then the slant distribution D_1 defines a totally geodesic foliation on M if and only if $g(\nabla^\perp_X mY, mRZ) = g(A_{mY} X + \mathfrak{P}_X Y, lZ) - g(A_{mlY} X + \mathfrak{P}_X lY, Z)$ and $g(A_{mY} X + \mathfrak{P}_X Y, BU) = g(\nabla^\perp_X mY, CU) - g(\nabla^\perp_X mlY - \mathfrak{P}_X lY, U)$ for any $X, Y \in \Gamma(D_1), Z = PZ + RZ \in \Gamma(D \oplus D_2)$ and $U \in \Gamma(T^\perp L)$.*

Proof. For any $X, Y \in \Gamma(D_1), Z = PZ + RZ \in \Gamma(D \oplus D_2)$, we have

$$\begin{aligned} g(\overline{\nabla}_X Y, Z) &= g(\overline{\nabla}_X JY, JZ) - g((\overline{\nabla}_X J)Y, JZ) \\ &= g(\overline{\nabla}_X lY, JZ) + g(\overline{\nabla}_X mY, JZ) - g(\mathfrak{P}_X Y, JZ) \\ &= -g(\overline{\nabla}_X l^2 Y, Z) - g(\overline{\nabla}_X mlY, Z) + g((\overline{\nabla}_X J)lY, Z) \\ &\quad + g(\overline{\nabla}_X mY, lPZ + lRZ + mRZ) - g(\mathfrak{P}_X Y, JZ), \end{aligned}$$

where equations (2.1), (2.6), (2.8) and (2.9) have been used. Again, using equations (2.5), (3.1), Lemma 3.2 and the fact that $lPZ + lRZ = lZ, mPZ = 0$ in the above equation, we find

$$\begin{aligned} g(\overline{\nabla}_X Y, Z) &= \cos^2\theta_1 g(\overline{\nabla}_X Y, Z) + g(A_{mlY} X, Z) - g(A_{mY} X, lPZ + lRZ) \\ &\quad + g(\nabla^\perp_X mY, mRZ) + g(\mathfrak{P}_X lY, Z) - g(\mathfrak{P}_X Y, lZ), \end{aligned}$$

which becomes

(i)

$$\begin{aligned} \sin^2\theta_1 g(\overline{\nabla}_X Y, Z) &= g(A_{mlY} X + \mathfrak{P}_X lY, Z) \\ &\quad + g(\nabla^\perp_X mY, mRZ) - g(A_{mY} X + \mathfrak{P}_X Y, lZ). \end{aligned}$$

Similarly,

(ii)

$$\begin{aligned} \sin^2\theta_1 g(\overline{\nabla}_X Y, Z) &= -g(\nabla^\perp_X mlY - \mathfrak{P}_X lY, U) - g(A_{mY} X \\ &\quad + \mathfrak{P}_X Y, BU) + g(\nabla^\perp_X mY, CU). \end{aligned}$$

Thus, from (i) and (ii), we have the assertions. □

In a similar argument to Theorem 5.2, we can conclude the following:

Corollary 5.1. *Let L be a proper quasi bi-slant submanifold of a nearly Kaehler manifold M. Then the slant distribution D_2 defines a totally geodesic foliation on M if and only if $g(\nabla^\perp_X mY, mQZ) = g(A_{mY} X + \mathfrak{P}_X Y, lZ) - g(A_{mlY} X + \mathfrak{P}_X lY, Z)$ and $g(A_{mY} X + \mathfrak{P}_X Y, BU) = g(\nabla^\perp_X mY, CU) - g(\nabla^\perp_X mlY - \mathfrak{P}_X lY, U)$ for any $X, Y \in \Gamma(D_2), Z \in \Gamma(D \oplus D_1)$ and $U \in \Gamma(T^\perp L)$.*

6. Examples

6.1. **Example.** Consider a 14-dimensional differentiable manifold $\overline{M} = \mathbb{R}^{14}$ such that

$$\overline{M} = \{(x_i, y_i) = (x_1, x_2, \ldots, x_7, y_1, y_2, \ldots, y_7) \in \mathbb{R}^{14};\ i = 1, 2, \ldots, 7\}.$$

We choose the vector fields

$$e_i = \frac{\partial}{\partial y_i}, e_{7+i} = \frac{\partial}{\partial x_i}, for\, i = 1, 2, \ldots, 7.$$

Let g be a Hermitian metric defined by

$$g = (dx_1)^2 + (dx_2)^2 + \ldots + (dx_7)^2 + (dy_1)^2 + (dy_2)^2 + \ldots + (dy_7)^2.$$

Here $\{e_1, e_2, \ldots, e_{14}\}$ forms an orthonormal basis. We define $(1,1)$-tensor field J as

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \forall\ i, j = 1, 2, \ldots, 7.$$

By using the linearity of J and g, we can verify that

$$J^2 = -I, \quad g(JX, JY) = g(X, Y), \forall\ X, Y \in \Gamma(T\overline{M}).$$

By straight forward calculations, we can easily show that the manifold $(\overline{M}, J, g)$ is a nearly Kaehler manifold of dimension 14.

Now, we consider a submanifold M of $\overline{M}$ defined by immersion f as follows:

$$f(u, v, w, r, s, t) = (u, w, 0, s, 0, 0, 0, v, r\text{cos}\theta_1, r\text{sin}\theta_1, t\text{cos}\theta_2, 0, 0, t\text{sin}\theta_2),$$

where θ_1 and θ_2 are constants. By direct computation, it is easy to check that the tangent bundle of M is spanned by a linearly independent set $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$$Z_1 = \frac{\partial}{\partial x_1},\ Z_2 = \frac{\partial}{\partial y_1},\ Z_3 = \frac{\partial}{\partial x_2},$$

$$Z_4 = \cos\theta_1 \frac{\partial}{\partial y_2} + \sin\theta_1 \frac{\partial}{\partial y_3},\ Z_5 = \frac{\partial}{\partial x_4},$$

$$Z_6 = \cos\theta_2 \frac{\partial}{\partial y_4} + \sin\theta_2 \frac{\partial}{\partial y_7}.$$

Now, we define the almost complex structure J of $\overline{M}$ as:

$$JZ_1 = \frac{\partial}{\partial y_1},\ JZ_2 = -\frac{\partial}{\partial x_1},\ JZ_3 = \frac{\partial}{\partial y_2},$$

$$JZ_4 = -\left(\cos\theta_1 \frac{\partial}{\partial x_2} + \sin\theta_1 \frac{\partial}{\partial x_3}\right), \; JZ_5 = \frac{\partial}{\partial y_4},$$

$$JZ_6 = -\left(\cos\theta_2 \frac{\partial}{\partial x_4} + \sin\theta_2 \frac{\partial}{\partial x_7}\right).$$

Let us define the distributions $D = span\{Z_1, Z_2\}$, $D_1 = Span\{Z_3, Z_4\}$, $D_2 = span\{Z_5, Z_6\}$. Then it is easy to see that D is invariant, D_1 and D_2 are slant distributions with slant angles θ_1 and θ_2, respectively. Hence, f defines a proper 6-dimensional quasi bi-slant submanifold M in $\overline{M}$.

6.2. **Example.** Let $\overline{M}$ be a nearly Kaehler manifold defined in Example 6.1. Consider a submanifold N of $\overline{M}$ defined by immersion ψ as follows:

$$\psi(u, v, w, r, s, t) = \left(\frac{u}{\sqrt{2}}, w, 0, \sqrt{3}s, 0, 0, \frac{u}{\sqrt{2}}, \frac{v}{\sqrt{2}}, r, r, t, s, 0, \frac{v}{\sqrt{2}}\right).$$

By direct computations, it is easy to check that the tangent bundle of N is spanned by a linearly independent set $\{X_1, X_2, X_3, X_4, X_5, X_6\}$, where

$$X_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_7}\right), \; X_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_7}\right),$$

$$X_3 = \frac{\partial}{\partial x_2}, \; X_4 = \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3},$$

$$X_5 = \sqrt{3}\frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_5}, \; X_6 = \frac{\partial}{\partial y_4}.$$

Define the almost complex structure J of $\overline{M}$ as follows:

$$JX_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_7}\right), JX_2 = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_7}\right),$$

$$JX_3 = \frac{\partial}{\partial y_2}, JX_4 = -\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right),$$

$$JX_5 = \sqrt{3}\frac{\partial}{\partial y_4} - \frac{\partial}{\partial x_5}, JX_6 = -\frac{\partial}{\partial x_4}.$$

Let the distributions $D = Span\{X_1, X_2\}$, $D_1 = Span\{X_3, X_4\}$, $D_2 = Span\{X_5, X_6\}$. Then it is easy to very that D is invariant, D_1 and D_2 are slant distributions with slant angles $\frac{\pi}{4}$ and $\frac{\pi}{3}$, respectively. Hence ψ defines a proper 6-dimensional quasi bi-slant submanifold N in $\overline{M}$.

6.3. **Example.** Consider $\mathbb{R}^{2n}$ with standard coordinates $(x_1, x_2, x_3, x_4, \dots, x_{2n-1}, x_{2n})$. We can canonically choose an almost complex structure J on $\mathbb{R}^{2n}$ as follows:

$$J\left(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3} + a_4\frac{\partial}{\partial x_4} + \dots + a_{2n-1}\frac{\partial}{\partial x_{2n-1}} + a_{2n}\frac{\partial}{\partial x_{2n}}\right)$$
$$= \left(a_1\frac{\partial}{\partial x_2} - a_2\frac{\partial}{\partial x_1} + a_3\frac{\partial}{\partial x_4} - a_4\frac{\partial}{\partial x_3} + \dots + a_{2n-1}\frac{\partial}{\partial x_{2n}} - a_{2n}\frac{\partial}{\partial x_{2n-1}}\right),$$

where $a_1, a_2, a_3, \dots, a_{2n}$ are C^∞ functions defined on $\mathbb{R}^{2n}$.

Let M be a submanifold of $\mathbb{R}^{10}$ defined by

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = \Big(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}}, x_3, x_4\cos\theta_1, x_5, x_6\cos\theta_2, 0,$$
$$x_4\sin\theta_1, 0, x_6\sin\theta_2\Big),$$

where θ_1 and θ_2 are constants.

By direct computations, it is easy to check that the tangent space at each point of M is spanned by a linearly independent set $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$$Z_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right),\ Z_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right),$$
$$Z_3 = \frac{\partial}{\partial x_3},\ Z_4 = \cos\theta_1\frac{\partial}{\partial x_4} + \sin\theta_1\frac{\partial}{\partial x_8},$$
$$Z_5 = \frac{\partial}{\partial x_5},\ Z_6 = \cos\theta_2\frac{\partial}{\partial x_6} + \sin\theta_2\frac{\partial}{\partial x_{10}}.$$

Let g be a Hermitian metric on $\mathbb{R}^{10}$ such that

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right) = 1;\ for\ 1 \le i \le 10,$$

and

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = 0\ , i \neq j\ for\ 1 \le i, j \le 10.$$

Then, using the canonical Hermitian structure of $\mathbb{R}^{10}$, we have

$$JZ_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right),\ JZ_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right),$$
$$JZ_3 = \frac{\partial}{\partial x_4},\ JZ_4 = -\cos\theta_1\frac{\partial}{\partial x_3} - \sin\theta_1\frac{\partial}{\partial x_7},$$
$$JZ_5 = \frac{\partial}{\partial x_6},\ Z_6 = -\cos\theta_2\frac{\partial}{\partial x_5} - \sin\theta_2\frac{\partial}{\partial x_9}.$$

Let $D = Span\{Z_1, Z_2\}$, $D_1 = Span\{Z_3, Z_4\}$ and $D_2 = Span\{Z_5, Z_6\}$. Then it is easy to see that D is invariant and D_1, D_2 are slant distributions with slant angles θ_1 and θ_2, respectively.

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