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GENERALIZED CONTINUOUS K-WEAVING FRAMES

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Abstract. Motivated with the study of discrete weaving frames by Bemrose et al. in 2015, we study generalized continuous K-weaving frames in Hilbert spaces and prove some new basic properties. Also, we prove a sufficient condition for generalized continuous K-frame to be woven. Further, we prove that generalized continuous K-weaving frames remain woven under invertible operator. Finally, we give Paley-Wiener type perturbation results for generalized continuous K-weaving frames.

1. Introduction

Frames for Hilbert spaces were formally introduced by Duffin and Schaeffer [13] who used frames as a tool in the study of non-harmonic Fourier series. Daubechies, Grossmann and Meyer [11], reintroduced frames and observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$. As we know frames are more flexible tools to convey information than bases, and so they are suitable replacement for bases in a Hilbert space $\mathcal{H}$. Finding a representation of $x \in \mathcal{H}$ as a linear combination of vectors of a frame, is the main goal of discrete frame theory. But in case of a continuous frame, which is a natural generalization of the discrete case, this property of frame is not straightforward. However, one of the applications of frames is in wavelet theory. In fact, the practical implementation of the wavelet transform in signal processing requires the selection of a discrete set of points in the transformed space. Keeping applications in mind, various generalizations of frames were introduced and studied namely Frames of subspaces in Hilbert spaces were first

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introduced and studied by Casazza and Kutyniok [8] and then in [4, 22]; Pseudo frames were introduced by Li and Ogawa [23]; Oblique frames were first introduced and studied first by Eldar [14] and then by Christensen and Eldar [10]; Outer frames were introduced and studied by Aldourbi, Cabrelli and Molter [1] and Bounded quasi-projectors were studied by Fornasier [15]. Sun [28] introduced a more general concept called G-frames and pointed out that most of the above generalizations of frames may be regarded as a special cases of G-frames and many of their basic properties can be derived within this more general setup. Rahimi [24] studied Multipliers of generalized frames in Hilbert spaces and Rahimi and Balazs [25] studies Multipliers for p-Bessel sequences in Banach spaces.

Another generalization of frames was proposed by Kaiser [20] and independently by Ali Tawreque, Antoine and Gazeau [2] who named it as continuous frames while Kaiser used the terminology generalized frames. Recently, Gabardo and Han [16] studied continuous frames and use the terminology (Ω, μ)-frame. Discrete and continuous frames arise in many applications in both pure and applied mathematics and, in particular, they play important roles in digital signal processing and scientific computations. Alizadeh, Faroughi, and Rahmani [3] studied continuous $K-G$-frames in Hilbert spaces. Continuous frames were further studied in [5, 12, 21]. In 2015, notion of discrete weaving frames appeared for first time in [7] by Bemrose, Casazza, Grochenig, Lammers and Lynch. In [19, 26], authors introduced and studied Near exact operator Banach frames, Λ-Banach frames and O-frames. Duals of K-operator frames in Hilbert spaces is also discussed in [27]. For a nice introduction to frames an interested reader may refer to [9] and references therein.

In this paper, we define the notion of generalized continuous K- weaving frame in a Hilbert space and we prove that if the sets of lower frame bounds of K-frames for a Hilbert space are bounded below, then the corresponding generalized continuous K-frames are woven. Also, we give a sufficient condition for generalized continuous K-frame to be woven. Further, we prove that generalized continuous K-weaving frames remain woven under invertible operator. Finally, we give Paley-Wiener type perturbation results for generalized continuous.

2. Preliminaries

Through this paper $\mathcal{H}$ denotes seperable Hilbert spaces. L. Găvruţa [17, 18] recently introduced a frame with respect to a bounded linear operator K in a Hilbert space $\mathcal{H}$, which is called K-frame, to reconstruct the elements in the range of K (range of K is denoted by $R(K)$). In fact, they gave the following definition:

Definition 2.1. A system $\{f_k\} \subset \mathcal{H}$ is called K-frame for $\mathcal{H}$ if there exists two positive constants $A, B > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{k\in\mathbb{N}} |\langle x, f_k\rangle|^2 \leq B\|x\|^2, \quad \text{for all} \quad x \in \mathcal{H} \tag{2.1}$$

We call A,B the lower frame bound and the upper frame bound for K-frame $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ respectively. If only the upper inequality in(2.1) is satisfied, then $\{f_k\}_{k\in\mathbb{N}}$ is called Bessel sequence.

Găvruţa [17] also proved the following result:

Theorem 2.2. *([17]) Let $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$and $K \in B(\mathcal{H})$. Then the following statements are equivalent:*

(i) *$\{f_k\}_{k\in\mathbb{N}}$ is an atomic system for K;*
(ii) *$\{f_k\}_{k\in\mathbb{N}}$ is a K-frame for $\mathcal{H}$;*
(iii) *there exists a Bessel sequence $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ such that*

$$Kx = \sum_{k\in\mathbb{N}} \langle x, g_k\rangle f_k, \quad \forall x \in \mathcal{H}.$$

We call the Bessel sequence $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ as the K-dual frame of the K-frame $\{f_k\}_{k\in\mathbb{N}}$.

Theorem 2.3. *([6]) Let $\mathcal{H}$ be a Hilbert space and $S, K \in B(\mathcal{H})$. Then the following statements are equivalent:*

(i) *$R(K) \subseteq R(S)$.*
(ii) *$\lambda KK^* \leq SS^*$for some $\lambda > 0$.*
(iii) *$K = SQ$ for some $Q \in B(\mathcal{H})$.*

Let $B_{\mathcal{H}}$ be the collection of all Bessel sequences in a Hilbert space $\mathcal{H}$. Let I be an at most countable index set. The following definition of a generalized continuous frame introduced and studied in [12] .

Definition 2.4. Let $\mathcal{H}$ be a complex Hilbert space, $K \in B(\mathcal{H})$ and (Ω, μ) be a measure space with positive measure μ. A mapping $F : \Omega \to B_{\mathcal{H}}; \omega \to \{f_i(\omega)\}_{i\in I}$ is called a *generalized continuous frame* with respect to (Ω, μ) if:

(i) F is weakly measurable, i.e., for all $f \in \mathcal{H}, i \in I, \omega \to \langle f, f_i(\omega)\rangle$ is a measurable function on Ω;

(ii) there exist positive constants A, B such that

$$A\|f\|^2 \leq \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2, \ \forall \ f \in \mathcal{H}. \tag{2.2}$$

The positive constants A and B are called generalized continuous frame bounds. F is called *A-tight generalized continuous frame* if condition (i) holds and

$$A\|f\|^2 = \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2, \quad f \in \mathcal{H}.$$

The mapping F is called *Bessel* if the upper inequality in (2.2) holds. In this case, B is called the Bessel bound. If the cardinality of I is one, F is a continuous frame, and if, further, μ is a counting measure and $\Omega := \mathbb{N}$, F is called a discrete frame.

Next we give the definition of generalized continuous K-frames.

Definition 2.5. Let $\mathcal{H}$ be a complex Hilbert space, $K \in B(\mathcal{H})$ and (Ω, μ) be a measure space with positive measure μ. A mapping $F : \Omega \to B_{\mathcal{H}}; \omega \to \{f_i(\omega)\}_{i\in I}$ is called a generalized continuous K-frame with respect to (Ω, μ) if:

(i) F is weakly measurable, i.e., for all $f \in \mathcal{H}, i \in I, \omega \to \langle f, f_i(\omega)\rangle$ is a measurable function on Ω;

(ii) there exist positive constants A and B such that

$$A\|K^*f\|^2 \leq \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2, \quad \text{for all} \quad f \in \mathcal{H}. \tag{2.3}$$

The positive constants A and B are called generalized continuous K-frame bounds. F is called A-tight generalized continuous K-frame if condition (i) holds and

$$A\|K^*f\|^2 = \int_\Omega \sum_{i\in I} |\langle f, f_i(\omega)\rangle|^2, \quad \text{for all} \quad f \in \mathcal{H}.$$

The mapping F is called Bessel if the upper inequality in (2.3) holds. In this case, A is called the Bessel bound. If the cardinality of I is one, F is a continuous K-frame, and if, further, μ is a counting measure and $\Omega := \mathbb{N}$, F is called a discrete K-frame. Let λ be a counting measure and F be a Bessel sequence with bound B.

The analysis operator associated with F is defined by

$$U_F : \mathcal{H} \to \; L^2(\Omega \times I, \mu \times \lambda), \quad U_F f(\omega, i) = \langle f, f_i(\omega)\rangle$$

and the synthesis operator associated with F is defined as

$$U_F^* : L^2(\Omega \times I, \mu \times \lambda) \to \mathcal{H}, \quad U_F^*\phi = \int_\Omega \sum_{i\in I} \phi(\omega, i) f_i(\omega) d\mu(\omega).$$

The frame operator for generalized continuous K-frame is defined as

$$S_F f = \int_\Omega \sum_{i\in I} \langle f, f_i(\omega)\rangle f_i(\omega) d\mu(\omega), \quad \text{for all} \quad f \in \mathcal{H}.$$

3. Main result

Definition 3.1. A family of generalized continuous K-frame

$$\{\{F_i(x)\}_{x\in\Omega} \; : i \in [m]\} = \left\{\{\{f_k^i(x)\}_{k\in I}\}_{x\in\Omega} : i \in [m]\right\},$$

for$\mathcal{H}$ w.r.t. μ is said to be woven, if there exists universal positive constants A and B such that for any partition $\{\sigma_i\}_{i\in[m]}\}$ of Ω, the family

$$\cup_{i\in[m]}\{F_i(x)\}_{x\in\sigma_i} = \cup_{i\in[m]}\{\{f_k^i(x)\}_{k\in I}\}_{x\in\sigma_i}$$

is a generalized continuous K-frame for $\mathcal{H}$ with lower and upper frames bounds A and B respectively.

Theorem 3.2. *Suppose that* $\{F_i(x)\}_{x\in\Omega} = \{\{f_k^i(x)\}_{k\in I}\}_{x\in\Omega}$ *is a generalized continuous Bessel sequence in* $\mathcal{H}$ *w.r.t.* μ *and with Bessel bound* $B_i (i \in [m])$. *Then for any partition* $\{\sigma_i\}_{i\in[m]}$ *of* Ω, *the family* $\cup_{i\in[m]} \{F_i(x)\}_{x\in\sigma_i}$ *is a generalized continuous Bessel sequence in* $\mathcal{H}$ *with Bessel bound* $\sum_{i\in[m]} B_i$.

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of Ω. Then for all $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in[m]}\int_{\sigma_i}\sum_{k\in I}|\langle f, f_k^i(x)\rangle|^2 d\mu(x) &\leq \sum_{i\in[m]}\int_{\Omega}\sum_{k\in I}|\langle f, f_k^i(x)\rangle|^2 d\mu(x)\\ &\leq \left(\sum_{i\in[m]} B_i\right)\|f\|^2, \quad f\in\mathcal{H}.\end{aligned}$$

□

Theorem 3.3. *Let $\{F_i(x)\}_{x\in\Omega} = \{\{f_k^i(x)\}_{k\in I}\}_{x\in\Omega}$ be a generalized continuous K-frame for $\mathcal{H}$ w.r.t. μ $(i \in [m])$. For each $x \in \Omega$, $i \in [m]$, assume that $\{f_k^i(x)\}_{k\in i}$ is a K-frame for $\mathcal{H}$ with lower bounds A_x^i. If the set $\{A_x^i : x \in \Omega\}(i \in [m])$ is bounded below with positive lower bound, then the family $\{\{F_i(x)\}_{x\in\Omega} : i \in [m]\}$ is woven in $\mathcal{H}$.*

Proof. Let C_i be a positive lower bound of the set $\{A_x^i : x \in \Omega\}, i \in [m]$. For any partition $\{\sigma_i\}_{i\in[m]}$ of Ω, we compute

$$\begin{aligned}\sum_{i\in[m]}\int_{\sigma_i}\sum_{k\in I}|\langle f, f_k^i(x)\rangle|^2 d\mu(x) &\geq \sum_{i\in[m]}\int_{\sigma_i} A_x^i ||K^*f||^2 d\mu(x)\\ &\geq \sum_{i\in[m]}\int_{\sigma_i} C_i ||K^*f||^2 d\mu(x)\\ &\geq \min\{C_i : i \in [m]\}\sum_{i\in[m]}\mu(\sigma_i)||K^*f||^2\\ &= (min\{C_i : i\in[m]\}\mu(\Omega))||K^*f||^2, \text{for all} \quad f\in\mathcal{H}.\end{aligned}$$

Hence, the family $\{\{F_i(x)\}_{x\in\Omega} : i \in [m]\}$ is woven in $\mathcal{H}$. □

Now, we give an example to show that the condition of positive lower bound on the sets of lower frame bounds given in Theorem 3.3 is only sufficient but not necessary.

Example 3.4. Let $\mathcal{H} = \ell^2(\mathbb{N}), \Omega = (0, 1), K \in B(\mathcal{H})$ and μ be a Lebesgue measure. Let $\{e_k\}$ be an orthonormal basis for $\mathcal{H}$. For $x \in \Omega$, define $\{f_k^1(x)\}_{k\in\mathbb{N}}$ as $f_k^1(x) = \sqrt{x}Ke_k$. Then $\{f_k^1(x)\}_{k\in\mathbb{N}}$ is a tight K-frame for $\mathcal{H}$ with frame bounds $A_x = B_x = x$. Also define $\{f_k^2(x)\}$ as $f_k^2(x) = \sqrt{2x}K(e_k + e_{k+1})$ and $K : \mathcal{H} \to \mathcal{H}$ by $Kx = \sum_{k\in\mathbb{N}}\langle x, e_k\rangle(e_k + e_{k+1})$. Then, $\{f_k^2(x)\}_{k\in\mathbb{N}}$ is a tight K-frame for $\mathcal{H}$ with frame bounds $A_x = B_x = 2x$. The sets of lower frames bounds $\{A_x^1 : x \in \Omega = (0, 1)\}$ and $\{A_x^2 : x \in \Omega\} = (0, 2)$ have no positive lower bounds. But the family $\{f_k^1(x)\}_{k\in\Omega}$ and

$\{f_k^2(x)\}_{k\in\Omega}$ are generalized continuous K-woven frames for $\mathcal{H}$ with universal bounds $\frac{1}{2}$ and 1.

Theorem 3.5. *Let $\{F_i(x)\}_{x\in\Omega} = \{\{f_k^i(x)\}_{k\in I}\}_{x\in\Omega}$ be a generalized continuous K-frame for $\mathcal{H}$ w.r.t. μ ($i \in [m]$). The following statements are equivalent :*

(i) $\{\{f_k^i(x)\}_{k\in I,x\in\Omega} : i \in [m]\}$ *is K-woven.*

(ii) $\{\{Uf_k^i(x)\}_{k\in I,x\in\Omega} : i \in [m]\}$ *is UK-woven for all bounded linear operator $U \in B(\mathcal{H})$.*

Proof. $(i) \Rightarrow (ii)$ Let A and B be universal generalized continuous K-frame bounds for the family $\{\{F_i(x)\}_{x\in\Omega} : i \in [m]\}$. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of Ω. Then for any $f \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i\in[m]}\int_{\sigma_i}\sum_{k\in I}|\langle f, Uf_k^i(x)\rangle|^2 d\mu(x) &= \sum_{i\in[m]}\int_{\sigma_i}\sum_{k\in I}|\langle U^*f, f_k^i(x)\rangle|^2 d\mu(x)\\
&\leq B||U^*f||^2\\
&\leq B||U^*||^2||f||^2
\end{aligned}$$

Similarly, for any $f \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i\in[m]}\int_{\sigma_i}\sum_{k\in I}|\langle f, Uf_k^i(x)\rangle|^2 d\mu(x) &= \sum_{i\in[m]}\int_{\sigma_i}\sum_{k\in I}|\langle U^*f, f_k^i(x)\rangle|^2 d\mu(x)\\
&\geq A||K^*U^*f||^2\\
&= A||(UK)^*f||^2
\end{aligned}$$

Hence the family $\{\{Uf_k^i(x)\}_{k\in I,x\in\Omega} : i \in [m]\}$ is UK-woven with universal generalized K-frame bounds A and $B||U^*||^2$.

$(ii) \Rightarrow (i)$ Choose $U = I$, the identity operator on $\mathcal{H}$. Then, the family $\{\{f_k^i(x)\}_{k\in I,x\in\Omega} : i \in [m]\}$ is K-woven. □

In the next result we prove the sufficient condition for generalized continuous K-frames.

Theorem 3.6. *Let $\{f_k^i(x)\}_{x\in\Omega}, i \in [m]$ be a generalized continuous K-frame for $\mathcal{H}$ w.r.t. μ with frame bounds A_i and B_i. Suppose that there $\gamma > 0$ such that*

$$\int_{\mathcal{J}} \sum_{k=1}^{\infty} |\langle f, f_k^i(x) - g_k^i(x)\rangle|^2 d\mu(x) \leq \gamma \min\Big\{ \int_{\mathcal{J}} \sum_{k=1}^{\infty} |\langle f, f_k^i(x)\rangle|^2 d\mu(x), \int_{\mathcal{J}} \sum_{k=1}^{\infty} |\langle f, f_k^i(x)\rangle|^2 d\mu(x) \Big\}$$

for all $f \in \mathcal{H}$ and for all measurable subsets $\mathcal{J} \subseteq \Omega$. Then the family of generalized continuous K- frames $\{f_k^i(x)\}_{x\in\Omega} : i \in [m]\}$ is a woven with universal frame bounds $\frac{A_1+A_2+\ldots+A_m}{2(m-1)(k+1)+1}$ and $\sum_{i\in[m]} B_i$.

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of Ω. Clearly, the family $\cup_{i\in[m]}\{f_k^i(x)\}_{x\in\sigma_i}$ is generalized Bessel sequence with universal upper frame bound $\sum_{i\in[m]} B_i$. For the lower frame inequlity, we compute

$$\begin{aligned}
(A_1 + A_2 + \ldots + A_m)||K^*f||^2 &\leq \int_{\Omega} \sum_{k=1}^{\infty} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) \\
&+ \int_{\Omega} \sum_{k=1}^{\infty} |\langle f, f_k^2(x)\rangle|^2 d\mu(x) + \cdots \\
&+ \int_{\Omega} \sum_{k=1}^{\infty} |\langle f, f_k^m(x)\rangle|^2 d\mu(x) \\
= \Big(\int_{\sigma_1} \sum_{k=1}^{\infty} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) &+ \int_{\sigma_2} \sum_{k=1}^{\infty} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) + \cdots \\
&+ \int_{\sigma_m} \sum_{k=1}^{\infty} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) \Big) + \cdots \\
+ \Big(\int_{\sigma_1} \sum_{k=1}^{\infty} |\langle f, f_k^m(x)\rangle|^2 d\mu(x) &+ \int_{\sigma_2} \sum_{k=1}^{\infty} |\langle f, f_k^m(x)\rangle|^2 d\mu(x) + \cdots \\
&+ \int_{\sigma_m} \sum_{k=1}^{\infty} |\langle f, f_k^m(x)\rangle|^2 d\mu(x) \Big) \\
&\leq \Big[\int_{\sigma_1} \sum_{k=1}^{\infty} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) \\
&+2\Big(\int_{\sigma_2} \sum_{k=1}^{\infty} |\langle f, f_k^1(x) - f_k^2(x)\rangle|^2 d\mu(x)
\end{aligned}$$

$$+\int_{\sigma_2}\sum_{k=1}^{\infty}|\langle f,f_k^2(x)\rangle|^2d\mu(x)\Big)+\cdots+2\Big(\int_{\sigma_m}\sum_{k=1}^{\infty}|\langle f,f_k^1(x)-f_k^m(x)\rangle|^2d\mu(x)$$

$$+\int_{\sigma_m}\sum_{k=1}^{\infty}|\langle f,f_k^m(x)\rangle|^2d\mu(x)\Big)\Big]+\cdots$$

$$+\Big[2\Big(\int_{\sigma_1}\sum_{k=1}^{\infty}|\langle f,f_k^m(x)-f_k^1(x)\rangle|^2d\mu(x)+\int_{\sigma_1}|\langle f,f_k^1(x)\rangle|^2d\mu(x)\Big)+\cdots$$

$$+2\Big(\int_{\sigma_{m-1}}\sum_{k=1}^{\infty}|\langle f,f_k^m(x)-f_k^{(m-1)}(x)\rangle|^2d\mu(x)$$

$$+\int_{\sigma_{m-1}}\sum_{k=1}^{\infty}|\langle f,f_k^{(m-1)}(x)\rangle|^2d\mu(x)\Big)+\cdots$$

$$+\int_{\sigma_m}\sum_{k=1}^{\infty}|\langle f,f_k^m(x)\rangle|^2d\mu(x)\Big]$$

$$\leq\Big[\int_{\sigma_1}\sum_{k=1}^{\infty}|\langle f,f_k^1(x)\rangle|^2d\mu(x)+2\Big(\gamma\int_{\sigma_2}\sum_{k=1}^{\infty}|\langle f,f_k^2(x)\rangle|^2d\mu(x)$$

$$+\int_{\sigma_2}\sum_{k=1}^{\infty}|\langle f,f_k^2(x)\rangle|^2d\mu(x)\Big)+\cdots$$

$$+2\Big(\gamma\int_{\sigma_m}\sum_{k=1}^{\infty}|\langle f,f_k^m(x)\rangle|^2d\mu(x)+\int_{\sigma_m}\sum_{k=1}^{\infty}|\langle f,f_k^m(x)\rangle|^2d\mu(x)\Big)\Big]=\cdots$$

$$+\Big[2\Big(\gamma\int_{\sigma_1}|\langle f,f_k^1(x)\rangle|^2d\mu(x)+\int_{\sigma_1}\sum_{k=1}^{\infty}|\langle f,f_k^1(x)\rangle|^2d\mu(x)\Big)+\cdots$$

$$+2\Big(\gamma\int_{\sigma_{m-1}}\sum_{k=1}^{\infty}|\langle f,f_k^{(m-1)}(x)\rangle|^2d\mu(x)+\int_{\sigma_{m-1}}\sum_{k=1}^{\infty}|\langle f,f_k^{(m-1)}(x\rangle|^2d\mu(x)\Big)$$

$$+\int_{\sigma_m}\sum_{k=1}^{\infty}|\langle f,f_km(x)\rangle|^2d\mu(x)\Big]$$

$$=\Big[2(m-1)(\gamma+1)+1\Big]\Big(\int_{\sigma_1}\sum_{k=1}^{\infty}|\langle f,f_k1(x)\rangle|^2d\mu(x)+\cdots$$

$$+\int_{\sigma_m}\sum_{k=1}^{\infty}|\langle f,f_km(x)\rangle|^2d\mu(x)\Big)\quad\text{for all}\quad f\in\mathcal{H}.$$

Therefore for all $f \in \mathcal{H}$ we have

$$\frac{A_1 + A_2 + ... + A_m}{2(m-1)(\gamma+1)+1}||f||^2 \leq \int_{\sigma_1} \sum_{k=1}^{n} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) + ... + \int_{\sigma_m} \sum_{k=1}^{n} |\langle f, f_k^m(x)\rangle|^2 d\mu(x) \leq \left(\sum_{i\in[m]} B_i \right) ||f||^2$$

This completes the proof.

□

Theorem 3.7. *Assume that for $i \in [m]$ the family $\{F_i(x)\}_{x\in\Omega} = \{\{f_k^i(x)\}_{k\in I}\}_{x\in\Omega}$ of a generalized continuous K-frame for $\mathcal{H}$ is woven with universal bounds A and B and let $\mathcal{Q} : \mathcal{H} \to \mathcal{H}$ be a bounded bijective operator such that $K^*(\mathcal{Q}^{-1})^* = (\mathcal{Q}^{-1})^* K^*$. Then the family of generalized continuous K-frames $\{\mathcal{Q}\{f_k^i(x)\}_{k\in I, x\in\Omega} : i \in [m]\}$ is woven with universal bounds $A||\mathcal{Q}^{-1}||^{-2} and B||\mathcal{Q}||^2$. Furthermore, if A_1, B_1 and A_2, B_2 are optimal universal frame bounds for $\{\{f_k^i(x)\}_{k\in I, x\in\Omega} : i \in [m]\}$ and $\{\{\mathcal{Q}f_k^i(x)\}_{k\in I, x\in\Omega} : i \in [m]\}$, respectively then $A_1||\mathcal{Q}^{-1}||^{-2} \leq A_2 \leq A_1||\mathcal{Q}||^2$ and $B_1||\mathcal{Q}^{-1}||^{-2} \leq B_2 \leq B_1||\mathcal{Q}||^2$.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of Ω. For any $f \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i\in[m]} \int_{\sigma_i} \sum_{k\in I} |\langle f, \mathcal{Q}f_k^i(x)\rangle|^2 d\mu(x) &= \sum_{i\in[m]} \int_{\sigma_i} \sum_{k\in I} |\langle \mathcal{Q}^* f, f_k^i(x)\rangle|^2 d\mu(x) \\
&\leq B||\mathcal{Q}||^2)||f||^2.
\end{aligned}$$

Again, let $f \in \mathcal{H}$. Then, there exists a $g \in \mathcal{H}$ such that $\mathcal{Q}g = f$. So we have

$$\begin{aligned}
||K^*f||^2 &= ||K^*(\mathcal{Q}\mathcal{Q}^{-1})^* Qg||^2 \\
&\leq ||\mathcal{Q}^{-1}||^2 ||K^*\mathcal{Q}^*\mathcal{Q}g||^2 \\
&\leq ||\mathcal{Q}^{-1}||^2/A \sum_{i\in[m]} \int_{\sigma_i} \sum_{k\in I} |\langle \mathcal{Q}^*\mathcal{Q}g, f_k^i(x)\rangle|^2 d\mu(x) \\
&= ||\mathcal{Q}^{-1}||^2/A \sum_{i\in[m]} \int_{\sigma_i} \sum_{k\in I} |\langle \mathcal{Q}g, \mathcal{Q}f_k^i(x)\rangle|^2 d\mu(x) \\
&= ||\mathcal{Q}^{-1}||^2/A \sum_{i\in[m]} \int_{\sigma_i} \sum_{k\in I} |\langle f, \mathcal{Q}f_k^i(x)\rangle|^2 d\mu(x)
\end{aligned}$$

Hence the family of generalized continuous K-frames $\{\{Qf_k^i(x)\}_{k\in I, x\in\Omega} : i \in [m]\}$ for $\mathcal{H}$ is woven.

□

Theorem 3.8. *Suppose the family* $\{\{f_k^i(x)\}_{k\in I, x\in\Omega}: i \in [m]\}]$ *of a generalized continuous K-frames for H w.r.t.* μ *is a woven with universal bounds A and B. If there exists* $0 < \delta < A$ *and a measurable subset* $\mathcal{N}$ *of* Ω *and* $n \in [m]$ *such that*

$$\sum_{i\in[m]\setminus\{n\}} \int_{\Omega\setminus\mathcal{N}} \sum_{k\in I} |\langle f, f_k^i(x)\rangle|^2 d\mu(x) \le \delta||K^* f||^2, \textit{for all } \ f \in \mathcal{H}$$

then for any partition $\{\sigma_i\}_{i\in[m]}$ *of* $\mathcal{L}$*, the family* $\cup_{i\in[m]}\{f_k^i(x)\}_{k\in I, x\in\sigma_i}$ *is a generalized continuous K-frame for* $\mathcal{H}$ *with frame bounds* $A-\delta$ *and* B.

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathcal{N}$. Let $\{\tau_i\}_{i\in[m]}$ be any partition of $\Omega\setminus\mathcal{N}$. Then $\cup_{i\in[m]}\{f_k^i(x)\}_{k\in I, x\in\sigma_i\cup\tau_i}$ is a generalized continuous K-frame for $\mathcal{H}$. Therefore, for $f \in \mathcal{H}$, we have

$$\int_{\mathcal{N}=\cup_{i\in[m]}\sigma_i} \sum_{k\in I} |\langle f, f_k^i(x)\rangle|^2 d\mu(x)$$

$$= \int_{\sigma_1} \sum_{k\in I} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) + \cdots + \int_{\sigma_m} \sum_{k\in I} |\langle f, f_k^m(x)\rangle|^2 d\mu(x)$$

$$\le \int_{\sigma_1\cup\tau_1} \sum_{k\in I} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) + \cdots + \int_{\sigma_m\cup\tau_m} \sum_{k\in I} |\langle f, f_k^m(x)\rangle|^2 d\mu(x)$$

$$\le B||f||^2.$$

Again, let $\{\pi_i\}_{i\in[m]}$ be any partition of $\Omega\setminus\mathcal{L}$ such that $\pi_n = \phi$. Then $\{\pi_i \cup \sigma_i\}_{i\in[m]}$ is a partition of Ω. We compute

$$\int_{\sigma_1} \sum_{k\in I} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) + \cdots + \int_{\sigma_n} \sum_{k\in I} |\langle f, f_k^n(x)\rangle|^2 d\mu(x) + \cdots + \int_{\sigma_m} \sum_{k\in I} |\langle f, f_k^m(x)\rangle|^2 d\mu(x)$$

$$= \sum_{i\in[m]\setminus\{n\}} \left(\int_{\sigma_i\cup\pi_i} \sum_{k\in I} |\langle f, f_k^i(x)\rangle|^2 d\mu(x) - \int_{\pi_i} \sum_{k\in I} |\langle f, f_k^i(x)\rangle|^2 d\mu(x) \right) + \int_{\sigma_n} \sum_{k\in I} |\langle f, f_k^n(x)\rangle|^2 d\mu(x)$$

$$\geq \sum_{i\in[m]\setminus\{n\}} \left(\int_{\sigma_i\cup\pi_i} \sum_{k\in I} |\langle f, f_k^i(x)\rangle|^2 d\mu(x) - \int_{\Omega\setminus\mathcal{N}} \sum_{k\in I} |\langle f, f_k^i(x)\rangle|^2 d\mu(x) \right)$$

$$+ \int_{\sigma_n} \sum_{k\in I} |\langle f, f_k^n(x)\rangle|^2 d\mu(x)$$

$$= \sum_{i\in[m]} \int_{\sigma_i\cup\pi_i} \sum_{k\in I} |< f, f_k^i(x) >|^2 d\mu(x) - \sum_{i\in[m]\setminus\{n\}} \int_{\Omega\setminus\mathcal{N}} \sum_{k\in I} |\langle f, f_k^i(x)\rangle|^2 d\mu(x)$$

$$\geq (A-\delta)||K^* f||^2, \text{for all } \ f\in\mathcal{H}.$$

□

Theorem 3.9. *For each $i \in [m]$, Let $\{F_i(x)\}_{x\in\Omega} = \{\{f_k^i(x)\}_{k\in I}\}_{x\in\Omega}$ is a generalized continuous K-frame for $\mathcal{H}$ w.r.t. μ with frame bounds A_i and B_i.If there exists a measurable subset $\mathcal{N}$ of Ω such that the family of generalized continuous K-frame $\{\{f_k^i(x)\}_{k\in I, x\in\mathcal{N}} : i \in [m]\}$ is a woven in H with universal bounds A and B, then the family $\{\{f_k^i(x)\}_{k\in I, x\in\Omega} : i \in [m]\}$ of generalized continuous K-frame for $\mathcal{H}$ w.r.t. μ is woven with universal bounds A and $\sum_{i\in[m]} B_i$.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of Ω. For any $f \in H$, we have

$$\int_{\sigma_1} \sum_{k\in I} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) + \cdots + \int_{\sigma_m} \sum_{k\in I} |\langle f, f_k^m(x)\rangle|^2 d\mu(x)$$

$$\leq \int_{\Omega} \sum_{k\in I} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) + \cdots + \int_{\Omega} \sum_{k\in I} |\langle f, f_k^m(x)\rangle|^2 d\mu(x)$$

$$\leq \Big(\sum_{i\in[m]} B_i\Big)||f||^2$$

The family $\cup_{i\in[m]}\{f_k^i(x)\}_{k\in I, x\in\Omega}$ satisfies the upper frame inequality with frame bound $\sum_{i\in[m]} B_i$.

Let $\{\sigma_i\}_{i\in[m]}$ be any partition of Ω, $\{\sigma_i \cap \mathcal{N}\}_{i\in[m]}$ is a partition of $\mathcal{N}$. Therefore, $\cup_{i\in[m]}\{f_k^i(x)\}_{k\in I, x\in\sigma_i\cap\mathcal{N}}$ is a generalized continuous K-frame for $\mathcal{H}$ w.r.t. μ and with lower frame bound A. This gives

$$\int_{\sigma_1} \sum_{k\in I} |\langle f, f_k^1(x)\rangle|^2 d\mu(x) + \cdots + \int_{\sigma_m} \sum_{k\in I} |\langle f, f_k^m(x)\rangle|^2 d\mu(x)$$

$$\geq \int_{\sigma_1 \cap \mathcal{N}} \sum_{k \in I} |\langle f, f_k^1(x) \rangle|^2 d\mu(x) + \cdots + \int_{\sigma_m \cap \mathcal{N}} \sum_{k \in I} |\langle f, f_k^m(x) \rangle|^2 d\mu(x)$$

$$\geq A||K^* f||^2$$

for all $f \in H$. This completes the proof. □

Theorem 3.10. *For each $i \in [m]$, Let $\{F_i(x)\}_{x\in\Omega} = \{\{f_k^i(x)\}_{k\in I}\}_{x\in\Omega}$ is a generalized continuous K-frame for $\mathcal{H}$ w.r.t. μ with frame bounds A_i and B_i. Assume for any partition $\{\tau_i\}_{i\in[m]}$ of a finite subset of Ω and for every $A > 0$ there exists a partition $\{\sigma_i\}_{i\in[m]}$ of the set $\Omega/(\tau_1 \cup \tau_2 \cup ... \cup \tau_m)$ s.t. $\cup_{i\in[m]}\{f_k^i(x)\}_{x\in\sigma_i\cup\tau_i}$ has a lower K-frame bound less than A. Then, there exists a partition $\{\pi_i\}_{i\in[m]}$ of Ω s.t. $\cup_{i\in[m]}\{F_i(x)\}_{x\in\pi_i}$ is not a generalized continuous K-frame for $\mathcal{H}$.*

Proof. Since (Ω, μ) is a σ - finite measurable space, $\Omega = \cup_{i\in N} Y_i$, where Y_i are disjoint measurable sets and $\mu(Y_i) < \infty$ for all $i \in N$. Suppose $\tau_{1i} = \phi$ for all $i \in [m]$ and $A = 1$. Then, there exists a partition $\{\sigma_{1i}\}_{i\in[m]}$ of Ω such that $\cup_{i\in[m]}\{f_k^i(x)\}_{x\in\sigma_{1i}\cup\tau_{1i}}$ has a lower bound less than 1. Therefore there exists a vector $h_1 \in \mathcal{H}$ such that

$$\sum_{i\in[m]} \int_{\sigma_{1i}\cup\tau_{1i}} \sum_{k\in I} |\langle h_1, f_k^i(x)\rangle|^2 d\mu(x) < ||K^* h_1||^2.$$

Since $\sum_{i\in[m]} \int_\Omega \sum_{k\in I} |\langle h_1, f_k^i(x)\rangle|^2 d\mu(x) < \infty$. there exists $r_1 \in N$ such that

$$\sum_{i\in[m]} \int_{\substack{\cup Y_i \\ i\geq k_1+1}} \sum_{k\in\mathbb{N}} |\langle h_1, f_k^i(x)\rangle|^2 d\mu(x) < ||K^* h_1||^2$$

Choose $\{\tau_{2i}\}_{i\in[m]} = \{\tau_{1i} \cup (\sigma_{1i} \cap (Y_1 \cup \ldots \cup Y_{k_1}))\}_{i\in[m]}$ a partition of $Y_1 \cup \ldots \cup Y_{k_1}$ and $A = \frac{1}{2}$. Then, there exists a partition $\{\sigma_{2i}\}_{i\in[m]}$ of $\Omega \setminus (Y_1 \cup \ldots \cup Y_{k_1})$ such that the family $\cup_{i\in[m]}\{f_k^i(x)\}_{\sigma_{2i}\cup\tau_{2i}}$ has a lower frame bound less than $\frac{1}{2}$. Therefore, there exists a vector $h_2 \in \mathcal{H}$ such that

$$\sum_{i\in[m]} \int_{\sigma_{1i}\cup\tau_{1i}} \sum_{k\in I} |\langle h_2, f_k^i(x)\rangle|^2 d\mu(x) < ||K^* h_2||^2.$$

Since $\sum_{i\in[m]} \int_\Omega \sum_{k\in I} |\langle h_2, f_k^i(x)\rangle|^2 d\mu(x) < \infty$. there exists $r_2 \in N$ s.t.

$$\sum_{i\in[m]} \int_{\substack{\cup Y_i \\ i\geq k_1+1}} \sum_{k\in\mathbb{N}} |\langle h_2, f_k^i(x)\rangle|^2 d\mu(x) < \frac{1}{2}||K^* h_2||^2$$

Proceeding in this manner, for $A = \frac{1}{p}$ and for a partition $\{\tau_{p_i}\}_{i\in[m]} = \left\{\tau_{(p-1)i} \cup \left(\sigma_{(p-1)i} \cap \left(Y_1 \cup Y_2 \ldots Y_{k_{(p-1)}}\right)\right)\right\}$ of $Y_1 \cup Y_2 \ldots Y_{k_{(p-1)}}$, we can find a partition of $\{\sigma_{p_i}\}_{i\in[m]}$ of $\Omega \setminus Y_1 \cup Y_2 \ldots Y_{k_{(p-1)}}$ such that $\cup_{i\in[m]}\{f_k^i(x)\}_{x\in\sigma_i\cup\pi_i}$ has a lower bound less than $\frac{1}{p}$. Thus there exists $h_p \in \mathcal{H}$ such that

$$\sum_{i\in[m]} \int_{\sigma_{p_i}\cup\tau_{p_i}} \sum_{k\in\mathbb{N}} |\langle h_p, f_k^i(x)\rangle|^2 d\mu(x) < \frac{1}{p}||K^* h_p||^2$$

and there exists $r_p > r_{p-1}$ such that

$$\sum_{i\in[m]} \int_{\substack{\cup Y_i \\ i\geq k_p+1}} \sum_{k\in\mathbb{N}} |\langle h_p, f_k^i(x)\rangle|^2 d\mu(x) < \frac{1}{p}||K^* h_p||^2$$

Choose a partition $\{\pi\}_{i\in[m]} = \{\cup_{j\in\mathbb{N}}\tau_{ij}\}_{i\in[m]}$ of Ω. Then the family $\cup_{i\in[m]}\{F_i(x)\}_{x\in\pi_i}$ is not generalized continuous K-frame for $\mathcal{H}$. Indeed, let $\cup_{i\in[m]}\{F_i(x)\}_{x\in\pi_i}$ be generalized continuous K-frame for $\mathcal{H}$ with frames bounds α and β, respectively. Then, by using the Archimedean Property there exists a $q \in \mathbb{N}$ such that $q > \frac{2}{\alpha}$. Then, we get

$$\int_{\pi_1} \sum_{k\in\mathbb{N}} |\langle h_q, f_k^i(x)\rangle|^2 d\mu(x) + \ldots + \int_{\pi_m} \sum_{k\in\mathbb{N}} |\langle h_q, f_k^i(x)\rangle|^2 d\mu(x)$$

$$\leq \sum_{i\in[m]} \int_{\tau_{q_i}\cup\sigma_{q_i}} \sum_{k\in\mathbb{N}} |\langle h_q, f_k^i(x)\rangle|^2 d\mu(x) + \sum_{i\in[m]} \int_{\substack{\cup Y_i \\ i\geq k_p+1}} \sum_{k\in\mathbb{N}} |\langle h_p, f_k^i(x)\rangle|^2 d\mu(x)$$

$$\leq \frac{1}{q} + \frac{1}{q} < \alpha\|h_q\|^2.$$

which is a contradiction. □

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