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BERNSTEIN TYPE INEQUALITIES FOR COMPOSITE POLYNOMIALS

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Abstract. Establishing the lower and upper bound estimates for the maximum modulus of the derivative of composition of polynomials $p(q(z))$, where $q(z)$ is a polynomial of degree m is an intriguing problem in geometric theory of polynomials. In this paper, the maximum modulus for composite polynomials of Bernstein type is taken up with constraints such as the given polynomial does not vanish in the disc $|z| < k$, where $k \geq 1$ which in particular yields some known inequalities of this type as special cases. In addition, the case when all the zeros of the underlying polynomial lie in $|z| \leq k$, where $k \leq 1$ is also considered.

1. Introduction

Let $\|f\|$ denote the supremum norm of a function f on $|z| = 1$. The Bernstein inequality asserts that

$$\|p'\| \leq n|z|^{n-1}\|p\| \tag{1.1}$$

holds for every polynomial $p(z) = \sum_{v=0}^{n} a_v z^v$ of degree n with complex coefficients. Various analogues of inequality are known in which the underlying intervals, the maximum norms, and the family of functions are replaced by more general sets, norms and families of functions such as hyperholomorphic functions respectively.

The inequality

$$|p'(z)| \geq n|z|^{n-1} \min_{|z|=1} |p(z)| \tag{1.2}$$

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holds for every polynomial p of degree at most n which has all zeros in $|z| \leq 1$. This inequality is ascribed to Aziz and Dawood [1]. Bernstein-type inequalities are known on various regions of the complex plane and n-dimensional Euclidean space such as Quaternions (see [5]), for various norms such as weighted L_p norms, and for many classes of functions such as polynomials with various constraints. Just to mention, Bernstein-type inequalities have their own intrinsic interest.

Both the inequalities (1.1) and (1.2) are sharp and equality holds when $p(z) = \alpha z^n$, for any complex number α, which has all its zeros at the origin, one would expect a relationship between the bound n and the distance of the zeros of the polynomial from the origin. This fact was observed as a refinement of Bernstein's inequality, conjectured by Erdös and later proved by Lax [7], and states that:

If $p(z)$ is a polynomial of degree n with complex coefficients having no zeros in $|z| < 1$, then

$$\|p'\| \leq \frac{n}{2}\|p\|. \tag{1.3}$$

This inequality is sharp and equality holds if p has all zeros on $|z| = 1$.

For polynomials of a complex variable, we also have the following more general result, due to Malik [8], which is one of the most known polynomial inequality after Bernstein inequality and will be useful in proving some of our results.

Theorem 1.1. [8] If $p(z)$ is a polynomial of degree n having no zeros in $|z| < k$, where $k \geq 1$ then

$$\|p'\| \leq \frac{n}{1+k}\|p\|. \tag{1.4}$$

In the context of bounds for the maximum modulus of $p'(z)$ taken over unit circle, our interest is centered upon the study of making these bounds notably sharper, with or without various constraints over polynomials.

The inequality

$$\|p'\| \leq \frac{n}{1+k}\{\|p\| - \min_{|z|=k} |p(z)|\} \tag{1.5}$$

is due to Govil [6] and it holds for all polynomials p of degree at most n with complex coefficients having no zeros in $|z| < k$, where $k \geq 1$. In this inequality the bound in comparison to the bound in (1.4) gets reduced by $\min_{|z|=k} |p(z)|$, which is a significant improvement.

Both the inequalities (1.4) and (1.5) are best possible and equality holds when $p(z) = (z+k)^n$.

Let χ_n represents the class of all polynomials $p(z)$ of degree at most n. Now consider a more general class of polynomials $f \circ g \in \chi_{mn}$ defined as $(f \circ g)(z) = f(g(z))$, where $g(z)$ is a polynomial of degree m. Dewan et al. [4] extended Theorem 1.1 and inequality (1.5) to the general class of polynomials $f \circ g \in \chi_{mn}$ and proved that the inequalities

$$\max_{|z|=1} |p'(q(z))| \leq \frac{n}{(1+k)m'} |z|^{mn-m} \|p \circ q\| \tag{1.6}$$

and

$$\max_{|z|=1} |p'(q(z))| \leq \frac{n}{(1+k)m'} |z|^{mn-m} \left\{ \|p \circ q\| - \min_{|z|=k} |p(q(z))| \right\} \tag{1.7}$$

holds for all polynomials $p \circ q \in \chi_{mn}$, such that $p(q(z)) \neq 0$ in $|z| < k$, where $k \geq 1$ and $q(z) \neq 0$ in $|z| \geq 1$ with $\min_{|z|=1} |q(z)| = m'$. Both the inequalities (1.6) and (1.7) are best possible and equality holds when $p(q(z)) = (z^m + k)^n$, here $p(z) = (z+k)^n$ and $q(z) = z^m$.

2. Auxiliary Results

For the proofs of our theorems we require the following lemmas.

Lemma 2.1. Let $p \circ q \in \chi_{mn}$. If $S(z) = p(q(z)) = c_{mn} z^{mn} + \sum_{v=0}^{mn-\mu} c_v z^v$, $1 \leq \mu \leq n$ such that $S(z) \neq 0$ in $|z| < k$, where $k \geq 1$, then for $|z| = 1$

$$k^{\mu} |S'(z)| \leq |R'(z)|,$$

where $R(z) = z^{mn} \overline{S(\frac{1}{\bar{z}})} = z^{mn} \overline{p(q(\frac{1}{\bar{z}}))}$.

Proof. Since $S(z) \neq 0$ in $|z| < k$, hence from Laguerre's Theorem [3] we have

$$\alpha S'(z) \neq z S'(z) - mn S(z) \tag{2.1}$$

for $|\alpha| < k$, $|z| < k$. Now choose the $\arg \alpha$ in (2.1) appropriately, then we get for any fixed z

$$|\alpha||S'(z)| \neq |zS'(z) - mnS(z)|.$$

This gives for $|\alpha| < k$ and $|z| < k$

$$|\alpha||S'(z)| < |zS'(z) - mnS(z)| \tag{2.2}$$

because otherwise the inequality is violated for sufficiently small values of $|\alpha|$. Letting $|\alpha| \to k$ in (2.2) we have

$$k|S'(kz)| \leq |kzS'(kz) - mnS(kz)| \tag{2.3}$$

for $|z| \leq 1$. Since $c_1 = c_2 = ... = c_{\mu-1} = 0$, from (2.3) we get

$$k^{\mu} \left| \sum_{v=\mu}^{mn} vc_v (kz)^{v-1} \right| \leq |kzS'(kz) - mnS(kz)| \tag{2.4}$$

for $|z| \leq 1$. In fact (2.4) also holds for $|z| = 1$, replace z by z/k in (3.4) we obtain for $|z| = 1$

$$k^{\mu} \left| \sum_{v=\mu}^{mn} vc_v (z)^{v-1} \right| \leq |zS'(z) - mnS(z)|.$$

It can be easily verified for $|z| = 1$ that

$$|R'(z)| = |zS'(z) - mnS(z)|.$$

Consequently $k^{\mu}|S'(z)| \leq |R'(z)|$ for $|z| = 1$. This completes the proof of Lemma 2.1. □

Lemma 2.2. Let $p \circ q \in \chi_{mn}$. If $S(z) = p(q(z)) = c_0 + \sum_{v=\mu}^{mn} c_v z^v$, $1 \leq \mu \leq n$ such that $S(z) = 0$ in $|z| \leq k$, where $k \leq 1$, then for $|z| = 1$

$$k^{\mu}|S'(z)| \geq |R'(z)|,$$

where $R(z) = z^{mn}\overline{S(\frac{1}{\bar{z}})}$.

Proof. Since $S(z) = 0$ in $|z| \leq k$ therefore all the zeros of the polynomial $R(z) = z^{mn}\overline{S(1/\bar{z})}$ lie in $|z| \geq \frac{1}{k}$. Applying Lemma 2.1 to the polynomial $R(z)$, we have

$$\frac{1}{k^{\mu}}|R'(z)| \leq |S'(z)|.$$

Consequently,

$$k^{\mu}|S'(z)| \geq |R'(z)|.$$

This completes the proof of Lemma 2.2. □

3. Main Results

The research on mathematical objects associated with polynomial inequalities has been active over a period; there are many research papers published in a variety of journals each year and different approaches have been taken for different purposes. The present article is concerned with Bernstein type inequalities for composite polynomial with restricted zeros. We also consider a class of polynomials $f \circ g \in \chi_{mn}$ having d-fold zeros at origin.

Theorem 3.1. For all $p \circ q \in \chi_{mn}$, such that $p(q(z)) \neq 0$ in $|z| < k$, where $k \geq 1$ except with d-fold zeros at origin and $q(z) \neq 0$ in $|z| \geq 1$ with $\min_{|z|=1} |q(z)| = A$, then

$$\max_{|z|=1} |p'(q(z))| \leq \left[\frac{n}{A(1+k)} + \frac{kd}{Am(1+k)}\right] \|p \circ q\|. \tag{3.1}$$

Proof. Let $p(q(z)) = z^d h(z)$, where $h(z)$ is a polynomial of degree $mn - d$ which has all its zeros in $|z| \geq k \geq 1$. Thus, by making use of Theorem 1.1 we have

$$|h'(z)| \leq \frac{mn-d}{1+k} \max_{|z|=1} |h(z))|. \tag{3.2}$$

Since $p(q(z)) = z^d h(z)$, therefore

$$p'(q(z)) = dz^{d-1}h(z) + z^d h'(z).$$

From this we get for $|z| = 1$

$$\begin{aligned} |p'(q(z))||q'(z)| &= |dh(z) + zh'(z)| \\ &\leq d|h(z)| + |h'(z)| \\ &= d|p(q(z))| + |h'(z)|. \end{aligned}$$

This gives in conjunction with (3.2)

$$|p'(q(z))||q'(z)| \leq \left[d + \frac{mn-d}{1+k}\right] |p(q(z))|.$$

Since all the zeros of $q(z)$ lie in $|z| < 1$, on using inequality (1.2) to the polynomial $q(z)$ with $\min_{|z|=1} |q(z))| = A$, we have

$$Am|p'(q(z))| \leq |p'(q(z))||q'(z)| \leq \left[\frac{mn + kd}{1 + k}\right] |p(q(z))|.$$

The above is equivalent to

$$\max_{|z|=1} |p'(q(z))| \leq \left[\frac{n}{A(1+k)} + \frac{kd}{Am(1+k)}\right] \|p \circ q\|.$$

This completes the proof of Theorem 3.1. □

Remark 1. *If we choose $d = 0$ in Theorem 3.1, we obtain inequality (1.6) as a special case.*

Remark 2. *If we set $d = 0$ and $q(z) = z$ then $q(z) = 0$ in $|z| < 1$ with $\min_{|z|=1} |q(z)| = A = 1$. Therefore from Theorem 3.1, we have Theorem 1.1 as a special case.*

Further we establish the refinement of Theorem 3.1 under the same hypothesis which in particular, yields inequality (1.7) as a special case. More specifically, we prove

Theorem 3.2. If for every $p \circ q \in \chi_{mn}$, such that $p(q(z)) \neq 0$ in $|z| < k$, where $k \geq 1$ except with d-fold zeros at origin and $q(z) \neq 0$ in $|z| \geq 1$ with $\min_{|z|=1} |q(z)| = A$, then

$$\max_{|z|=1} |p'(q(z))| \leq \frac{1}{Am(1+k)} \left\{(mn + kd)\|p \circ q\| - (mn - d) \min_{|z|=k} |p(q(z))|\right\}. \tag{3.3}$$

Proof. Using the analysis similar to used in the proof of Theorem 3.1 and applying inequality (1.5) to the polynomial $h(z)$ we have

$$|h'(z)| \leq \frac{mn - d}{1 + k} \left\{\max_{|z|=1} |h(z))| - \min_{|z|=k} |h(z))|\right\}. \tag{3.4}$$

Now it follows for $|z| = 1$ that

$$\begin{aligned} |p'(q(z))||q'(z)| &= |dh(z) + zh'(z)| \\ &\leq d|h(z)| + |h'(z)| \\ &= d|p(q(z))| + |h'(z)|. \end{aligned}$$

This gives with the help of (3.4)

$$|p'(q(z))||q'(z)| \leq \frac{1}{(1+k)}\left\{(mn+kd)\|p\circ q\| - (mn-d)\min_{|z|=k}|p(q(z))|\right\}.$$

Since all the zeros of $q(z)$ lie in $|z|<1$, on using inequality (1.2) to the polynomial $q(z)$ with $\min_{|z|=1}|q(z))| = A$, we have

$$Am|p'(q(z))| \leq |p'(q(z))||q'(z)| \leq \frac{1}{(1+k)}\left\{(mn+kd)\|p\circ q\| - (mn-d)\min_{|z|=k}|p(q(z))|\right\}.$$

The above is equivalent to (3.3) and this completes the proof of Theorem 3.2. □

Remark 3. *In particular Theorem 3.2 gives inequality (1.7) for $d=0$.*

Remark 4. *Consider the case when $d=0$ and $q(z)=z$, this implies that $\min_{|z|=1}|q(z)| = A = 1$ and we see that all the zeros of $q(z)$ lie in $|z|<1$. In this connection, Theorem 3.2 in particular gives inequality (1.5).*

In the sequel, the validity of the analogous results has been established well for the case when all the zeros of $p(z)$ lie in $|z|\leq k$, where $k\leq 1$. In the connection, the first and foremost result is ascribed to Malik [8, corollary] and it plays a very significant role in the theory of *Geometry of polynomials*. Consequently, we prove the following result for composite polynomials under the assumption that all its zeros lie in $|z|\leq k$, where $k\leq 1$.

Theorem 3.3. If for every $p\circ q\in\chi_{mn}$, such that $p(q(z)) = c_{mn}z^{mn} + \sum_{v=0}^{mn-\mu} c_v z^v$, $1\leq\mu\leq n$, $p(q(z))=0$ in $|z|\leq k$, where $k\leq 1$ and $q(z)$ is a polynomial of degree m with $\max_{|z|=1}|q(z)| = M$, then

$$\max_{|z|=1}|p'(q(z))| \geq \frac{n}{(1+k^{\mu})M}|z|^{mn-m}\|p\circ q\|. \tag{3.5}$$

Proof. Let $E(z)=p(q(z))$. Recall that $E(z)=0$ in $|z|\leq k$, $k\leq 1$. Consider the polynomial $L(z)=z^{mn}\overline{E(\frac{1}{\bar z})}$. From this it follows for $|z|=1$ that

$$|L(z)| = |E(z)|.$$

Also $E(z) = z^{mn}\overline{L(\frac{1}{\bar z})}$, which gives us for $|z|=1$,

$$|E'(z)| = |mnL(z) - zL'(z)|.$$

Clearly

$$mnL(z) = mnL(z) - zL'(z) + zL'(z)$$
$$|mnL(z)| = |mnL(z) - zL'(z) + zL'(z)|$$
$$|mnL(z)| \leq |mnL(z) - zL'(z)| + |zL'(z)|$$

that is,

$$mn|E(z)| \leq |E'(z)| + |zL'(z)|.$$

Since all the zeros of $L(z)$ lie in $|z| \geq \frac{1}{k}$ therefore in accordance with Lemma 2.2 we have for $|z| = 1$

$$mn|z|^{mn-1}|E(z)| \leq (1 + k^{\mu})|E'(z)|$$
$$|E'(z)| \geq \frac{mn}{1 + k^{\mu}}|z|^{mn-1}|E(z)|$$

that is,

$$|p'(q(z))||q'(z)| \geq \frac{mn}{1 + k^{\mu}}|z|^{mn-1}|p(q(z))|.$$

Since $q(z)$ is a polynomial of degree m with $\max\limits_{|z|=1} |q(z)| = M$, hence by using inequality (1.1) we get

$$mM|z|^{m-1}|p'(q(z))| \geq \frac{mn}{1 + k^{\mu}}|z|^{mn-1}|p(q(z))|,$$

which is (3.5) and the proof of Theorem 3.3 is thus complete. □

Remark 5. *In relation to $q(z)$, note that if we choose $q(z) = z$ then clearly $M = 1$. Therefore we have from Theorem 3.3 that, if $p(z) = c_n z^n + \sum_{v=0}^{n-\mu} c_v z^v$ has all its zeros in $|z| \leq k$, where $k \leq 1$ then for $|z| = 1$*

$$\|p'\| \geq \frac{n}{1 + k^{\mu}}\|p\|.$$

This result is due to Chan and Malik [2].

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