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GE-FILTER EXPANSIONS IN GE-ALGEBRAS

YOUNG BAE JUN(1) AND RAVIKUMAR BANDARU(2*)

Abstract. The notions of GE-filter expansion and ξ-primary GE-filter are introduced and their properties are investigated. Different ways to create a GE-filter expansion are provided. The notion of good GE-filter expansion is introduced and its properties investigated. The conditions for an image and an inverse image of a ξ-primary GE-filter of a GE-algebra to be a ξ-primary GE-filter are provided.

1. Introduction

Y. Imai and K. Iséki (see [8, 9]) introduced BCK-algebras in 1966 as the algebraic semantics for a non-classical logic with only implication. Various scholars have studied generalized notions of BCK-algebras since then. L. Henkin and T. Skolem introduced Hilbert algebras in the 1950s for research into intuitionistic and other non-classical logics. A. Diego demonstrated that Hilbert algebras constitute a locally finite variety (see [6]). Later, several researchers expanded on the theory of Hilbert algebras (see [5, 7, 10, 12]). The notion of BE-algebra was introduced by H. S. Kim and Y. H. Kim as a generalization of a dual BCK-algebra (see [13]). A. Rezaei et al. discussed relations between Hilbert algebras and BE-algebras (see [15]). Y. B. Jun (see [11]) introduced the notions of expansion of subalgebras (resp., ideals), σ-primary ideals, and residual divisions, and investigated related properties. In the study of algebraic structures, abstraction is an important methodology. As a generalization of Hilbert algebras, R. K. Bandaru et.al. introduced the notion of GE-algebras, and investigated several properties (see [1]). For the general development of GE-algebras,

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*Corresponding author.

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the filter theory plays an important role. With this motivation, R. K. Bandaru et. al. introduce the notion of belligerent GE-filters in GE-algebras and studied its properties (see [2]). A. Rezaei et.al. introduced the concept of prominent GE-filters in GE-algebras and discussed its properties (see [16]). R. K. Bandaru et.al. introduced the concept of bordered GE-algebra and investigated its properties (see [3]). Later, M. A. Öztürk et. al. introduced the concept of Strong GE-filters, GE-ideals of bordered GE-algebras and investigated its properties (see [14]). A. Borumand Saeid et. al. introduced the concept of voluntary GE-filters of GE-algebras and investigated its properties (see [4]). S. Z. Song et. al. introduced the concept of imploring GE-filters of GE-algebras and discussed its properties (see [18]).

In this paper, we introduce the notions of GE-filter expansion and ξ-primary GE-filter and investigate their properties. We provide different ways to create a GE-filter expansion. We introduce the notion of good GE-filter expansion and investigate its properties. Finally, we provide the conditions for an image and an inverse image of a ξ-primary GE-filter of a GE-algebra to be a ξ-primary GE-filter.

2. Preliminaries

Definition 2.1 ([1]). By a *GE-algebra* we mean a nonempty set X with a constant 1 and a binary operation "$*$" satisfying the following axioms:

(GE1) $u * u = 1$,

(GE2) $1 * u = u$,

(GE3) $u * (v * w) = u * (v * (u * w))$

for all $u, v, w \in X$.

Let $(X, *, 1)$ be a GE-algebra. Define a binary relation " $\leq$ " on X by $u \leq v$ if and only if $u * v = 1$. We can observe that $\leq$ is only a reflexive relation on X.

Definition 2.2 ([1, 2]). A GE-algebra X is said to be

- *transitive* if it satisfies:

$$(\forall u, v, w \in X)\,(u * v \leq (w * u) * (w * v))\,. \tag{2.1}$$

- *commutative* if it satisfies:

$$(\forall u, v \in X)\,((u * v) * v = (v * u) * u)\,. \tag{2.2}$$

Proposition 2.1 ([1]). *Every GE-algebra X satisfies the following properties.*

$$(\forall u \in X)\,(u * 1 = 1)\,. \tag{2.3}$$

$$(\forall u, v \in X)\,(u * (u * v) = u * v)\,. \tag{2.4}$$

$$(\forall u, v \in X)\,(u \leq v * u)\,. \tag{2.5}$$

$$(\forall u, v, w \in X)\,(u * (v * w) \leq v * (u * w))\,. \tag{2.6}$$

$$(\forall u \in X)\,(1 \leq u \;\Rightarrow\; u = 1)\,. \tag{2.7}$$

$$(\forall u, v \in X)\,(u \leq (v * u) * u)\,. \tag{2.8}$$

$$(\forall u, v \in X)\,(u \leq (u * v) * v)\,. \tag{2.9}$$

$$(\forall u, v, w \in X)\,(u \leq v * w \;\Leftrightarrow\; v \leq u * w)\,. \tag{2.10}$$

If X is transitive, then

$$(\forall u, v, w \in X)\,(u \leq v \;\Rightarrow\; w * u \leq w * v,\; v * w \leq u * w)\,. \tag{2.11}$$

$$(\forall u, v, w \in X)\,(u * v \leq (v * w) * (u * w))\,. \tag{2.12}$$

$$(\forall u, v, w \in X)\,(u \leq v,\; v \leq w \;\Rightarrow\; u \leq w)\,. \tag{2.13}$$

If X is commutative, then

$$(\forall u, v, w \in X)\,(u * (v * w) = v * (u * w))\,. \tag{2.14}$$

$$(\forall u, v, w \in X)\,(u * (v * w) = (u * v) * (u * w))\,. \tag{2.15}$$

Theorem 2.1. *If X is a commutative GE-algebra then X is antisymmetric and transitive GE-algebra.*

Proof. Let X be a commutative GE-algebra and $x, y, z \in X$. Suppose $x * y = 1$ and $y * x = 1$. Then, by (GE2) and (2.2), $x = 1 * x = (y * x) * x = (x * y) * y = 1 * y = y$. Hence X is antisymmetric GE-algebra. Also, by (2.15) and (2.5), $(x * y) * ((z * x) * (z * y)) = (x * y) * (z * (x * y)) = 1$. Hence X is transitive GE-algebra. □

Example 2.1. *Let* $\mathbb{N}$ *be the set of all natural numbers and* $*$ *be the binary operation on* $\mathbb{N}$ *defined by:*

$$x * y = \begin{cases} y & \text{if } x = 1; \\ 1 & \text{if } x \neq 1. \end{cases}$$

Then $(\mathbb{N}, *, 1)$ *is not a commutative GE-algebra, since* $(5*1)*1 = 1 \neq (1*5)*5 = 5$.

Example 2.2. *Let* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $*$ *be the binary operation on* $\mathbb{N}_0$ *defined by:*

$$x * y = \begin{cases} 0 & \text{if } y \leq x; \\ y - x & \text{if } x < y. \end{cases}$$

Then $(\mathbb{N}_0, *, 0)$ *is a commutative GE-algebra.*

Definition 2.3 ([1])**.** A subset F of a GE-algebra X is called a *GE-filter* of X if it satisfies:

$$1 \in F, \tag{2.16}$$

$$(\forall u, v \in X)(u \in F,\ u * v \in F \ \Rightarrow\ v \in F). \tag{2.17}$$

Lemma 2.1 ([1])**.** *In a GE-algebra* X*, every GE-filter* F *of* X *satisfies:*

$$(\forall u, v \in X)\left(u \leq v,\ u \in F \ \Rightarrow\ v \in F\right). \tag{2.18}$$

Definition 2.4 ([16])**.** Let X be a GE-algebra. Then a mapping $f : X \to X$ is called a *GE-endomorphism* if it satisfies:

$$(\forall x, y \in X)(f(x * y) = f(x) * f(y)).$$

Note that the kernel of f is given by $\ker(f) = \{x \in X \mid f(x) = 1\}$.

Definition 2.5 ([3])**.** If a GE-algebra X has a special element, say 0, that satisfies $0 \leq u$ for all $u \in X$, we call X a *bordered GE-algebra*.

For every element u of a bordered GE-algebra X, we denote $u * 0$ by u^0, and $(u^0)^0$ is denoted by u^{00}.

Definition 2.6 ([3])**.** If a bordered GE-algebra X satisfies the condition (2.1), we say that X is a *transitive bordered GE-algebra*.

3. GE-filter expansions

In what follows, X represents a GE-algebra and $\mathcal{F}(X)$ represents the set of GE-filters in X unless otherwise stated.

Definition 3.1. A *GE-filter expansion* of X is defined to be a self-map ξ on $\mathcal{F}(X)$ that satisfies:

$$(\forall A \in \mathcal{F}(X))(A \subseteq \xi(A)), \tag{3.1}$$

$$(\forall A, B \in \mathcal{F}(X))(A \subseteq B \ \Rightarrow\ \xi(A) \subseteq \xi(B)). \tag{3.2}$$

It is clear that the identity self-map ξ on $\mathcal{F}(X)$ is a GE-filter expansion in X.

Example 3.1. *(1) The constant map $\xi : \mathcal{F}(X) \to \mathcal{F}(X),\ A \mapsto X$, is a GE-filter expansion in X.*

(2) Given a GE-filter A of X, the self-map ξ on $\mathcal{F}(X)$ given as follows:

$$\xi(A) = \begin{cases} X & \text{if } A = X, \\ \bigcap \left\{ M_A \;\middle|\; \begin{array}{l} M_A \text{ is a maximal GE-filter of } X \text{, that is,} \\ \text{it is a proper GE-filter of } X \text{ which is not a} \\ \text{proper subset of any proper GE-filter of } X, \\ \text{which contains } A \end{array} \right\} & \text{if } A \neq X \end{cases}$$

is a GE-filter expansion in X.

(3) Let $X = \{1, a, b, c, d, e\}$ be a set with the binary operation "$*$" in the following Cayley Table:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	1	d	d
b	1	c	$1x$	c	d	d
c	1	b	b	1	d	e
d	1	a	b	c	1	b
e	1	c	1	c	1	1

Then $(X, *, 1)$ is a commutative GE-algebra and the set of all GE-filters of X is

$$\mathcal{F}(X) = \{F_1, F_2, F_3, F_4, F_5, F_6\}$$

where $F_1 = \{1\}$, $F_2 = \{1, b\}$, $F_3 = \{1, d\}$, $F_4 = \{1, a, b, c\}$, $F_5 = \{1, b, d, e\}$ *and* $F_6 = X$. *Let* ξ *be a self-map on* $\mathcal{F}(X)$ *defined by* $\xi(F_1) = \xi(F_2) = F_2$, $\xi(F_3) = \xi(F_5) = F_5$, $\xi(F_4) = F_4$ *and* $\xi(F_6) = F_6$. *It is easy to verify that* ξ *is a GE-filter expansion in* X.

Let us introduce a way to make a GE-filter expansion.

Lemma 3.1. *Every transitive bordered GE-algebra* X *satisfies:*

$$(\forall x, y \in X)((x * y)^{00} \leq x^{00} * y^{00}). \tag{3.3}$$

Proof. Let $x, y \in X$. Then $(x * y)^{00} \leq (x^{00} * y^{00})^{00}$ by (2.11) and (2.12). Using (GE1), (2.6), (2.9) and (2.11), we have

$$\begin{aligned} 1 &= (x^{00} * y^{00}) * (x^{00} * y^{00}) \\ &\leq x^{00} * ((x^{00} * y^{00}) * y^{00}) \\ &\leq x^{00} * (y^{0} * (x^{00} * y^{00})^{0}) \\ &\leq x^{00} * (y^{0} * (x^{00} * y^{00})^{000}) \\ &\leq x^{00} * ((x^{00} * y^{00})^{00} * y^{00}) \\ &\leq (x^{00} * y^{00})^{00} * (x^{00} * y^{00}), \end{aligned}$$

and so $(x^{00} * y^{00})^{00} * (x^{00} * y^{00}) = 1$ by (2.7), that is, $(x^{00} * y^{00})^{00} \leq x^{00} * y^{00}$. Now (3.3) follows from (2.13). □

Theorem 3.1. *Let* X *be a transitive bordered GE-algebra. Define a self-map* ξ *on* $\mathcal{F}(X)$ *as follows:*

$$\xi : \mathcal{F}(X) \to \mathcal{F}(X), \ A \mapsto A^{\diamond} \tag{3.4}$$

where $A^{\diamond} := \{x \in X \mid x^{00} \in A\}$. *Then* ξ *is a GE-filter expansion in* X.

Proof. Let $A, B \in \mathcal{F}(X)$. We first show that $A^{\diamond}$ is a GE-filter of X. Since $1^{00} = 1 \in A$, we have $1 \in A^{\diamond}$. Let $x, y \in X$ be such that $x \in A^{\diamond}$ and $x * y \in A^{\diamond}$. Then $x^{00} \in A$ and $(x * y)^{00} \in A$. Since $(x * y)^{00} \leq x^{00} * y^{00}$ by Lemma 3.1, it follows from Lemma 2.1 that $x^{00} * y^{00} \in A$. Hence $y^{00} \in A$, and so $y \in A^{\diamond}$. Therefore $A^{\diamond}$ is a GE-filter of X. Hence we know that the map ξ is well-defined. Let $x \in A$. Since $x \leq x^{00}$ by (2.9), we have $x^{00} \in A$ by Lemma 2.1, that is, $x \in A^{\diamond}$. Thus $A \subseteq A^{\diamond}$. Assume that

$A \subseteq B$. If $x \in A^\diamond$, then $x^{00} \in A \subseteq B$ and so $x \in B^\diamond$. Hence $\xi(A) = A^\diamond \subseteq B^\diamond = \xi(B)$. Consequently, ξ is a GE-filter expansion in X. □

The following example illustrates Theorem 3.1.

Example 3.2. *Let $X = \{0, 1, a, b, c, d, e\}$ be a set with the binary operation "$*$" in the following Cayley Table.*

$*$	0	1	a	b	c	d	e
0	1	1	1	1	1	1	1
1	0	1	a	b	c	d	e
a	0	1	1	b	e	d	e
b	0	1	1	1	c	d	c
c	d	1	1	1	1	d	1
d	e	1	1	b	e	1	e
e	d	1	1	1	1	d	1

Then $(X, *, 1)$ is a transitive bordered GE-algebra and all GE-filters of X are $F_1 = \{1\}$, $F_2 = \{1, a\}$, $F_3 = \{1, a, b\}$, $F_4 = \{1, a, d\}$, $F_5 = \{1, a, b, d\}$, $F_6 = \{1, a, b, c, e\}$ and $F_7 = X$. Hence $\mathcal{F}(X) = \{F_1, F_2, F_3, F_4, F_5, F_6, F_7\}$. Define a self-map ξ on $\mathcal{F}(X)$ as follows:

$$\xi : \mathcal{F}(X) \to \mathcal{F}(X),\ A \mapsto \begin{cases} F_3 & \text{if } A \in \{F_1, F_2, F_3\} \\ F_5 & \text{if } A \in \{F_4, F_5\}, \\ F_6 & \text{if } A = F_6, \\ F_7 & \text{if } A = F_7, \end{cases}$$

where $F_1^\diamond = F_2^\diamond = F_3^\diamond = F_3$, $F_4^\diamond = F_5^\diamond = F_5$, $F_6^\diamond = F_6$ and $F_7^\diamond = F_7$. It is routine to verify that ξ is a GE-filter expansion in X.

Theorem 3.2. *The intersection of two GE-filter expansions in X is also a GE-filter expansion in X, that is, given two GE-filter expansions α and β in X, the intersection $\xi := \alpha \cap \beta$ on $\mathcal{F}(X)$ given as follows:*

$$\xi := \alpha \cap \beta : \mathcal{F}(X) \to \mathcal{F}(X),\ A \mapsto \alpha(A) \cap \beta(A)$$

is a GE-filter expansion in X.

Proof. Assume that α and β are GE-filter expansions in X. Let $A \in \mathcal{F}(X)$. Then $A \subseteq \alpha(A)$ and $A \subseteq \beta(A)$. Hence $A \subseteq \alpha(A) \cap \beta(A) = \xi(A)$. Let $A, B \in \mathcal{F}(X)$ and $A \subseteq B$. Then $\alpha(A) \subseteq \alpha(B)$ and $\beta(A) \subseteq \beta(B)$. Thus

$$\xi(A) = \alpha(A) \cap \beta(A) \subseteq \alpha(B) \cap \beta(B) = \xi(B).$$

Therefore ξ is a GE-filter expansion in X. □

The union of two GE-filter expansions in X need not be a GE-filter expansion in X, that is, given two GE-filter expansions α and β in X, the union $\eta := \alpha \cup \beta$ on $\mathcal{F}(X)$ given as follows:

$$\eta := \alpha \cup \beta : \mathcal{F}(X) \to \mathcal{F}(X),\ A \mapsto \alpha(A) \cup \beta(A),$$

is not a GE-filter expansion in X.

Example 3.3. *Let $(X, *, 1)$ be a GE-algebra given in Example 3.1(3). Let α be the GE-filter expansion ξ, defined in the Example 3.1(3), and β is the GE-filter expansion defined by*

$$\beta(F_1) = \beta(F_3) = F_3, \beta(F_2) = \beta(F_5) = F_5, \beta(F_4) = \beta(F_6) = F_6.$$

But $\alpha \cup \beta$ is not a GE-filter expansion of X, since

$$(\alpha \cup \beta)(F_1) = \alpha(F_1) \cup \beta(F_1) = F_2 \cup F_3 = \{1, b, d\} \notin \mathcal{F}(X).$$

Definition 3.2. Let X be a GE-algebra and let ξ be a GE-filter expansion in X. Then a GE-filter A of X is said to be

- *first ξ-primary* if it satisfies:

(3.5) $$(\forall x, y \in X)(x \dotplus y \in A,\ x \notin A \ \Rightarrow\ y \in \xi(A))$$

- *second ξ-primary* if it satisfies:

(3.6) $$(\forall x, y \in X)(x \dotplus y \in A,\ y \notin A \ \Rightarrow\ x \in \xi(A))$$

- *ξ-primary* if it is both first ξ-primary and second ξ-primary

where $x \dotplus y := (x * y) * y$ for all $x, y \in X$.

It is clear that if X is commutative, then the notion of first ξ-primary coincides with that of second ξ-primary. In this case, it is only called ξ-primary.

Example 3.4. *Consider the GE-filter expansion ξ in Example 3.1(3). It is routine to verify that $F_4 := \{1, a, b, c\}$, $F_5 = \{1, b, d, e\}$ and $F_6 = X$ are ξ-primary GE-filters of X. But $F_1 = \{1\}$, $F_2 = \{1, b\}$ and $F_3 = \{1, d\}$ are not ξ-primary because of*

$$(a * d) * d = d * d = 1 \in F_1 \text{ and } a \notin F_1 \text{ but } d \notin \xi(F_1) = F_2,$$

$$(a * d) * d = d * d = 1 \in F_2 \text{ and } a \notin F_2 \text{ but } d \notin \xi(F_2) = F_2,$$

$$(b * c) * c = c * c = 1 \in F_3 \text{ and } b \notin F_3 \text{ but } c \notin \xi(F_3) = F_5.$$

Example 3.5. *Let $X = \{1, a, b, c, d, e\}$ be a set with the binary operation "$*$" in the following Cayley Table:*

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	c	1	e
b	1	d	1	c	d	1
c	1	a	1	1	a	1
d	1	1	1	c	1	e
e	1	a	b	c	d	1

Then $(X, *, 1)$ is a GE-algebra and the set of all GE-filters of X is

$$\mathcal{F}(X) = \{F_1, F_2, F_3, F_4, F_5\}$$

where $F_1 = \{1\}$, $F_2 = \{1, b, e\}$, $F_3 = \{1, b, c, e\}$, $F_4 = \{1, a, b, d, e\}$ and $F_5 = X$. Let ξ be a self-map on $\mathcal{F}(X)$ defined by $\xi(F_1) = \xi(F_2) = F_2$, $\xi(F_3) = F_3$, $\xi(F_4) = F_4$ and $\xi(F_5) = F_5$. It is easy to verify that ξ is a GE-filter expansion in X. Also, we can observe that F_3 is a first ξ-primary GE-filter of X and F_4 is a second ξ-primary GE-filter of X.

Theorem 3.3. *Let ξ and η be GE-filter expansions in X that satisfies:*

$$(\forall A \in \mathcal{F}(X))(\xi(A) \subseteq \eta(A)). \tag{3.7}$$

Then every first (resp., second) ξ-primary GE-filter is a first (resp., second) η-primary GE-filter.

Proof. Let B be a first ξ-primary GE-filter of X and let $x, y \in X$ be such that $x \dot{+} y \in B$ and $x \notin B$. Then $y \in \xi(B) \subseteq \eta(B)$ by (3.7). Hence B is a first η-primary

GE-filter of X. Similarly, if B is a second ξ-primary GE-filter, then it is a second η-primary GE-filter. □

Corollary 3.1. *If ξ and η are GE-filter expansions in X that satisfy (3.7), then every ξ-primary GE-filter is an η-primary GE-filter.*

Proposition 3.1. *Let ξ be a GE-filter expansion in X. Then every first ξ-primary GE-filter A of X satisfies:*

$$(\forall F, G \in \mathcal{F}(X))(F \dot{+} G \subseteq A,\ F \nsubseteq A \ \Rightarrow\ G \subseteq \xi(A)) \tag{3.8}$$

where $F \dot{+} G = \{x \dot{+} y \mid x \in F,\ y \in G\}$.

Proof. Let A be a first ξ-primary GE-filter of X and let $F, G \in \mathcal{F}(X)$ be such that $F \dot{+} G \subseteq A$ and $F \nsubseteq A$. If $G \nsubseteq \xi(A)$, then there exist $r \in F \setminus A$ and $s \in G \setminus \xi(A)$, such that $r \dot{+} s \in F \dot{+} G \subseteq A$. But $r \notin A$ and $s \notin \xi(A)$. This contradicts the assumption that A is ξ-primary. Hence A satisfies (3.8). □

Similarly, we have the following result.

Proposition 3.2. *Let ξ be a GE-filter expansion in X. Then every second ξ-primary GE-filter A of X satisfies:*

$$(\forall F, G \in \mathcal{F}(X))(F \dot{+} G \subseteq A,\ G \nsubseteq A \ \Rightarrow\ F \subseteq \xi(A)). \tag{3.9}$$

We provide one way to create a new GE-filter expansion using a given GE-filter expansion.

Theorem 3.4. *Let α be a GE-filter expansion in a commutative GE-algebra X. Define a self-map ξ_α on $\mathcal{F}(X)$ as follows:*

$$\xi_\alpha : \mathcal{F}(X) \to \mathcal{F}(X),\ A \mapsto \bigcap\{B \in \mathcal{F}(X) \mid\ A \subseteq B \text{ and } B \text{ is } \alpha\text{-primary}\}. \tag{3.10}$$

Then ξ_α is a GE-filter expansion in X.

Proof. It is clear that $A \subseteq \xi_\alpha(A)$ for all $A \in \mathcal{F}(X)$. Let $F, G \in \mathcal{F}(X)$ be such that $F \subseteq G$. Then

$$\begin{aligned}\xi_\alpha(F) &= \bigcap\{B \in \mathcal{F}(X) \mid\ F \subseteq B \text{ and } B \text{ is } \alpha\text{-primary}\}\\ &\subseteq \bigcap\{B \in \mathcal{F}(X) \mid\ G \subseteq B \text{ and } B \text{ is } \alpha\text{-primary}\}\\ &= \xi_\alpha(G).\end{aligned}$$

Therefore ξ_α is a GE-filter expansion in X. □

Given a nonempty subset A of a GE-algebra X and an element a in X, consider the following set:

(3.11) $$a^{-1}A := \{x \in X \mid (a * x) * x \in A\}.$$

It is clear that if A and B are nonempty subsets of X and $A \subseteq B$, then $a^{-1}A \subseteq a^{-1}B$ for all $a \in X$.

The following example shows that $a^{-1}A$ is not a GE-filter of X.

Example 3.6. *Let $X = \{1, a, b, c, d, e\}$ be a set with the binary operation "$*$" in the following Cayley Table:*

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	d	b
b	1	1	1	d	d	1
c	1	1	e	1	1	e
d	1	a	b	1	1	b
e	1	a	1	d	d	1

Then $(X, *, 1)$ is a GE-algebra. Let $A = \{1, a, b\}$ and $e \in X$. Then it is easy to see that $e^{-1}A = \{1, a, b, c, d\}$. But $e^{-1}A$ is not a GE-filter of X since $b * e = 1 \in e^{-1}A$ and $b \in e^{-1}A$ but $e \notin e^{-1}A$.

We provide conditions for the set $a^{-1}A$ to be a GE-filter.

Lemma 3.2. *Every commutative GE-algebra X satisfies:*

$$(\forall x, y \in X)(((x * y) * y) * y = x * y). \tag{3.12}$$

$$(\forall a, x, y \in X)(a \dot{+} (x * y) \leq (a \dot{+} x) * (a \dot{+} y)). \tag{3.13}$$

Proof. For every $x, y \in X$, we have $x * y \leq ((x * y) * y) * y$ by (GE1) and (2.14). On the other hand, we get

$$\begin{aligned} 1 &= (x * ((x * y) * y)) * ((((x * y) * y) * y) * (x * y)) \\ &= ((x * y) * (x * y)) * ((((x * y) * y) * y) * (x * y)) \\ &= 1 * ((((x * y) * y) * y) * (x * y)) \\ &= (((x * y) * y) * y) * (x * y) \end{aligned}$$

by (GE1), (GE2), (2.12) and (2.14), that is, $((x * y) * y) * y \leq x * y$. Since every commutative GE-algebra is antisymmetric, it follows that $((x * y) * y) * y = x * y$, i.e., (3.12) is valid. Let $a, x, y \in X$. Using (2.12), (2.14), (2.15) and (3.12), we have

$$\begin{aligned} &(a \dot{+} (x * y)) * ((a \dot{+} x) * (a \dot{+} y)) \\ &= ((x * y) \dot{+} a) * ((x \dot{+} a) * (y \dot{+} a)) \\ &= ((x * y) \dot{+} a) * ((x \dot{+} a) * ((y * a) * a)) \\ &= ((x * y) \dot{+} a) * ((y * a) * ((x \dot{+} a) * a)) \\ &= ((x * y) \dot{+} a) * ((y * a) * (((x * a) * a) * a)) \\ &= ((x * y) \dot{+} a) * ((y * a) * (x * a)) \\ &= (y * a) * (((x * y) \dot{+} a) * (x * a)) \\ &= (y * a) * (x * (((x * y) \dot{+} a) * a)) \\ &= (y * a) * (x * ((((x * y) * a) * a) * a)) \\ &= (y * a) * (x * ((x * y) * a)) \\ &= (y * a) * ((x * y) * (x * a)) \\ &= (x * y) * ((y * a) * (x * a)) = 1, \end{aligned}$$

which shows that (3.13) is valid. □

Theorem 3.5. *If A is a GE-filter of a commutative GE-algebra X, then $a^{-1}A$ is a GE-filter of X which contains A.*

Proof. Assume that A is a GE-filter of X. It is clear that $1 \in a^{-1}A$. Let $x, y \in X$ be such that $x * y \in a^{-1}A$ and $x \in a^{-1}A$. Then $a \dotplus (x * y) \in A$ and $a \dotplus x \in A$. Hence $(a \dotplus x) * (a \dotplus y) \in A$ by Lemma 2.1 and (3.13), which implies from (2.17) that $a \dotplus y \in A$, i.e., $y \in a^{-1}A$. Therefore $a^{-1}A$ is a GE-filter of X. Let $y \in A$. Since $y \leq (a * y) * y = a \dotplus y$ by (2.8), it follows from Lemma 2.1 that $a \dotplus y \in A$, that is, $y \in a^{-1}A$. Thus A is contained in $a^{-1}A$, and the proof is complete. □

Proposition 3.3. *Given a nonempty subset A of a GE-algebra X and an element a in X, we have*

(1) *If $a^{-1}A = X$, then $a \in A$.*
(2) *If A is a GE-filter of X, then*

$$a \in A \;\Rightarrow\; a^{-1}A = X. \tag{3.14}$$

(3) *If X is commutative and A is a GE-filter of X, then*

$$(\forall x, y \in X)(x \leq y \;\Rightarrow\; x^{-1}A \subseteq y^{-1}A). \tag{3.15}$$

Proof. (1) If $a^{-1}A = X$, then $a \in a^{-1}A$ and so $a = 1 * a = (a * a) * a \in A$ by (GE1) and (GE2).

(2) Assume that A is a GE-filter of X and let $a \in A$. It is clear that $a^{-1}A \subseteq X$. If $x \in X$, then $a \leq (a * x) * x$ by (2.9) and so $(a * x) * x \in A$ by Lemma 2.1, that is, $x \in a^{-1}A$. Thus $a^{-1}A = X$.

(3) Assume that X is commutative and A is a GE-filter of X. Now let $x, y \in X$ be such that $x \leq y$. Then $y * a \leq x * a$, and so

$$x \dotplus a = (x * a) * a \leq (y * a) * a = y \dotplus a \tag{3.16}$$

for all $a \in X$ by (2.11). If $z \in x^{-1}A$, then $x \dotplus z \in A$ which implies from Lemma 2.1 and (3.16) that $y \dotplus z \in A$, that is, $z \in y^{-1}A$. Hence $x^{-1}A \subseteq y^{-1}A$. □

Theorem 3.6. *Let X be a commutative GE-algebra. Given an element a of X, the self-map ξ_a on $\mathcal{F}(X)$ defined as follows:*

$$\xi_a : \mathcal{F}(X) \to \mathcal{F}(X),\ A \mapsto a^{-1}A \tag{3.17}$$

is a GE-filter expansion in X.

Proof. It is derived from Theorem 3.5. □

Lemma 3.3. *Every commutative GE-algebra X satisfies:*

$$(\forall x, y, z \in X)(x \leq y \ \Rightarrow\ x \dotplus z \leq y \dotplus z), \tag{3.18}$$

$$(\forall x, y, z \in X)(x \leq z,\ y \leq z \ \Rightarrow\ x \dotplus y \leq z). \tag{3.19}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $y * z \leq x * z$ by (2.11), and hence $x \dotplus z = (x*z)*z \leq (y*z)*z = y \dotplus z$ for all $x \in X$. This proves (3.18). Let $x, y, z \in X$ be such that $x \leq z$ and $y \leq z$. Then $x \dotplus z \leq z \dotplus z = z$ and $x \dotplus y = y \dotplus x \leq z \dotplus x = x \dotplus z$. Hence $x \dotplus y \leq z$ since X is commutative and hence transitive. Therefore (3.19) is valid. □

Lemma 3.4. *If X is a commutative GE-algebra, then $(X, \dotplus)$ is a semigroup.*

Proof. Let $x, y, z \in X$. Then $x * (x \dotplus y) = 1$ and $(x \dotplus y) * ((x \dotplus y) \dotplus z) = 1$ by (2.9). Hence $x \leq (x \dotplus y)$ and $x \dotplus y \leq (x \dotplus y) \dotplus z$. Since X is commutative, we get $x \leq (x \dotplus y) \dotplus z$. Since, by (2.8), $y*(x \dotplus y) = 1$, we get $(y \dotplus z)*((x \dotplus y) \dotplus z) = 1$ by (3.18). Thus $y \dotplus z \leq (x \dotplus y) \dotplus z$ which implies from Lemma 3.3 that $x \dotplus (y \dotplus z) \leq (x \dotplus y) \dotplus z$. Similarly, we can show that $(x \dotplus y) \dotplus z \leq x \dotplus (y \dotplus z)$. Since X is commutative, we get $x \dotplus (y \dotplus z) = (x \dotplus y) \dotplus z$. Therefore $(X, \dotplus)$ is a semigroup. □

Theorem 3.7. *Let ξ be a GE-filter expansion in a commutative GE-algebra X and let B be a ξ-primary GE-filter of X. If A is a GE-filter of X, then the set $\bigcap_{a \in A} a^{-1}B$ is a ξ-primary GE-filter of X. Also, if F is a GE-filter of X which is not contained in $\xi(B)$, then $B = \bigcap_{x \in F} x^{-1}B$.*

Proof. Let A be a GE-filter of X. Then $\bigcap_{a \in A} a^{-1}B$ is a GE-filter of X which contains B by Theorem 3.5. Assume that B is ξ-primary and let $x, y \in X$ be such that

$x \dot{+} y \in \bigcap_{a\in A} a^{-1}B$ and $x \notin \bigcap_{a\in A} a^{-1}B$. Then $a \dot{+} x \notin B$ for some $a \in A$ and so $(a \dot{+} x) \dot{+} y = a \dot{+} (x \dot{+} y) \in B$ by Lemma 3.4. Since B is ξ-primary and $a \dot{+} x \notin B$, we have $y \in \xi(B)$. Hence $y \in \xi(B) \subseteq \xi\left(\bigcap_{a\in A} a^{-1}B\right)$. Therefore $\bigcap_{a\in A} a^{-1}B$ is a ξ-primary GE-filter of X. Now, assume that F is a GE-filter of X which is not contained in $\xi(B)$. It is clear that $B \subseteq \bigcap_{x\in F} x^{-1}B$ by Theorem 3.5. Let $z \in F \dot{+} \bigcap_{x\in F} x^{-1}B$. Then $z = a \dot{+} y$ for some $a \in F$ and $y \in \bigcap_{x\in F} x^{-1}B$. It follows that $z = a \dot{+} y \in B$. Hence

$$F \dot{+} \bigcap_{x\in F} x^{-1}B \subseteq B.$$

Since $F \nsubseteq \xi(B)$, we have $\bigcap_{x\in F} x^{-1}B \subseteq B$ by Proposition 3.2. Therefore $B = \bigcap_{x\in F} x^{-1}B$. □

Lemma 3.5 ([16, 17]). *Given a GE-morphism $f : X \to Y$, we have*

(1) *If G is a GE-filter of Y, then $f^{-1}(G)$ is a GE-filter of X.*

(2) *If f is onto and F is a GE-filter of X, then $f(F)$ is a GE-filter of Y.*

Definition 3.3. Let X be a GE-algebra and $f : X \to X$ a GE-endomorphism. A GE-filter expansion ξ in X is called *f-good* (or good with respect to f) if

$$(\forall A \in \mathcal{F}(X))(\xi(f^{-1}(A)) = f^{-1}(\xi(A))). \tag{3.20}$$

Example 3.7. *(1) The identity self-map ξ on $\mathcal{F}(X)$ is a good GE-filter expansion in X.*

(2) Let $X = \{1, a, b, c, d, e\}$ be a set with the binary operation "$*$" in the following Cayley Table:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	d	d	1
b	1	1	1	c	c	e
c	1	b	b	1	1	1
d	1	a	b	1	1	e
e	1	a	b	c	c	1

Then $(X, *, 1)$ is a GE-algebra and the set of all GE-filters of X is

$$\mathcal{F}(X) = \{F_1, F_2, F_3, F_4, F_5\}$$

where $F_1 = \{1\}$, $F_2 = \{1, e\}$, $F_3 = \{1, c, d, e\}$, $F_4 = \{1, a, b, e\}$ *and* $F_5 = X$. *Define a self-map* $f : X \to X$ *defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \in \{1, e\}, \\ a & \text{if } x \in \{a, b\}, \\ d & \text{if } x \in \{c, d\}. \end{cases}$$

Then we can observe that f *is a GE-endomorphism. Also,* $f^{-1}(F_1) = f^{-1}(F_2) = F_2$, $f^{-1}(F_3) = F_3$, $f^{-1}(F_4) = F_4$ *and* $f^{-1}(F_5) = F_5$. *Let* ξ *be a self-map on* $\mathcal{F}(X)$ *defined by* $\xi(F_1) = \xi(F_2) = F_2$, $\xi(F_3) = F_3$, $\xi(F_4) = F_4$ *and* $\xi(F_5) = F_5$. *It is routine to verify that* ξ *is a* f*-good GE-filter expansion in* X.

Theorem 3.8. *Let* $f : X \to X$ *be a GE-endomorphism and* ξ *an* f*-good GE-filter expansion in* X. *If* A *is a first (resp., second)* ξ*-primary GE-filter of* X, *then so is* $f^{-1}(A)$.

Proof. Assume that ξ is a f-good GE-filter expansion in X. If A is a first ξ-primary GE-filter of X, then A is a GE-filter of X and so $f^{-1}(A)$ is a GE-filter of X by Lemma 3.5(1). Let $x, y \in X$ be such that $x \dot{+} y \in f^{-1}(A)$ and $x \notin f^{-1}(A)$. Then $f(x) \notin A$ and

$$\begin{aligned} f(x) \dot{+} f(y) &= (f(x) * f(y)) * f(y) \\ &= f(x * y) * f(y) \\ &= f((x * y) * y) \\ &= f(x \dot{+} y) \in f(f^{-1}(A)) \subseteq A. \end{aligned}$$

Since A is a first ξ-primary, it follows that $f(y) \in \xi(A)$. Hence $y \in f^{-1}(\xi(A)) = \xi(f^{-1}(A))$, and therefore $f^{-1}(A)$ is a first ξ-primary GE-filter of X. Similarly, if A is a second ξ-primary GE-filter of X, then $f^{-1}(A)$ is a second ξ-primary GE-filter of X. □

Corollary 3.2. *Let* $f : X \to X$ *be a GE-endomorphism and* ξ *an* f*-good GE-filter expansion in a commutative GE-algebra* X. *If* A *is a* ξ*-primary GE-filter of* X, *then so is* $f^{-1}(A)$.

Lemma 3.6. *Let $f : X \to X$ be a GE-endomorphism. If A is a GE-filter of X that contains the kernel of f, then $f^{-1}(f(A)) = A$.*

Proof. It is sufficient to show that $f^{-1}(f(A)) \subseteq A$. If $x \in f^{-1}(f(A))$, then $f(x) \in f(A)$ and so $f(x) = f(y)$ for some $y \in A$. Hence $f(y * x) = f(y) * f(x) = 1$, which implies that $y * x \in \ker(f) \subseteq A$. Thus $x \in A$. This shows that $f^{-1}(f(A)) \subseteq A$, and the proof is complete. □

Theorem 3.9. *Let $f : X \to X$ be a GE-endomorphism and ξ an f-good GE-filter expansion in X. Let A be a GE-filter of X that contains the kernel of f.*

(1) *If $f(A)$ is a first (resp., second) ξ-primary GE-filter of X, then so is A.*

(2) *If f is a one-to-one and onto GE-morphism, that is, f is a GE-automorphism, and A is a first (resp., second) ξ-primary GE-filter of X, then $f(A)$ is also a first (resp., second) ξ-primary GE-filter of X.*

Proof. (1) Assume that $f(A)$ is a first (resp., second) ξ-primary GE-filter of X. Then $A = f^{-1}(f(A))$ is a first (resp., second) ξ-primary GE-filter of X by Theorem 3.8 and Lemma 3.6.

(2) Assume that f is a one-to-one and onto GE-morphism and let A be a first ξ-primary GE-filter of X. Then $f(A)$ is a GE-filter of X by Lemma 3.5. Using (3.20) and Lemma 3.6, we have

$$\xi(A) = \xi(f^{-1}(f(A))) = f^{-1}(\xi(f(A))),$$

and so

$$f(\xi(A)) = f(f^{-1}(\xi(f(A)))) = \xi(f(A)).$$

Let $x, y \in X$ be such that $x \dotplus y \in f(A)$ and $x \notin f(A)$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. Hence

$$f(a \dotplus b) = f(a) \dotplus f(b) = x \dotplus y \in f(A),$$

and so $a \dotplus b \in f^{-1}(f(A)) = A$. Since $f(a) = x \notin f(A)$, we have $a \notin A$. Since A is first ξ-primary, we have $b \in \xi(A)$ and so $y = f(b) \in f(\xi(A)) = \xi(f(A))$. Therefore $f(A)$ is a first ξ-primary GE-filter of X. Similarly, if A is a second ξ-primary GE-filter of X, then $f(A)$ is also a second ξ-primary GE-filter of X. □

4. Conclusion

We have introduced the notions of GE-filter expansion and ξ-primary GE-filter and investigated their properties. We have provided different ways to create a GE-filter expansion. We have introduced the concept of good GE-filter expansion and investigated its properties. We have provided the conditions for an image and an inverse image of ξ-primary GE-filter of a GE-algebra to be a ξ-primary GE-filter.

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(1) Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

Email address: skywine@gmail.com

(2)Department of Mathematics, GITAM (Deemed to be University), Hyderabad Campus, Telangana-502329, India

Email address: ravimaths83@gmail.com