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STABLE AND UNSTABLE MANIFOLDS OF THE TWO-DIMENSIONAL PIECEWISE-LINEAR NORMAL FORM MAPS.

ABDELLAH MENASRI

Abstract. The stable and unstable sets or stable and unstable manifolds give a formal mathematical definition to the general notions embodied in the idea of an attractor. In this paper, we will determine the stable and unstable manifolds of the normal form of two-dimensional piecewise-linear maps in the neighborhood of a fixed point at the border using stable manifold theorem, this is an important result about the structure of the set of orbits approaching to a hyperbolic fixed point.

1. Introduction

Stable and unstable manifolds are most easily introduced in the context of a saddle fixed point of a two dimensional maps. They are the natural extensions of the linear eigenvectors of the stability analysis of the fixed point into the nonlinear regime. Many papers have been published in this field, for example, in [6] they studied the stable and unstable manifolds of unstable periodic orbits as well as the Hamiltonian chaos generated by the dynamics of passive tracers moving in a two-dimensional fluid flow and describe the complex structure formed in a chaotic layer that separates a vortex region of the shear flow. In [16] a numerical procedure is described to compute the successive images of a curve under an R^N diffeomorphism. Given a tolerance ε, they showed how to rigorously guarantee that each point of the calculated curve is not more than a distance ε from the "true" image curve. they applied the method to compute one-dimensional stable and unstable manifolds of Hénon's and Ikeda's maps,

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as well as a Poincaré map for the forced-damped pendulum. In [1] two precise and fast algorithms have been developed for the computation of stable and unstable manifolds of hyperbolic trajectories of two-dimensional, aperiodically time-dependent vector fields. These methods are then applied to the computation of stable and unstable manifolds of hyperbolic trajectories of several temporally aperiodic variables variants of Duffing's equation.

The piecewise-linear non-reversible on both sides[8] or on one side [9] has been studied in a number of papers in the literature, for example, in [9] an important theorem gives the conditions of the existence of a robust chaotic attractor for the normal form of non-reversible maps on one side.

In this paper, we will determine the stable and unstable manifolds of the normal form of the two-dimensional piecewise-linear maps in the neighborhood of a fixed point on the border. We will use the idea in [9] to give a new theorem of the exisitence of a robust chaotic attractor for two-dimensional piecewise-linear maps reversible on both sides.

The normal form of the two-dimensional piecewise-linear maps at the neighborhood of a fixed point on the border [12, 15] is given by the form:

$$f_\mu(x,y) = \begin{cases} \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & x \leq 0 \\ \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & x > 0 \end{cases} \tag{1.1}$$

Where μ is a bifurcation parameter and τ_L, τ_R, δ_L, δ_R are the traces and determinants for the two matrices (1.2), (1.3).

$$J_L = \begin{pmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{pmatrix} \tag{1.2}$$

$$J_R = \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}, \tag{1.3}$$

evaluated to fixed points. System (1.1) has two fixed points $P_L(\frac{\mu}{1-\tau_L+\delta_L}, \frac{\mu\delta_L}{1-\tau_L+\delta_L})$ and $P_R(\frac{\mu}{1-\tau_R+\delta_R}, \frac{\mu\delta_R}{1-\tau_R+\delta_R})$ in two subregions,

$$R_L := \left\{(x, y) \in \mathbb{R}^2 : x \leq 0,\, y \in \mathbb{R}\right\} \tag{1.4}$$

$$R_R := \left\{(x, y) \in \mathbb{R}^2 : x > 0,\, y \in \mathbb{R}\right\} \tag{1.5}$$

Therefore, P_L and P_R exist for $\frac{\mu}{1-\tau_L+\delta_L} \leq 0$, $\frac{\mu}{1-\tau_R+\delta_R} > 0$ respectively. The two eigenvalues of Jacobian matrices (1.2), (1.3) in (1.4), (1.5) respectively are $\lambda_{1L,2L} = \frac{1}{2}\left(\tau_L \pm \sqrt{\tau_L^2 - 4\delta_L}\right)$, $\lambda_{1R,2R} = \frac{1}{2}\left(\tau_R \pm \sqrt{\tau_R^2 - 4\delta_R}\right)$. We will study the case where the map (1.1) is reversible on both sides, i.e., δ_L and δ_R are nonzero. Hence, the inverse of map (1.1) is defined by the form (1.6),

$$f_\mu^{-1}(x, y) = \begin{cases} \begin{pmatrix} 0 & -\frac{1}{\delta_L} \\ 1 & \frac{\tau_L}{\delta_L} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & x \leq 0 \\ \begin{pmatrix} 0 & -\frac{1}{\delta_R} \\ 1 & \frac{\tau_R}{\delta_R} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & x > 0 \end{cases} \tag{1.6}$$

The stable (unstable) manifold of P_L and P_R are defined by:

$$\begin{cases} W_L^s(P_L) := \{(x, y) \in R_L : f^n(x, y) \to P_L,\, n \to \infty\ \} \\ W_R^s(P_R) := \{(x, y) \in R_R : f^n(x, y) \to P_R,\, n \to \infty\} \\ W_L^u(P_L) := \{(x, y) \in R_L : f^{-n}(x, y) \to P_L,\, n \to \infty\} \\ W_R^u(P_R) := \{(x, y) \in R_R : f^{-n}(x, y) \to P_R,\, n \to \infty\} \end{cases} \tag{1.7}$$

2. The stable and unstable manifolds for the system (1.1)

To determine the stable and unstable manifolds [16] for system (1.1), we must first mention the case of the stability and instability of the fixed points P_L and P_R. The stability of each of them is determined by eigenvalues $\lambda_{1L,2L}$, $\lambda_{1R,2R}$ and to $\delta_{L,R}$. We will consider the case where the system (1.1) is dessipative, i.e., $|\delta_L| < 1$ and $|\delta_R| < 1$. We have the following cases:

Case.1: For $\delta_{L,R} < \frac{\tau_{L,R}^2}{4}$, the eigenvalues $\lambda_{1L,2L}$, $\lambda_{1R,2R}$ are real.

1.: if $2\sqrt{\delta_{L,R}} < \tau_{L,R} < (1 + \delta_{L,R})$ then, $0 < \lambda_{1L,2L} < 1$ and then, $0 < \lambda_{1R,2R} < 1$.
The fixed points are a regular attractor.

2.: if $-(1+\delta_{L,R}) < \tau_{L,R} < -2\sqrt{\delta_{L,R}}$ then, $\lambda_{1L,2L} < 0$ and then, $\lambda_{1R,2R} < 0$. The fixed points are a flip attractor.

3.: if $\tau_{L,R} > (1+\delta_{L,R})$ then, $\lambda_{1L,1R} > 1$ and then, $0 < \lambda_{2L,2R} < 1$. The fixed points are a regular saddle.

4.: if $\tau_{L,R} < -(1+\delta_{L,R})$ then, $\lambda_{2L,2R} < -1$ and then, $-1 < \lambda_{1L,1R} < 0$. The fixed points are a flip saddle.

Case.2: For $\delta_{L,R} < 0$, the eigenvalues are always real and spiralling orbits cannot exist, thus there can be only two types of fixed points.

1.: if $-(1+\delta_{L,R}) < \tau_{L,R} < (1+\delta_{L,R})$ then $0 < \lambda_{1L,2L} < 1$ and $-1 < \lambda_{1R,2R} < 0$, or $-1 < \lambda_{1L,2L} < 0$ and $0 < \lambda_{1R,2R} < 1$ which means that the fixed points are a flip attractor.

2.: if $\tau_{L,R} > (1+\delta_{L,R})$ then, $\lambda_{1L,1R} > 1$ and then, $-1 < \lambda_{2L,2R} < 0$, the fixed points are a flip saddle.

3.: if $\tau_{L,R} < -(1+\delta_{L,R})$ then, $\lambda_{2L,2R} < -1$ and then, $0 < \lambda_{1L,1R} < 1$, the fixed points are again a flip saddle.

Case.3: For $\delta_{L,R} > \frac{\tau_L^2}{4}$, the eigenvalues $\lambda_{1L,2L}$, $\lambda_{1R,2R}$ are complex, the fixed points are spirally attracting.

1.: if $\tau_{L,R} > 0$, the fixed points are a clockwise spiral.

2.: if $\tau_{L,R} < 0$, the fixed points are a spiralling motion is counter-clockwise. We note that if, eigenvalues are real, invariant manifolds of the fixed points exist and play an important role in deciding the system dynamics [12].

Lemma 2.1. *the system (1.1) is stable, if it is stable on the two subregions R_L and R_R, and unstable, if it is unstable in one of the two regions R_L, R_R.*

We can write the system (1.1) in the form (2.1),

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{cases} \begin{pmatrix} \tau_L x_n + y_n + \mu \\ -\delta_L x_n \end{pmatrix}, & x_n \leq 0 \\ \begin{pmatrix} \tau_R x_n + y_n + \mu \\ -\delta_R x_n \end{pmatrix}, & x_n > 0 \end{cases} \tag{2.1}$$

Accordingly, we consider the system equations (2.2) on subregion R_L:

$$
\begin{cases} x_{n+1} = \tau_L x_n + y_n + \mu, & x_n \leq 0 \\ y_{n+1} = -\delta_L x_n \end{cases} \tag{2.2}
$$

After, substituting the second equation into the first equation, we obtain the equation (2.3):

$$
x_{n+2} - \tau_L x_{n+1} + \delta_L x_n - \mu = 0, \quad x_n \leq 0 \tag{2.3}
$$

So the system (2.2) is equivalent to the system (1.1) on the subregion R_L:

$$
\begin{cases} x_{n+2} - \tau_L x_{n+1} + \delta_L x_n - \mu = 0, & x_n \leq 0 \\ y_{n+1} = -\delta_L x_n \end{cases} \tag{2.4}
$$

Remark 1. *With the same method we can show that, the system (2.5) equivalent to the system (1.1) on the subregion R_R,*

$$
\begin{cases} x_{n+2} - \tau_R x_{n+1} + \delta_R x_n - \mu = 0, & x_n > 0 \\ y_{n+1} = -\delta_R x_n \end{cases} \tag{2.5}
$$

To find stable manifolds of system (1.1) on subregion R_L, we must first search all $(x_n, y_n)_{n\in\mathbb{N}}$, which converge and verify two equation of system (2.2). We can easily demonstrate that the solution of system (2.2) is:

$$
\begin{cases} x_n = q_1 r_{1L}^n + q_2 r_{2L}^n + \frac{\mu}{1-\tau_L+\delta_L}, \; x_n \leq 0 \\ y_n = -\delta_L q_1 r_{1L}^{n-1} - \delta_L q_2 r_{2L}^{n-1} - \frac{\delta_L \mu}{1-\tau_L+\delta_L} \end{cases} , \tau_L \neq 1 + \delta_L \tag{2.6}
$$

where q_1, q_2 are constants and r_{1L}, r_{2L} are roots of quadratic equation (2.7),

$$
r_L^2 - \tau_L r + \delta_L = 0 \tag{2.7}
$$

Therefore the stability and unstability of system (2.2) on the subregion R_L depends on r_L.We note that, if r_{1L}, r_{2L} are complex, the system solution of (2.2) will be in the form (2.8),

$$
\begin{cases} x_n = q r^n(\cos(n\theta) + \sin(n\theta)) + \frac{\mu}{1-\tau_L+\delta_L}, \; x_n \leq 0 \\ y_n = -\delta_L q r^{n-1}(\cos((n-1)\theta) + \sin((n-1)\theta)) - \frac{\delta_L \mu}{1-\tau_L+\delta_L} \end{cases} , \tau_L \neq 1 + \delta_L \tag{2.8}
$$

where $q = |r_L|$ and $\theta = \arg(r_L)$. We have the following cases:

Case.1: $\delta_L < \frac{\tau_L^2}{4}$.

The equation (2.7) has two real solutions,

$$r_{1L,\,2L} = \lambda_{1L,2L} = \frac{1}{2}\left(\tau_L \pm \sqrt{\tau_L^2 - 4\delta_L}\right) \tag{2.9}$$

Hence, we have

$|\lambda_{1L}| = \frac{1}{2}\left|\tau_L + \sqrt{\tau_L^2 - 4\delta_L}\right| < 1$, if $\tau_L - 1 < \delta_L < \min(-(\tau_L+1), \frac{\tau_L^2}{4}), \tau_L < -2$, $\tau_L - 1 < \delta_L < \frac{\tau_L^2}{4}, -2 \leq \tau_L < 2$.

$|\lambda_{2L}| = \frac{1}{2}\left|\tau_L - \sqrt{\tau_L^2 - 4\delta_L}\right| < 1$, if $-\tau_L - 1 < \delta_L < \min(\tau_L - 1, \frac{\tau_L^2}{4})$, $\tau_L > 2$, $-\tau_L - 1 < \delta_L < \frac{\tau_L^2}{4}, -2 \leq \tau_L \leq 2$.

Remark 2. *We have the same cases on the subregion R_R. Therefore stable manifolds of the system (1.1) in the two subregion R_L, R_R exist when, $0 < \delta_{L,R} < \frac{\tau_{L,R}^2}{4}, -2 \leq \tau_{L,R} < 2$. And their form is given by:*

- For $\mu > 0$

1.: $W_L^s(P_L) = \{(x,y) \in R_L : y = 0, \ -\mu \leq x \leq 0\}$, $W_R^s(P_R) =$
: $\{(x,y) \in R_R : y = 0, \ 0 < x < \mu\}$, As shown in Figure (1).
2.: $W_L^s(P_L) = \left\{(x,y) \in R_L : y = \frac{\delta_L}{\lambda_{1L}}x, \ -\mu \leq x \leq 0\right\}$, $W_R^s(P_R) =$
: $\left\{(x,y) \in R_R : y = -\frac{\delta_R}{\lambda_{1R}}x, 0 < x < \mu\right\}$, as shown in Figure (2).

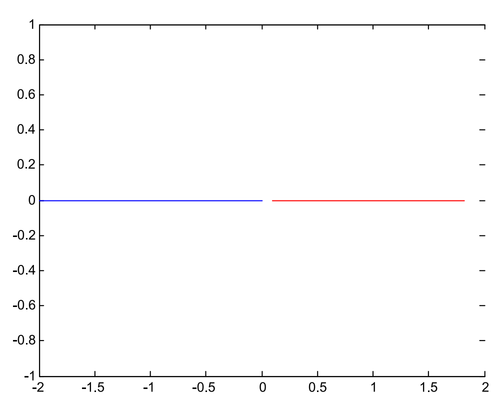

FIGURE 1. The stable manifold in the two region R_L and R_R for $\mu > 0$, $0 < \delta_{L,R} < \frac{\tau_{L,R}^2}{4}, -2 \leq \tau_{L,R} < 2$.

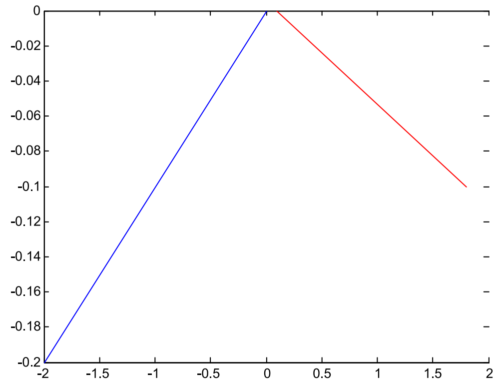

FIGURE 2. The stable manifold in the two region R_L and R_R for $\mu < 0$. $0 < \delta_{L,R} < \frac{\tau_{L,R}^2}{4}$, $-2 \leq \tau_{L,R} < 2$.

- For $\mu < 0$

$W_L^s(P_L) = \{(x, y) \in R_L : y = 0,\ \mu \leq x \leq 0\}$, $W_R^s(P_R) = \left\{(x, y) \in R_R : y = \frac{\delta_R}{\lambda_{1R}}x,\ 0 < x < \mu\right\}$, as shown in Figure (3).

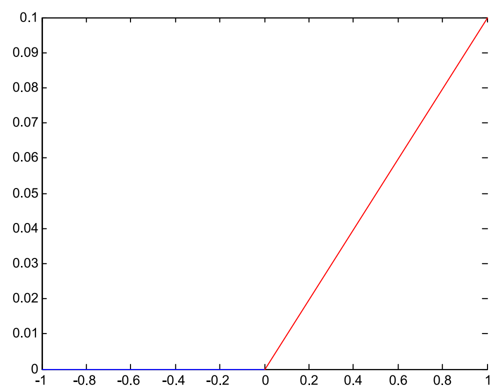

FIGURE 3. The stable manifold in the two region R_L and R_R for $\mu < 0$, $0 < \delta_{L,R} < \frac{\tau_{L,R}^2}{4}$, $-2 \leq \tau_{L,R} < 2$.

- $|\lambda_{1L}| = \frac{1}{2}\left|\tau_L + \sqrt{\tau_L^2 - 4\delta_L}\right| > 1$, if

1.: $\delta_L < \frac{\tau_L^2}{4}$, $\tau_L \geq 2$.

2.: $\delta_L < \min(\tau_L - 1,\ \frac{\tau_L^2}{4})$, $\tau_L \leq 2$.

3.: $-(\tau_L + 1) < \delta_L < \frac{\tau_L^2}{4}, \tau_L \leq -2$.

  - $|\lambda_{2L}| = \frac{1}{2}\left|\tau_L - \sqrt{\tau_L^2 - 4\delta_L}\right| > 1$, if

1.: $\tau_L < -2$, $\delta_L < \frac{\tau_L^2}{4}$.

2.: $\delta_L < \min(-(\tau_L+1), \frac{\tau_L^2}{4}),\ \tau_L \geq -2.$

3.: $\tau_L - 1 < \delta_L < \frac{\tau_L^2}{4}, \tau_L \geq 2.$

Remark 3. *We have the same cases on the subregion R_R. Hence, the unstable manifolds of the system (1.1) in the two subregions R_L, R_R exist in previous cases, and their form is given by:*

- For $\mu < 0$

1.: $W_L^u(P_L) = \{(x,y) \in R_L : y = 0,\ \mu \leq x \leq 0\}$. as shown in Figure (4).

2.: $W_L^u(P_L) = \{(x,y) \in R_L : y = 0,\ \mu \leq x \leq 0\}$, $W_R^u(P_R) =$
: $\{(x,y) \in R_R : y = 0,\ 0 < x < -2\mu\}$, as shown in Figure (5).

3.: $W_L^u(P_L) = \{(x,y) \in R_L : y = 0,\ \mu \leq x \leq 0\}$, $W_R^u(P_R) =$
: $\left\{(x,y) \in R_R : y = \frac{\delta_R}{\lambda_{2R}}x,\ 0 < x < -2\mu\right\}$, as shown in Figure (6).

4.: $W_L^u(P_L) = \{(x,y) \in R_L : y = 0,\ 3\mu \leq x \leq 0\}$, $W_R^u(P_R) =$
: $\{(x,y) \in R_R : y = 0,\ 0 < x < -2\mu\}$, as shown in Figure (7).

5.: $W_L^u(P_L) = \{(x,y) \in R_L : y = 0,\ 3\mu \leq x \leq 0\}$, $W_R^u(P_R) =$
: $\left\{(x,y) \in R_R : y = \frac{\delta_R}{\lambda_{2R}}x,\ 0 < x < -2\mu\right\}$, as shown in Figures (8), (9).

- For $\mu > 0$

1.: $W_R^u(P_R) = \{(x,y) \in R_R : y = 0,\ 0 \leq x \leq \mu\} \cup$
: $\left\{(x,y) \in R_R : y = -\frac{\delta_R}{\lambda_{2R}}x + \frac{\delta_R}{\lambda_{2R}},\ \mu < x < 4\mu\right\}$, as shown in Figure (10).

2.: $W_R^u(P_R) = \{(x,y) \in R_R : y = 0,\ 0 < x < \mu\ \}$, as shown in Figure (11).

3.: $W_L^u(P_L) = \{(x,y) \in R_L : y = x,\ -2\mu \leq x \leq 0\}$, $W_R^u(P_R) =$
: $\{(x,y) \in R_R : y = 0,\ 0 < x < \mu\}$, as shown in Figure (12).

4.: $W_R^u(P_R) = \{(x,y) \in R_R : y = 0,\ 0 < x < \mu\} \cup$
: $\left\{(x,y) \in R_R : y = -\frac{\delta_R}{\lambda_{2R}}x + \frac{\delta_R}{\lambda_{2R}},\ \mu < x < 4\mu\right\}$, as shown in Figure (13).

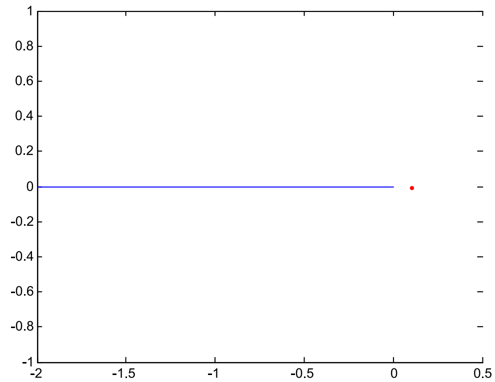

FIGURE 4. The unstable manifolds for the system for $\mu < 0$, $\tau_L + 1 < \delta_L < \frac{\tau_L^2}{4}, \tau_L \leq -2$ and $\tau_L - 1 < \delta_L < \frac{\tau_L^2}{4}, \tau_L \geq 2$.

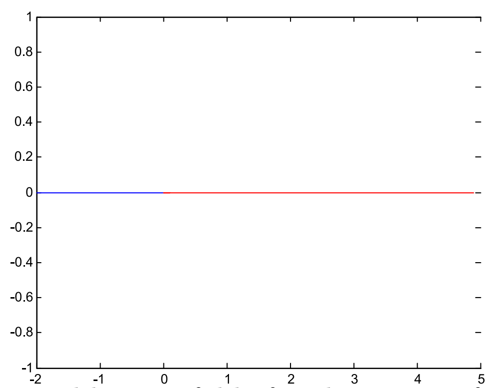

FIGURE 5. The unstable manifolds for the system for $\mu < 0, \tau_L - 1 < \delta_L < \frac{\tau_L^2}{4}, \tau_L \geq 2$ and $\tau_R < -2$, $\delta_R < \frac{\tau_R^2}{4}$.

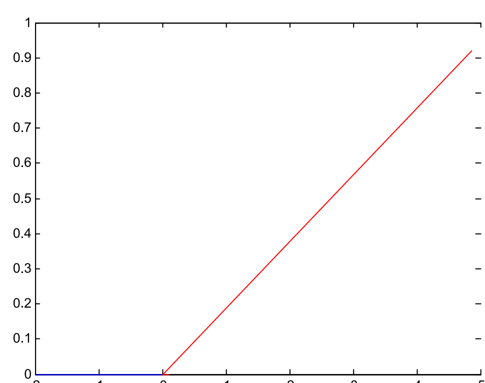

FIGURE 6. The unstable manifolds for the system for $\mu < 0, \tau_L - 1 < \delta_L < \frac{\tau_L^2}{4}, \tau_L \geq 2$ and $\tau_R < -2$, $\delta_R < \frac{\tau_R^2}{4}$.

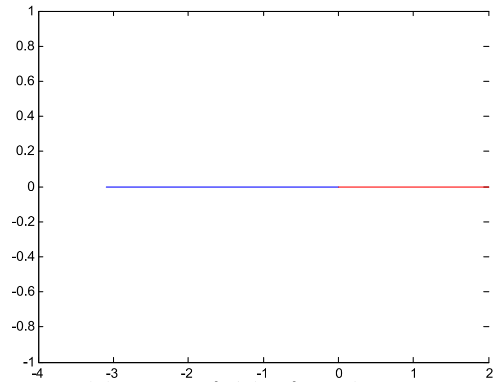

FIGURE 7. The unstable manifolds for the system for $\mu < 0, \delta_L < \frac{\tau_L^2}{4}, \tau_L > 2$ and $\tau_R < -2,$ $\delta_R < \frac{\tau_R^2}{4}$.

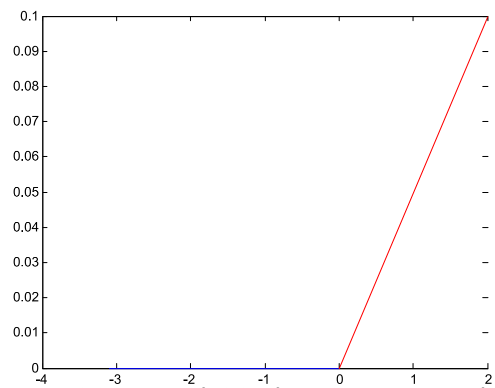

FIGURE 8. The unstable manifolds for the system for $\mu < 0, \tau_L - 1 < \delta_L < \frac{\tau_L^2}{4}, \tau_L \geq 2$ and $\tau_R < -2,\ \delta_R < \frac{\tau_R^2}{4}$.

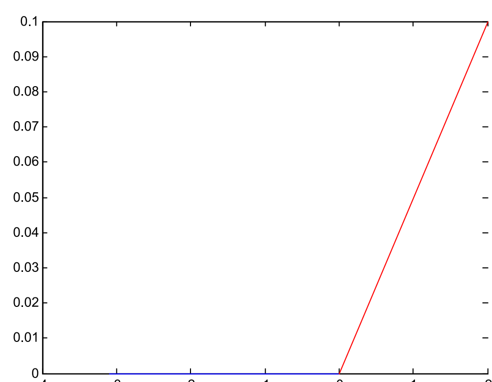

FIGURE 9. The unstable minifolds for $\mu < 0,\ \delta_L < \frac{\tau_L^2}{4}, \tau_L > 2$ and $\delta_L < \frac{\tau_L^2}{4}, \tau_L > 2$ and $\tau_R < -2, \delta_R < \frac{\tau_R^2}{4}$.

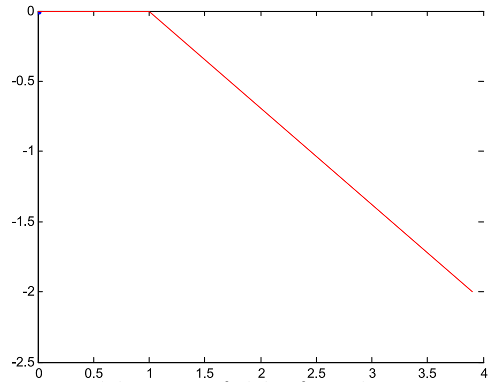

FIGURE 10. The unstable manifolds for the system for $\mu > 0, \delta_L < \frac{\tau_L^2}{4}, \tau_L > 2$ and $\tau_R < -2$, $\delta_R < \frac{\tau_R^2}{4}$.

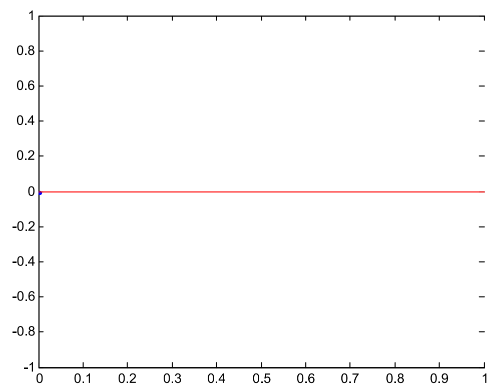

FIGURE 11. The unstable manifolds for the system for $\mu > 0, \quad \delta_L < \frac{\tau_L^2}{4}, \tau_L > 2$ and $\tau_R > 2$, $\delta_R - 1 < \delta_R < \frac{\tau_R^2}{4}$.

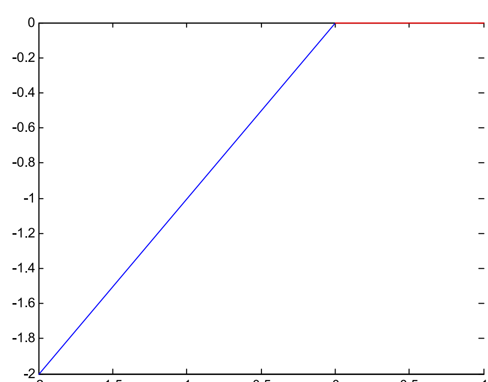

FIGURE 12. The unstable manifolds for the system for $\mu > 0, \delta_L < \frac{\tau_L^2}{4}, \tau_L > 2$ and $\tau_R < -2$, $\delta_R < \frac{\tau_R^2}{4}$.

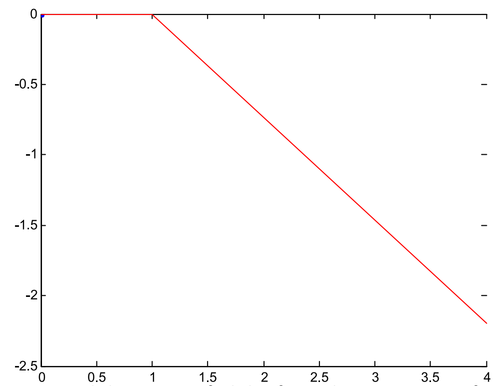

FIGURE 13. The unstable manifolds for the system for $\mu > 0, -\tau_L - 1 < \delta_L < \frac{\tau_L^2}{4}, \tau_L \leq -2$ and $\tau_R > 2$, $\delta_R - 1 < \delta_R < \frac{\tau_R^2}{4}$.

Case.2: $\tau_L^2 - 4\delta_L < 0$.

The equation (2.7) has two complex conjugate solutions,

$$r_{1L,\,2L} = \lambda_{1L,2L} = \frac{1}{2}\left(\tau_L \pm i\sqrt{4\delta_L - \tau_L^2}\right). \tag{2.10}$$

Hence

$$|\lambda_{1L,\,2L}| = \frac{1}{2}\left|\tau_L \pm i\sqrt{4\delta_L - \tau_L^2}\right| < 1, \text{if } \frac{\tau_L^2}{4} < \delta_L < 1, -2 < \tau_L < 2.$$

Remark 4. *We have the same cases in the subregion R_R. Therefore, the stable manifolds of system (1.1) in the two subregion R_R, R_L is define by the form:*

- For $\mu > 0$, $W_L^s(P_L) = \{(x, y) \in R_L : y = x, \ -\mu \leq x \leq 0\}$, $W_R^s(P_R) =$ **:** $\{(x, y) \in R_R : y = 0, \ 0 < x < \mu\}$. as shown in Figure (14).
- For $\mu < 0$, $W_L^s(P_L) = \{(x, y) \in R_L : y = 0, \ \mu \leq x \leq 0\}$, $W_R^s(P_R) =$ **:** $\{(x, y) \in R_R : y = x, \ 0 < x < -\mu\}$. as shown in Figure (15).

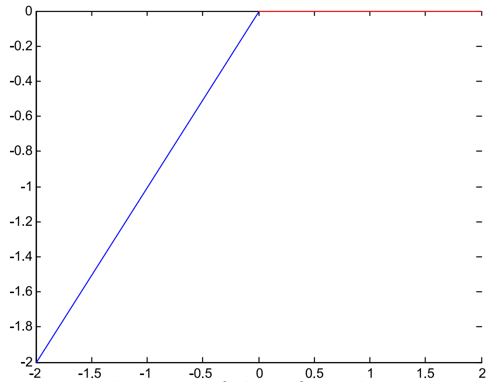

FIGURE 14. The unstable manifolds for the system for $\mu < 0, \delta_L < \min(\tau_L - 1, \frac{\tau_L^2}{4}), \tau_L \leq 2$ and $\tau_R \geq 2$, $\tau_L - 1 < \delta_R < \frac{\tau_R^2}{4}$.

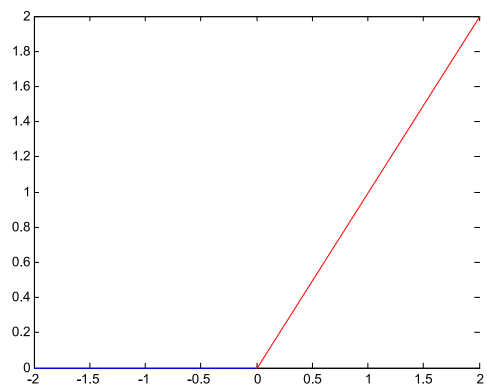

FIGURE 15. The stable manifolds for the system (1) for $\mu > 0, \frac{\tau_L^2}{4} < \delta_L < 1, -2 < \tau_L < 2$.

$|r_{1L,\,2L}| = \frac{1}{2}\left|\tau_L \pm i\sqrt{4\delta_L - \tau_L^2}\right| > 1$, if $\delta_L > \max(\frac{\tau_L^2}{4}, 1)$. The unstable manifolds for the system (1.1) is defined by:

- For $\mu > 0$, $W_L^u(P_L) = \{(x, y) \in R_L : y = x, \ -\mu \leq x \leq 0\}$, $W_R^u(P_R) =$

  : $\{(x, y) \in R_R : y = 0, \ 0 < x < \mu\}$. as shown in Figure (16).

- For $\mu < 0$, $W_L^u(P_L) = \{(x, y) \in R_L : y = 0, \ \mu \leq x \leq 0\}$, $W_R^u(P_R) =$

  : $\{(x, y) \in R_R : y = x, \ 0 < x < -\mu\}$. as shown in Figure (17).

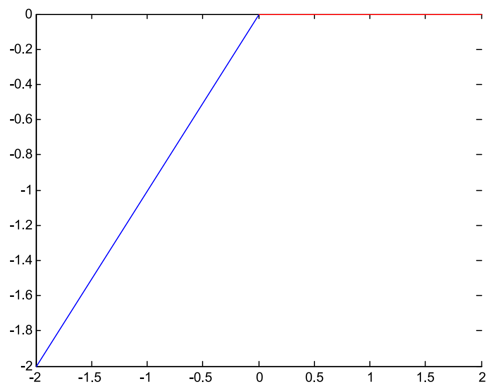

FIGURE 16. The stable manifolds for the system (1) for $\mu > 0, \frac{\tau_L^2}{4} < \delta_L < 1, -2 < \tau_L < 2$.

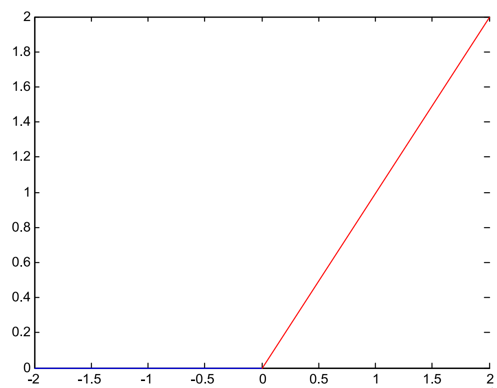

FIGURE 17. The unstable manifolds for the system (1) for $\mu < 0, \delta_L > \max(\frac{\tau_L^2}{4}, 1)$.

2.1. **The chaotic manifolds of the system (1.1).** Now, we examine the previous cases where the system (1.1) has manifolds with chaotic behaviour [12], [14], [15]. We note that, if

$$\tau_L > 1 + \delta_L, \text{ and } \tau_R < 1 + \delta_R, \tag{2.11}$$

then, there is no fixed point for $\mu < 0$, and there are two fixed points, one each in R_L and R_R, for $\mu > 0$, and if

$$\tau_L < 1 + \delta_L, \text{ and } \tau_R > 1 + \delta_R, \tag{2.12}$$

then, there is no fixed point for $\mu > 0$, and there are two fixed points, one each in R_L and R_R, for $\mu < 0$. The two fixed points are born on the border to $\mu = 0$. Therefore, we have a border-collision pair of bifurcations as μ is reduced to zero. The system (1.1) is dissipative on both subregion R_L and R_R, i.e., $0 < \delta_L < 1$, and $0 < \delta_R < 1$. Therefore, we will study the chaotic situation of the system (1.1) under the variation of μ, and $0 < \delta_L < 1$, $0 < \delta_R < 1$. We assume that the border-collision bifurcation from fixed points P_L, P_R is exhibited by map (1.1) under the variation of μ [8] , [9] . We consider the point $A = [\mu, 0]$ situated on the x-axis.

- For $\mu > 0$, the point A belongs to the subregion R_R. The image of A by map (1.1) is the point $B = [(\tau_R + 1)\mu, -\delta_R\mu]$ and the image of B is the point $C = [(\tau_L\tau_R + \tau_L - \delta_R + 1)\mu, -\delta_L(\tau_R + 1)\mu]$. So we can define each point of two subregions R_L and R_R onto a portion of the segment BO that situated in the subregion R_L. The origin O is moving to the point A, and by continuity, the image of BO by map (1.1) is the segment AC, if

$$\tau_L < \frac{\delta_R - 1}{1 + \tau_R} \tag{2.13}$$

  AC does not cross the y-axis. Then, the point A is moving to point B. And, If the following holds

$$\tau_L(\tau_R + 1) - \delta_L(1 + \frac{1}{\delta_R}) - \delta_R < 0 \tag{2.14}$$

$$\tau_L(\tau_R + 1) - \delta_L(1 + \frac{1}{\delta_R}) - \delta_R + 1 > 0, \tag{2.15}$$

the point C is moving into the inside of the segment BA. Hence, the point B is a pre-image of C, so the piecewise-linear continuous invariant segment BAC is an invariant set that must contain all the long-term dynamics, which gives an attractor in the subregion R_L as shown in the Figures (21).

- For $\mu < 0$, the point A belongs to the subregion R_L. The image of A by the map (1.1) is the point $B = [(\tau_L + 1)\mu, -\delta_L\mu]$ and the image of B is the point $C = [(\tau_R\tau_L + \tau_R - \delta_L + 1)\mu, -\delta_R(\tau_L + 1)\mu]$. Also, we can define each point of the two subregions R_L and R_R onto a portion of the segment BO wich situated in the subregion R_R this time. The origin O is mapped to the point

A, and by continuity, the image of BO by map (1.1) is the segment AC, If

$$\tau_R > \frac{\delta_L - 1}{1 + \tau_L} \tag{2.16}$$

AC does not cross the y-axis. Then, point A is mapped to point B. And If the following holds

$$\tau_R(\tau_L + 1) - \delta_R(1 + \frac{1}{\delta_L}) - \delta_L < 0 \tag{2.17}$$

$$\tau_R(\tau_L + 1) - \delta_R(1 + \frac{1}{\delta_L}) - \delta_L + 1 > 0 \tag{2.18}$$

The point C is mapped into the inside of the segment BA. Therefore, the point B is a pre-image of C, so the piecewise-linear continuous invariant segment BAC is an invariant set that must contain all the long-term dynamics, which gives an attractor in the subregion R_R as shown in the Figures (19). We observe a switching between R_L and R_R, so a period of two points must be, one on BO and one on AC. Within two points can be either stable or unstable according to the quantity $(\tau_L\tau_R - \delta_L)$ or $(\tau_L\tau_R - \delta_R)$, if

$$|\tau_L\tau_R - \delta_L| > 1 \tag{2.19}$$

The period of two points in the region R_L (global attractor of the system) is unstable. Therefore, more interesting scenario can be observed. We can easily show that there is no other stable periodic points in this set. Hence, this attractor is chaotic and robust [9]. But, if

$$|\tau_L\tau_R - \delta_R| > 1 \tag{2.20}$$

We have the same situation in the region R_R.

Theorem 2.1. *If we assume that the border-collision bifurcation from the fixed points P_L and P_R are exhibited by map (1.1) with the variation of μ by the followings:*

- For $\mu > 0$, if the conditions (2.13), (2.14) and (2.15) are satisfied then, there exists an attractor in the subregion R_L that lies the piecewise-linear continuous invariant segment ABC, where $A = [\mu, 0]$, $B = [(\tau_R + 1)\mu, -\delta_R\mu]$ and $C = [(\tau_L\tau_R + \tau_L - \delta_R + 1)\mu, -\delta_L(\tau_R + 1)\mu]$.

- For $\mu < 0$, if the the conditions (2.16), (2.17) and (2.18) are satisfied then, there exists an attractor in the subregion R_R that lies the piecewise-linear continuous invariant segment ABC, where $A = [\mu, 0]$, $B = [(\tau_L + 1)\mu, -\delta_L\mu]$ and $C = [(\tau_L\tau_R + \tau_R - \delta_L + 1)\mu, -\delta_R(\tau_L + 1)\mu]$.
- In the two previous cases, if the conditions (2.19), (2.20) are also satisfied respectively, then this attractor is chaotic and robust.

2.2. **The numerical simulation.** The numerical simulation [1] shows that, if the theorem (2.1) conditions are satisfied then, for the all initial conditions and with variation of μ, if $-1 < \tau_{L,R} < 0$, $0 < \tau_{L,R} < 1$, $0 < \delta_{L,R} < 1$, the system (1.1) has a unique chaotic attractor, this one can not be destroyed by small changes in the parameters. Since, a small changes in the parameters can only cause small change in the Lyapunov exponents [12], where the chaotic attractor is stable and robust.

- For $\mu < 0$, if $-1 < \tau_{L,R} < 1$, $0 < \delta_{L,R} < 1$, this attractor appears in the subregion R_R with the stable manifold on the subregion R_L as shown in the Figures (18), (19).

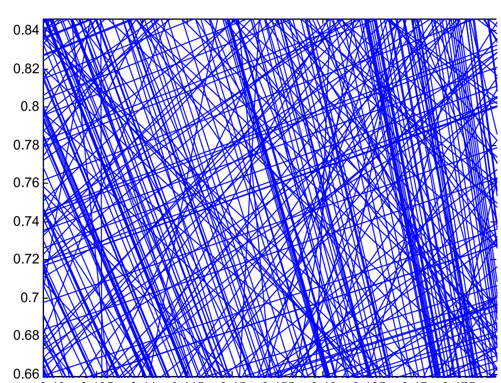

FIGURE 18. The robust chaotic attractor of the system (1) for $\tau_L = -1$, $\tau_R = 0.8$, $\delta_L = 0.1$, $\delta_R = 1$, $\mu = -2$.

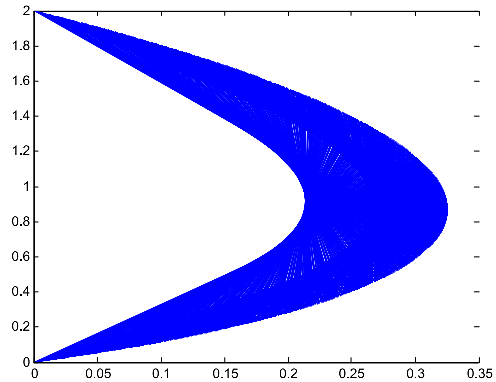

FIGURE 19. The robust chaotic attractor of the system (1) for $\tau_L = -1$, $\tau_R = 0.8$, $\delta_L = 0.1$, $\delta_R = 1$, $\mu = -2$.

- For $\mu > 0$, if $-1 < \tau_{L,R} < 1$, $0 < \delta_{L,R} < 1$, the same chaotic attractor appears in the subregion R_L, with the stable manifold in subregion R_R as shown in the Figures (20), (21).

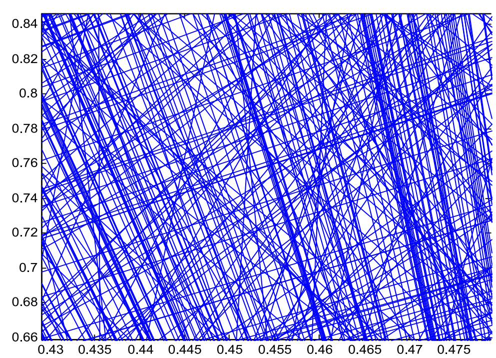

FIGURE 20. The robust chaotic attractor of the system (1) of $\tau_L = 0.8$, $\tau_R = -1$, $\delta_L = 0.1$, $\delta_R = 0.2$, $\mu = 2$.

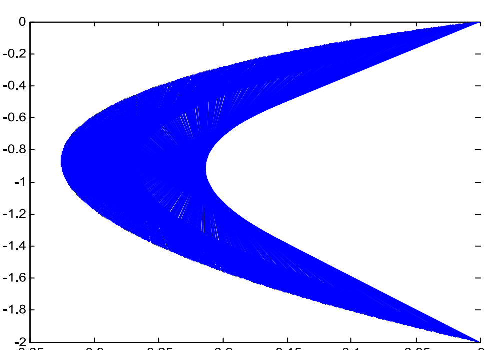

FIGURE 21. The robust chaotic attractor of the system (1) of $\tau_L = 0.8$, $\tau_R = -1$, $\delta_L = 0.1$, $\delta_R = 0.2$, $\mu = 2$.

Conclusion

In this paper, we have given the possible cases of stable and unstable manifolds for a two-dimensional piecewise-linear normal form maps, we also presented a new theorem which allows us to know the existence of chaotic attractors for a two-dimensional piecewise-linear normal form maps, it gives us an important idea about the evolution and the nature of the attractors of two-dimensional piecewise-linear maps.

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Higher National School of Forests, Khenchela, Algeria.

Email address: abdellah.menasri70@gmail.com