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FUZZY HYPER PSEUDO BCK-IDEALS OF HYPER PSEUDO BCK-ALGEBRAS

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Abstract. In this paper, by considering notion of fuzzy set, we define the 6 types of fuzzy hyper pseudo BCK-ideals denoted by, $F_1, F_2, ..., F_6$ and strong fuzzy hyper pseudo BCK-ideal on hyper pseudo BCK-algebras. Then investigate their numerous properties. Also describe the relationship between fuzzy hyper pseudo BCK-ideals and hyper pseudo BCK-ideals of hyper pseudo BCK-algebras. Also will obtained the relationship between the fuzzy hyper pseudo BCK-ideals. This relationship is shown in a lattice diagram.

1. Introduction

The study of BCK-algebra initiated by Y.Imai and Iseki [8] in 1966 as a generalization of the concept of set theoretic difference and calculi. Pseudo BCK-algebras were introduce by Georgescu and Iorgulescu [5] as a generalization of BCK-algebra in order to give a structure corresponding to pseudo MV-algebras. Since the bounded commutative BCK-algebras to correspond MV-algebras. Hyper structure (called also multi algebras) was introduced in 1934 by F. Marty [13] at the 8th congress of Scandinavian Mathematicians. Since then many researchers have worked on algebraic hyper structures and developed them. Corsini and Leoreanu in [2] presented some of the numerous applications of algebraic hyper structure, especially those from last fifteen years, to the following subjects: geometry, hyper graphs, binary relations, lattices, fuzzy sets and rough sets, automate, cryptography, cods, median algebras, relation algebras, artificial intelligence and probabilities. Hyper structures have many

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applications to several sectors of both pure and applied sciences. In [1, 9], R.A. Borzooei et al. applied the hyper structures to (pseudo) BCK- algebra which is generalization of (pseudo) BCK-algebra and investigated some related properties. In his pionearing paper [14], Zadeh introduced the notion of a fuzzy set in a set X as a function from X into the closed interval $[0, 1]$. After reading Jon's article[10, 11], I was motivated to generalize the concept of fuzzy sets on hyper pseudo BCK-algebra. By doing this, I was able to achieve interesting results. Thit article is the result of this effort. In this paper by considering notion of fuzzy set, I define the 6 types of fuzzy hyper pseudo BCK-ideals denote by, $F_1, F_2, ..., F_6$, strong fuzzy hyper pseudo BCK-ideal on hyper pseudo BCK-algebras. Then investigate their numerous properties. Also describe the relationship between fuzzy hyper pseudo BCK-ideals and hyper pseudo BCK-ideals of hyper pseudo BCK-algebras. I have also obtained the relationship between the fuzzy hyper pseudo BCK-ideals. This relationship is also shown in their lattice diagram.

2. Preliminaries

Definition 2.1. [5] A pseudo BCK-algebra is a structure $X = (X, *, \diamond, 0)$, where " $*$ " and " $\diamond$ " are binary operations on X and "0" is a constant element of X, that satisfies the following;

(a1) $(x * y) \diamond (x * z) \preceq z * y$, $(x \diamond y) * (x \diamond z) \preceq z \diamond y$,

(a2) $x * (x \diamond y) \preceq y$, $x \diamond (x * y) \preceq y$,

(a3) $x \preceq x$,

(a4) $0 \preceq x$,

(a5) $x \preceq y$, $y \preceq x$ implies $x = y$,

(a6) $x \preceq y \Leftrightarrow x * y = 0 \Leftrightarrow x \diamond y = 0$,

for all $x, y, z \in X$.

Definition 2.2. [1] A hyper pseudo BCK-algebra is a structure $(H; \circ, *, 0)$ where " $\circ$ " and" $*$ " are hyper operations on H and "0" is a constant element that satisfies the following axioms:

(PHK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$, $\quad (x * z) * (y * z) \ll x * y$,

(PHK2) $(x \circ y) * z = (x * z) \circ y$, (PHK3) $x \circ y \ll x, \qquad x * y \ll x$,

(PHK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$.

Proposition 2.1. [1] *In any hyper pseudo BCK-algebra H, the following holds:*

(i) $0 \circ 0 = 0, \quad 0 * 0 = 0, \quad x \circ 0 = x, \quad x * 0 = x$,

(ii) $0 \ll x, \quad x \ll x, \quad A \ll A$,

(iii) $0 \circ x = 0, \quad 0 * x = 0, \quad 0 \circ A = \ 0\ , \quad 0 * A = 0$,

(iv) $A \subseteq B$ *implies* $A \ll B$,

(v) $A \ll 0$ *implies* $A = \{0\}$,

(vi) $y \ll z$ *implies* $x \circ z \ll x \circ y$ *and* $x * z \ll x * y$,

(vii) $x \circ y = \{0\}$ *implies* $(x \circ z) \circ (y \circ z) = \{0\}$, *that is,* $x \circ z \ll y \circ z$; $\quad x * y = \{0\}$ *implies* $(x * z) * (y * z) = \{0\}$, *that is,* $x * z \ll y * z$,

(viii) $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$, *and* $A * \{0\} = \{0\}$ *implies* $A = \{0\}$,

(ix) $(A \circ c) \circ (B \circ c) \ll A \circ B, \quad (A * c) * (B * c) \ll A * B$

for all $x, y, z \in H$.

Remark 1. [1, 6], *Let H be a hyper pseudo BCK-algebra. For any subset I of H and any element $y \in H$, we denote,*

(1) $*(y, I)^{\ll} = \{x \in H | x * y \ll I\}$, *(2)* $*(y, I)^{\subseteq} = \{x \in H | x * y \subseteq I\}$,

(3) $\circ(y, I)^{\ll} = \{x \in H | x \circ y \ll I\}$, *(4)* $\circ(y, I)^{\subseteq} = \{x \in H | x \circ y \subseteq I\}$,

(5) $*(y, I)^{\cap} = \{x \in H | x * y \cap I \neq \emptyset\}$, *(6)* $\circ(y, I)^{\cap} = \{x \in H | x \circ y \cap I \neq \emptyset\}$.

Definition 2.3. [1]Let H be a hyper pseudo BCK-algebra, $\emptyset \neq I \subseteq H$ and $0 \in I$. Then I is said to be a hyper pseudo BCK-ideal of

(i1) type (1), if for any $y \in I, *(y, I)^{\ll} \subseteq I$ and $\circ(y, I)^{\ll} \subseteq I$;

(i2) type (2), if for any $y \in I, *(y, I)^{\subseteq} \subseteq I$ and $\circ(y, I)^{\ll} \subseteq I$;

(i3) type (3), if for any $y \in I, *(y, I)^{\ll} \subseteq I$ and $\circ(y, I)^{\subseteq} \subseteq I$;

(i4) type (4), if for any $y \in I, *(y, I)^{\subseteq} \subseteq I$ and $\circ(y, I)^{\subseteq} \subseteq I$;

(i5) type (5), if for any $y \in I, *(y, I)^{\ll} \subseteq I$ or $\circ(y, I)^{\ll} \subseteq I$;

(i6) type (6), if for any $y \in I, *(y, I)^{\subseteq} \subseteq I$ or $\circ(y, I)^{\ll} \subseteq I$;

(i7) type (7), if for any $y \in I, *(y, I)^{\ll} \subseteq I$ or $\circ(y, I)^{\subseteq} \subseteq I$;

(i8) type (8), if for any $y \in I, *(y,I)^{\subseteq} \subseteq I$ or $\circ(y,I)^{\subseteq} \subseteq I$;

(i9) type (9), if for any $y \in I, *(y,I)^{\ll} \cap \circ(y,I)^{\ll} \subseteq I$;

(i10) type (10), if for any $y \in I, *(y,I)^{\subseteq} \cap \circ(y,I)^{\ll} \subseteq I$;

(i11) type (11), if for any $y \in I, *(y,I)^{\ll} \cap \circ(y,I)^{\subseteq} \subseteq I$;

(i12) type (12), if for any $y \in I, *(y,I)^{\subseteq} \cap \circ(y,I)^{\subseteq} \subseteq I$.

Remark 2. . *The relationship among all of types of a hyper pseudo BCK-ideals of hyper pseudo BCK-algebras is given by the figure 1: (see* [1].

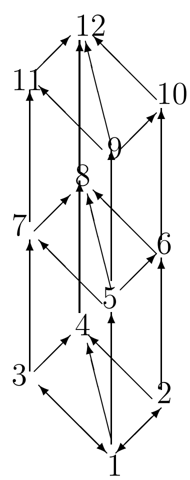

FIGURE 1. Lattice of hyper pseudo-BCK Ideals of hyper pseudo BCK-algebra

Definition 2.4. [6] Let H be a hyper pseudo BCK algebra, $I \subseteq H$ and $0 \in I$. Then I is called a strong hyper pseudo BCK-ideal of H if for any $y \in I, *(y,I)^{\cap} \subseteq I$ and $\circ(y,I)^{\cap} \subseteq I$.

Theorem 2.1. [6] Let H be a hyper pseudo BCK-algebra and $I \subseteq H$. Then I is a strong hyper pseudo BCK-algebra of H if and only if the following hold;

(i) $0 \in I$,

(ii) for any $y \in I, *(y,I)^{\cap} \subseteq I$ or for any $y \in I, \circ(y,I)^{\cap} \subseteq I$.

Proposition 2.2. [6] *Let H be a hyper pseudo BCK-algebra. then for all nonempty subset A and I of H, If I is a hyper pseudo BCK-ideal of type 1,2,3,5 or 9 such that $A \ll I$, then $A \subseteq I$.*

Remark 3. *From now on, in this paper we let H be a hyper pseudo BCK-algebra.*

3. fuzzy Hyper Pseudo BCK-ideals

Definition 3.1. A fuzzy set μ in H is called a fuzzy Hyper pseudo BCK-ideal of type F_1 of H if;

(i) $x \ll y \Rightarrow \mu(x) \geq \mu(y)$,
(ii) $\mu(x) \geq min\{inf_{u \in x*y}\mu(u), \mu(y)\}$,
(iii) $\mu(x) \geq min\{inf_{u \in x \circ y}\mu(u),\ \mu(y)\}$
for all $x, y \in H$.

Theorem 3.1. Let μ be a fuzzy set in H. Then μ is a fuzzy Hyper pseudo BCK-ideal of type F_1 of H if and only if $\mu_t = \{x \in H : \mu(x) \geq t\}$ is a hyper pseudo BCK-ideal of type 1, for all $t \in Im\mu$.

Proof. Let μ be a fuzzy Hyper pseudo BCK-ideal of type F_1 of H and $t \in Im\mu$. Then there exist $x \in H$ such that $\mu(x) = t$. Since $0 \ll x$ we get $\mu(0) \geq \mu(x)$. That is, $0 \in \mu_t$. Moreover let $x \in *(y, \mu_t)^{\ll}$ $(x \in \circ(y, \mu_t)^{\ll})$ and $y \in \mu_t$. As a result $x*y \ll \mu_t$ $(x \circ y \ll \mu_t)$ and so for any $u \in x*y$ $(u \in x \circ y)$ there exist $s \in \mu_t$ such that $u \ll s$ thus $\mu(u) \geq \mu(s) \geq t$. Hence $inf\{\mu(u)|u \in x*y\} \geq t$ $(inf\{\mu(u)|u \in x \circ y\} \geq t)$. Therefore $min\{inf_{u \in x*y}\mu(u),\ \mu(y)\} \geq t$ $(min\{inf_{u \in x \circ y}\mu(u),\ \mu(y)\} \geq t)$. Since μ is a fuzzy Hyper pseudo BCK-ideal of type F_1, we get $\mu(x) \geq t$ thus $x \in \mu_t$. So $*(y,\ \mu_t)^{\ll} \subseteq \mu_t$ $(\circ(y,\ \mu_t)^{\ll} \subseteq \mu_t)$ and this implies that, μ_t is a hyper pseudo BCK-ideal of type 1.

Conversely, let $x \ll y$ and $\mu(y) = t$. Therefore $y \in \mu_t$. Combining $x \ll y$ and $y \in \mu_t$, we get $\{x\} \ll \mu_t$. Since μ_t is a hyper pseudo BCK-ideal of type 1 by Proposition 2.2, we get $x \in \mu_t$. Hence $\mu(x) \geq t = \mu(y)$. At the end, let $x, y \in H$ and put $t = min\{inf_{u \in x*y}\mu(u),\ \mu(y)\}$. Then $y \in \mu_t$ and for all u$\in x * y$, $\mu(u) \geq t$. Hence $x * y \subseteq \mu_t$. Since μ_t is a hyper pseudo BCK-ideal of type 1 and $y \in \mu_t$ we get $x \in \mu_t$ and so $\mu(x) \geq min\{inf_{u \in x*y}\mu(u),\ \mu(y)\}$. By the similar way we can show that

$\mu(x) \geq min\{inf_{u\in x\circ y}\mu(u),\, \mu(y)\}$. Which shows μ is a fuzzy hyper pseudo BCK-ideal of type F_1 .

□

Definition 3.2. A fuzzy set μ in H is called a fuzzy Hyper pseudo BCK-ideal of type F_2 of H if;

(i) $\mu(0) \geq \mu(x) \geq min\{inf_{u\in x*y}\mu(u),\, \mu(y)\}$,

(ii) $\mu(0) \geq \mu(x) \geq min\{inf_{u\in x\circ y}\mu(u),\, \mu(y)\}$

for all $x, y \in H$.

Proposition 3.1. *Let μ be a fuzzy set in H. Then μ is a fuzzy Hyper pseudo BCK-ideal of type F_2 of H if and only if μ_t is a hyper pseudo BCK-ideal of type 4 of H, for all $t \in Im\mu$.*

Proof. Let $t \in Im\mu$. Therefore there exist $x \in H$ such that $\mu(x) = t$. Since μ is a fuzzy hyper pseudoo BCK-ideal of type F_2, we get $\mu(0) \geq \mu(x) = t$ hence $0 \in \mu_t$. Moreover, let $x, y \in H$ and $x * y \subseteq \mu_t$ therefore for all $a \in x * y$, $\mu(a) \geq t$ and so $\inf\{\mu(a)|a \in x * y\} \geq t$. Thus $\min\{inf_{a\in x*y}\mu(a),\, \mu(y)\} \geq t$. Since μ is a fuzzy hyper pseudo BCK-ideal of type F_2, we get $\mu(x) \geq t$ and so $x \in \mu_t$ that is, $*(y, \mu_t)^{\subseteq} \subseteq \mu_t$. In the similar way it is shown that $\circ(y, \mu_t)^{\subseteq} \subseteq \mu_t$. This implies that μ_t is a hyper psedo BCK-ideal of type 4. The proof of converse similar to the proof of Theorem 3.1. □

Proposition 3.2. *Every fuzzy Hyper pseudo BCK-ideal of type F_1 of H is a fuzzy Hyper pseudo BCK-ideal of type F_2 of H.*

Proof. The proof is straightforward. □

In the following examples at first we give an example of fuzzy hyper pseudo BCK-ideal of type F_1 and then we show that; the converse of proposition 3.2, is not correct in general.

Example 3.1. (i) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H are defined as follows;

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. Define μ as follows;

(i) Define;

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/2 & \text{if } x = a \\ 0 & \text{if } x = b. \end{cases}$$

Then μ, is a fuzzy hyper pseudo BCK-ideal of type F_1 .

(ii) Define;

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/2 & \text{if } x = b \\ 0 & \text{if } x = a. \end{cases}$$

Then μ, is a fuzzy hyper pseudo BCK-ideal of type F_2 of H but, it is not a fuzzy hyper pseudo BCK-ideal of type F_1, since $a \ll b$, while $0 = \mu(a) \ngeq \mu(b) = 1/2$.

Definition 3.3. A fuzzy set μ in H is called a fuzzy Hyper pseudo BCK-ideal of type F_3 of H if;

(i) $\forall x, y \in H$, $x \ll y \Rightarrow \mu(x) \geq \mu(y)$,

(ii) $\forall x, y \in H$, $\mu(x) \geq min\{inf_{u \in x*y}\mu(u), \mu(y)\}$

or

(iii) $\forall x, y \in H, \mu(x) \geq min\{inf_{u \in x \circ y}\mu(u), \mu(y)\}$.

Theorem 3.2. Let μ be a fuzzy set in H. Then μ is a fuzzy Hyper pseudo BCK-ideal of type F_3 of H if and only if μ_t is a hyper pseudo BCK-ideal of type 5, for all $t \in Im\,\mu$.

Proof. Assume that μ be a fuzzy Hyper pseudo BCK-ideal of type F_3 and $t \in Im\mu$ therefore there exist $x \in H$ such that $\mu(x) = t$. Since $0 \ll x$ by Definition 3.3(i), $\mu(0) \geq \mu(x) = t$. That is, $0 \in \mu_t$. In the following, we will show that, if μ holds the condition (ii) in Definition 3.3, then for every $y \in \mu_t, *(y, \mu_t)^{\ll} \subseteq \mu_t$ and if μ holds the condition (iii) in Definition 3.3, then for every $y \in \mu_t, \circ(y, \mu_t)^{\ll} \subseteq \mu_t$. Without loss of generality, we assume that μ holds the condition (ii) in Definition 3.3. Let $x \in *(y, \mu_t)^{\ll}$ where, $y \in \mu_t$. Then $x * y \ll \mu_t$. Therefore for any $a \in x * y$, there exist $u \in \mu_t$ such that $a \ll u$. By condition (i) in Definition 3.3, $\mu(a) \geq \mu(u) \geq t$. Hence $inf_{a \in x*y}\mu(a) \geq t$. Since $\mu(y) \geq t$, we get $min\{inf_{u \in x*y}\mu(u), \mu(y)\} \geq t$. By condition (ii) in Definition 3.3, we have $\mu(x) \geq t$. So $x \in \mu_t$ hence μ_t is a hyper pseudo BCK-ideal of type 5.

Conversely, let μ_t is a hyper pseudo BCK-ideal of typy 5 of H for all $t \in Im\mu$. Suppose that $x, y \in H, x \ll y$ and $\mu(y) = t$. Combining $x \ll y, y \in \mu_t$, we get $\{x\} \ll \mu_t$. Since μ_t is a hyper pseudo BCK-ideal of type 5 by Proposition 2.2, $\{x\} \subseteq \mu_t$. Therefore $x \in \mu_t$ and so $\mu(x) \geq \mu(y)$. Without loss of generality we assume that $\forall y \in \mu_t, *(y, \mu_t)^{\ll} \subseteq \mu_t$ and we show that $\forall x \in H, \mu(x) \geq min\{inf_{u \in x*y}\mu(u), \mu(y)\}$. For this, let $x, y \in H$ and put $t = min\{inf_{a \in x*y}\mu(a), \mu(y)\}$. Then $y \in \mu_t$ and for any u$\in x * y$, $\mu(u) \geq t$. Hence $x * y \subseteq \mu_t$. Since μ_t is a hyper pseudo BCK-ideal of type 5 and $y \in \mu_t$ we get $x \in \mu_t$ and so $\mu(x) \geq min\{inf_{u \in x*y}\mu(u),\ \mu(y)\}$. Which shows μ is a fuzzy hyper pseudo BCK-ideal of type F_3. □

Proposition 3.3. *Every fuzzy Hyper pseudo BCK-ideal of type F_1 of H is a fuzzy Hyper pseudo BCK-ideal of type F_3 of H.*

Proof. By Definitions 3.1 and 3.3, the proof is straightforward. □

In the following examples we show that;

(i) A fuzzy hyper pseudo BCK-ideal of type F_3 is not fuzzy hyper pseudo BCK-ideal of type F_1, nor of type F_2 in general,

(ii)A fuzzy hyper pseudo BCK-ideal of type F_2 is not fuzzy hyper pseudo BCK-ideal of type F_3 in general.

Example 3.2. (i) Let $H = \{0, a, b, c, d\}$. Hyper operations "$\circ$" and "$*$" on H given by the following tables:

$\circ$	0	a	b	c	d
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$	$\{0, a, b\}$	$\{0, a, b\}$
c	$\{c\}$	$\{c, b\}$	$\{b, d\}$	$\{0, a, b, d\}$	$\{b, d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{0, d\}$	$\{0, d\}$

$*$	0	a	b	c	d
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, a, b\}$	$\{0, b\}$	$\{0, a, b\}$
c	$\{c\}$	$\{c\}$	$\{a, c\}$	$\{0, c\}$	$\{a, b, c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{0, d\}$	$\{0, d\}$

Then $(H, *, \circ, 0)$ is a hyper pseudo BCK-algebra. Define fuzzy set μ as follow:

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } a \\ 1/2 & \text{if } x = b \\ 1/3 & \text{if } x = d \\ 0 & \text{if } x = c. \end{cases}$$

It is easy to see that μ, is a fuzzy hyper pseudo BCK-ideal of type F_3 and it is not fuzzy hyper pseudo BCK-ideal of type F_1, nor of type F_2. Becuase, $c \circ b = \{b, d\}$ and

$$\mu(c) = 0 \not\geq \min\{inf\{\mu(b), \mu(d)\}, \mu(b)\} = \{inf\{1/2, 1/3\}, 1/2\} = 1/3$$

(ii) fuzzy set μ in Example 3.1 (ii), is a fuzzy hyper pseudo BCK-ideal of type F_2 and it is not fuzzy hyper pseudo BCK-ideal of type F_3.

Definition 3.4. A fuzzy set μ in H is called a fuzzy Hyper pseudo BCK-ideal of type F_4 of H if;

(i) $\forall x \in H, \mu(0) \geq \mu(x) \geq min\{inf_{u \in x * y}\mu(u), \mu(y)\}$

or

(ii) $\forall x \in H, \mu(0) \geq \mu(x) \geq min\{inf_{u \in x \circ y}\mu(u), \mu(y)\}$.

Theorem 3.3. Let μ be a fuzzy set in H. Then μ is a fuzzy Hyper pseudo BCK-ideal of type F_4 of H if and only if μ_t is a hyper pseudo BCK-ideal of type 8, for all $t \in Im\mu$.

Proof. The proof is similar to the proof of theorem 3.2, by some modification.

□

Proposition 3.4. *Every fuzzy Hyper pseudo BCK-ideal of type F_1, F_2, F_3 of H is a fuzzy Hyper pseudo BCK-ideal of type F_4 of H.*

Proof. The proof is straightforward. □

In the following example, we show that the converse of Proposition 3.4, is not correct in general.

Example 3.3. Let H be a hyper pseudo BCK-algebra defind in Example 3.2, Define the fuzzy subset μ of H by;

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } a \\ 1/3 & \text{if } x = b \\ 1/2 & \text{if } x = d \\ 0 & \text{if } x = c. \end{cases}$$

It is easy to see that μ, is a fuzzy hyper pseudo BCK-ideal of type F_4 and it is not a fuzzy hyper pseudo BCK-ideal of type F_1, nor of type F_2. Because,

$$\mu(c) = 0 \ngeq \min\{inf\{\mu(b), \mu(d)\}, \mu(b)\} = \{inf\{1/2, 1/3\}, 1/2\} = 1/3.$$

Since $b \ll d, \mu(d) \geq \mu(b)$ we get μ is not a fuzzy hyper pseudo BCK-ideal of type F_3.

Definition 3.5. A fuzzy set μ in H is called a fuzzy Hyper pseudo BCK-ideal of type F_5 of H if;

(i) $\forall x, y \in H,\ x \ll y \Rightarrow \mu(x) \geq \mu(y)$,

(ii) $\forall x, y \in H,\ \mu(x) \geq min\{min\{inf_{a\in x*y}\mu(a), inf_{b\in x\circ y}\mu(b)\}, \mu(y)\}$

Theorem 3.4. Let μ be a fuzzy set in H. Then μ is a fuzzy Hyper pseudo BCK-ideal of type F_5 of H if and only if μ_t is a hyper pseudo BCK-ideal of type 9 of H, for all $t \in Im\mu$.

Proof. Let μ be a fuzzy Hyper pseudo BCK-ideal of type F_5 and $t \in Im\mu$. Therefore there exist $x \in H$ such that $\mu(x) = t$. By Definition 3.5(i), and $\mu(0) \geq \mu(x) = t$ we get, $0 \in \mu_t$. Now, let $x \in *(y, \mu_t)^{\ll} \cap \circ(y, \mu_t)^{\ll}$, where $y \in \mu_t$ then $x*y \ll \mu_t, x\circ y \ll \mu_t$. Therefore for any $a \in x * y$, there exist $u \in \mu_t$ such that $a \ll u$ and for any $b \in x \circ y$ there exist $v \in \mu_t$ such that $b \ll v$. regarding condition (i) in Definition 3.5, $\mu(a) \geq \mu(u) \geq t, \mu(b) \geq \mu(v) \geq t$ for all, $a \in x * y, b \in x \circ y$. Hence $inf_{a\in x*y}\mu(a) \geq t, inf\,\mu(b)_{b\in x\circ y} \geq t$ and so $min\{inf_{a\in x*y}\mu(a), inf_{b\in x\circ y}\mu(b)\} \geq t$. Since $\mu(y) \geq t$, we get $min\{min\{inf_{a\in x*y}\mu(a), inf_{b\in x\circ y}\mu(b)\}, \mu(y)\} \geq t$. By condition (ii) in Definition 3.5, we have $\mu(x) \geq t$. Therefore $x \in \mu_t$ and this implies that μ_t is a hyper pseudo BCK-ideal of type 9.

Conversely, let for all $t \in Im\mu$, μ_t is a hyper pseudo BCK-ideal of typy 9 of H.

at first, let $x, y \in H, x \ll y$ and $\mu(y) = t$. Combining $x \ll y, y \in \mu_t$, we get $\{x\} \ll \mu_t$. Since μ_t is a hyper pseudo BCK-ideal of type 9 by Proposition 2.2, $\{x\} \subseteq \mu_t$. Therefore $x \in \mu_t$ and so $\mu(x) \geq \mu(y) = t$. Let $x, y \in H$ and put $t = min\{min\{inf_{a\in x*y}\mu(a), inf_{b\in x\circ y}\mu(b)\}, \mu(y)\}$, since $\mu(y) \geq t$ we get $y \in \mu_t$. Also for every $a \in x * y$, and $b \in x \circ y$, we have $\mu(a) \geq t$, $\mu(b) \geq t$. Hence $x * y, x \circ y \subseteq \mu_t$ Therefore $x * y, x \circ y \ll \mu_t$ Thus $x \in *(y, \mu_t) \cap \circ(y, \mu_t)$. Since μ_t is a hyper pseudo BCK-ideal of type 9 and $y \in \mu_t$ we get $x \in \mu_t$ and so $min\{min\{inf_{a\in x*y}\mu(a), inf_{b\in x\circ y}\mu(b)\}, \mu(y)\}$. Which shows μ is a fuzzy hyper pseudo BCK-ideal of type F_5. □

Proposition 3.5. *Every fuzzy Hyper pseudo BCK-ideal of type F_3 of H is a fuzzy Hyper pseudo BCK-ideal of type of type F_5 of H.*

Proof. Let μ be a fuzzy hyper pseudo BCK-ideal of type F_3. Then it is clear that condition (i) in Definition 3.5, holds. Let (ii), in Definition 3.5, is not correct, that is, there exist $x \in H$ such that $\mu(x) < min\{min\{inf_{a\in x*y}\mu(a), inf_{b\in x\circ y}\mu(b)\}, \mu(y)\}$. Therefore $\mu(x) < min\{inf_{a\in x*y}\mu(a), inf_{b\in x\circ y}\mu(b)\}$ and $\mu(x) < \mu(y)$. Hence $\mu(x) < inf_{a\in x*y}\mu(a)$, $\mu(x) < inf_{b\in x\circ y}\mu(b)$ and so $\mu(x) < min\{inf_{a\in x*y}\mu(a), \mu(y)\}$ and $\mu(x) < min\{inf_{a\in x\circ y}\mu(a), \mu(y)\}$. This implies that μ is not a fuzzy hyper pseudo BCK-ideal of type F_3, which is contradiction. This contradiction shows that any fuzzy hyper pseudo BCK-ideal of type F_3 is a fuzzy hyper pseudo BCK-ideal of type F_5. □

In the following example we show that a fuzzy hyper pseudo BCK-ideal of type F_5 is not fuzzy hyper pseudo BCK-ideal of type F_3 in general.

Example 3.4. Let $H = \{0, a, b, c\}$ and operations " $*$ " and " $\circ$ " on H are defined as follows;

$\circ$	0	a	b	c
0	{0}	{0}	{0}	{0}
a	{a}	{0}	{a}	{0}
b	{b}	{b}	{0}	{0}
c	{c}	{b}	{c}	{0}

$*$	0	a	b	c
0	{0}	{0}	{0}	{0}
a	{a}	{0}	{a}	{0}
b	{b}	{b}	{0}	{0}
c	{c}	{c}	{a}	{0}

Then $(H, *, \circ, 0)$ is a hyper pseudo BCK-algebra. Define fuzzy set μ as follows;

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/2 & \text{if } x = a \text{ or } b \\ 0 & \text{if } x = c. \end{cases}$$

Then μ, is a fuzzy hyper pseudo BCK-ideal of type F_5 and it is not of type F_3. Because,

$$\mu(c) = 0 \ngeq min\{inf_{t\in c\circ a}\mu(t), \mu(a)\} = min\{\mu(b), \mu(a)\} = 1/2$$

and

$$\mu(c) = 0 \ngeq min\{inf_{t\in c*b}\mu(t), \mu(b)\} = min\{\mu(a), \mu(b)\} = 1/2.$$

Definition 3.6. A fuzzy set μ in H is called a fuzzy Hyper pseudo BCK-ideal of type F_6 of H if $\forall x, y \in H,\ \mu(0) \geq \mu(x) \geq min\{min\{inf_{a\in x*y}\mu(a), inf_{b\in x\circ y}\mu(b)\}, \mu(y)\}$.

Theorem 3.5. Let μ be a fuzzy set in H. Then μ is a fuzzy Hyper pseudo BCK-ideal of type F_6 of H if and only if μ_t is a hyper pseudo BCK-ideal of type 12, for all $t \in Im\mu$.

Proof. Let μ be a fuzzy Hyper pseudo BCK-ideal of type F_6 and $t \in Im\mu$. Therefore, there exist $x \in H$ such that $\mu(x) = t$. Since $\mu(0) \geq \mu(x) = t$, we get $0 \in \mu_t$. Now let $x \in *(y, \mu_t)^{\subseteq} \cap \circ(y, \mu_t)^{\subseteq}$ then $x * y \subseteq \mu_t$, $x \circ y \subseteq \mu_t$. Hence for all $a \in x * y, b \in x \circ y, \mu(a)$, $\mu(b) \geq t$. Therefore $inf_{a\in x*y}\mu(a) \geq t$, $inf_{b\in x\circ y}\mu(b) \geq t$ and so $min\{inf\,\mu(a)_{a\in x*y},\ inf_{b\in x\circ y}\mu(b)\} \geq t$. Since $\mu(y) \geq t$ we get

$$min\{min\{inf_{a\in x*y}\mu(a), inf_{b\in x\circ y}\mu(b)\},\ \mu(y)\} \geq t.$$

Therefore $\mu(x) \geq t$ and so $x \in \mu_t$. That is μ_t is a hyper pseudo BCK-ideal of type 12.

Conversely, the proof is similar to the proof of Theorem 3.4, by some modification. □

Proposition 3.6. *(i) Every fuzzy Hyper pseudo BCK-ideal of type F_4 of H is a fuzzy Hyper pseudo BCK-ideal of type of type F_6 of H.*

(ii) Every fuzzy Hyper pseudo BCK-ideal of type F_5 of H is a fuzzy Hyper pseudo BCK-ideal of type of type F_6 of H.

Proof. (i)The proof of this proposition is similar to the proof of Proposition 3.5, by some modification.
(ii) The proof is straightforward. □

In the following examples we show that the converse of Proposition 3.6, is not correct in general.

Example 3.5. (i) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H are defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. Define μ as follows

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } b \\ 1/2 & \text{if } x = a. \end{cases}$$

Then μ, is a fuzzy hyper pseudo BCK-ideal of type F_6 and it is not of type F_5. Because, $a \ll b$ and $\mu(a) = 1/2 \ngeq \mu(b) = 1$.
(ii) Consider hyper pseudo BCK-algebra H and fuzzy hyper pseudo BCK-ideal μ in Example 3.4, then μ is a fuzzy hyper pseudo BCK-ideal of type F_6 and it is not hyper pseudo BCK-ideal of type F_4.

In the figure 2, we show the relationship among all types of fuzzy hyper pseudo BCK-ideals.

Definition 3.7. A fuzzy set μ in H is called a fuzzy Hyper pseudo BCK-ideal of

(i) type, F_{3_*} of H if, $\forall x \in H, \mu(0) \geq \mu(x) \geq min\{inf_{u \in x*y}\mu(u), \mu(y)\}$,

(ii) type, $F_{3_\circ}$ of H if, $\forall x \in H, \mu(0) \geq \mu(x) \geq min\{inf_{u \in x \circ y}\mu(u), \mu(y)\}$,

Theorem 3.6. let μ be a fuzzy set on H. If,

(i) μ is a fuzzy Hyper pseudo BCK-ideal of type F_{3_*} of H then μ_t is a hyper pseudo BCK-ideal of type 6.

(ii) μ is a fuzzy Hyper pseudo BCK-ideal of type $F_{3_\circ}$ of H then μ_t is a hyper pseudo BCK-ideal of type 7. for all $t \in Im\mu$

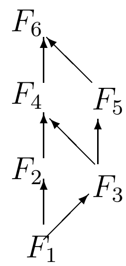

FIGURE 2. Lattice of fuzzy hyper pseudo BCK- Ideals of hyper pseudo BCK-algebra

Proof. The proof of this theorem is similar to the proof of Theorem 3.2, by some modification. □

Example 3.6. *(i)* Let H and μ be a hyper pseudo BCK-algebra and fuzzy hyper pseudo BCK-ideal, defind in Example 3.2. It is easy to check that μ, is a fuzzy hyper pseudo BCK-ideal of type F_{3_*}. Since $c \circ b = \{b, d\}$ we get

$$\mu(c) = 0 \ngeq \min\{inf\{\mu(b), \mu(d)\}, \mu(b)\} = \{inf\{1/2, 1/3\}, 1/2\} = 1/3.$$

and so μ, is not fuzzy hyper pseudo BCK-ideal of type $F_{3_\circ}$

(ii) Let $H = \{0, a, b, c, d\}$. By replacing Hyper operations "$*$" and "$\circ$" in Example 3.2. We can see that,

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } a \\ 1/2 & \text{if } x = b \\ 1/3 & \text{if } x = d \\ 0 & \text{if } x = c. \end{cases}$$

Is a fuzzy hyper pseudo BCK-ideal of type$F_{3_\circ}$. Since $c * b = \{b, d\}$ we get

$$\mu(c) = 0 \ngeq \min\{inf\{\mu(b), \mu(d)\}, \mu(b)\} = min\{inf\{1/2, 1/3\}, 1/2\} = 1/3.$$

This implies that μ, is not fuzzy hyper pseudo BCK-ideal of type F_{3_*}

Definition 3.8. A fuzzy set μ in H is called a fuzzy strong hyper pseudo BCK-ideal of H if,

(i) $\forall x, y \in H, \mu(0) \geq \mu(x) \geq min\{sup_{u \in x*y}\mu(u), \mu(y)\}$

or

(ii) $\forall x, y \in H, \mu(0) \geq \mu(x) \geq min\{sup_{u \in x \circ y} \mu(u), \mu(y)\}$.

Proposition 3.7. *Let μ be a fuzzy strong hyper pseudo BCK-ideal of H. Then for all $x, y \in H$;*

(i) $x \ll y$ *implies that* $\mu(y) \leq \mu(x)$,

(ii) $inf_{a \in x*x} \mu(a) \geq \mu(x)$ *for all* $x \in H$,

(iii) $inf_{a \in x \circ x} \mu(a) \geq \mu(x)$ *for all* $x \in H$,

(iv) $\mu(x) \geq min\{\mu(a), \mu(y)\}$ *for all* $a \in x \circ y, a \in x * y$.

Proof. *(i)* Let $x \ll y$ then $0 \in x * y$. combining $\mu(0) \geq \mu(x)$ and $0 \in x * y$ we get $\mu(0) = sup_{u \in x \circ y} \mu(u)$. Therefore $min\{sup_{u \in x*y} \mu(u),\ \mu(y)\} = min\{\mu(0), \mu(y)\} = \mu(y)$. By Definition 3.8(i), we get, $\mu(x) \geq \mu(y)$.

(ii) Since $x * x \ll \{x\}, \forall a \in x * x, a \ll x$. and so $\forall a \in x * x, \mu(a) \geq \mu(x)$. Therefore, $inf_{a \in x*x} \mu(a) \geq \mu(x)$, which implies that (ii) is true .

(iii) The proof of (iii) is similar to the proof of (ii).

(iv) Let $x, y \in H$. Since for all $a \in x*y$ $(a \in x \circ y)$, $\mu(x) \geq min\{sup_{u \in x*y} \mu(u), \mu(y)\}$ $(\mu(x) \geq min\{sup_{u \in x \circ y} \mu(u), \mu(y)\})$ we get (iv), is true.

□

Corollary 3.1. *Every fuzzy strong hyper pseudo BCK-ideal is a fuzzy hyper pseudo BCK-ideal of type F_1.*

Proof. By Preoposition 3.7 (iv), the proof is straightforward. □

Theorem 3.7. Let μ be a fuzzy strong hyper pseudo BCK-ideal. Then μ_t is a strong hyper pseudo BCK-ideal for all $t \in Im\mu$.

Proof. Let $t \in Im\mu$. Therefore there exist $x \in H$ such that $\mu(x) = t$. Since $\mu(0) \geq \mu(x)$ we get $0 \in \mu_t$. Now, we show that $*(y, \mu_t)^{\cap} \subseteq \mu_t$ for all $y \in \mu_t$. For this, let $a \in x*y \cap \mu_t$. Then $a \in x*y, a \in \mu_t$. Since $\mu(a) \geq t$ we conclude that, sup $\mu(b)_{b \in x*y} \geq t$. combining $sup_{b \in x*y} \mu(b) \geq t$ and $\mu(y) \geq t$, we get min $\{sup_{b \in x*y} \mu(b), \mu(y)\} \geq t$. According to μ is a fuzzy strong hyper pseudo BCK-ideal we get $\mu(x) \geq t$ and so $x \in \mu_t$. Therefore $*(y, \mu_t)^{\cap} \subseteq \mu_t$. By Theorem 2.1, μ_t is a strong hyper pseudo BCK-ideal. □

In the following examples at first we give fuzzy strong hyper pseudo BCK- ideal on H. Then with an example we show that any fuzzy hyper pseudoBCK- ideal of type F_1 is not fuzzy strong hyper pseudoBCK- ideal in general.

Example 3.7. *(i)* Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H are defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK–algebra. Define fuzzy set μ as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } a \\ 0 & \text{if } x = b. \end{cases}$$

It is easy to chek that μ, is a fuzzy strong hyper pseudo BCK-ideal.

(ii) Let $H = \{0, a, b, c, d, e\}$. Hyperoperations "$*$" and "$\circ$" given by the following tables:

$\circ$	0	a	b	c	d	e
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$	$\{0, a, b\}$	$\{0, a, b\}$	$\{0, b\}$
c	$\{c\}$	$\{c, b\}$	$\{b, d\}$	$\{0, a, b, d\}$	$\{b, d\}$	$\{0, a, b, d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{0, d\}$	$\{0, d\}$	$\{0, d\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e, d\}$	$\{e, d\}$	$\{0, e\}$

$*$	0	a	b	c	d	e
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, a, b\}$	$\{0, b\}$	$\{0, a, b\}$	$\{0, b\}$
c	$\{c\}$	$\{c\}$	$\{a, c\}$	$\{0, c\}$	$\{a, b, c\}$	$\{0, c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{0, d\}$	$\{0, d\}$	$\{0, d\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e, d\}$	$\{e, d\}$	$\{0, e\}$

Then $(H, *, \circ, 0)$, is a hyper pseudo BCK-algebra. Define fuzzy set μ as follow:

$$\mu(x) = \begin{cases} 1 & \text{if } x \neq e \\ 0 & \text{if } x = e. \end{cases}$$

Then μ, is a fuzzy hyper pseudo BCK-ideal of type 1, but it is not fuzzy strong hyper pseudo BCK-ideal. Becuase $e * d = \{e, d\}$ and

$$0 = \mu(e) \ngeq \min\{sup\{\mu(e), \mu(d)\}, \mu(d)\} = \min\{sup\{0, 1\}, 1\} = 1$$

.

Conclusion

To conclude, having known that hyper pseudo BCK-algebra is generalization of hyper BCK-algebra and pseudo BCK-algebra, we aim to generalize the notion of fuzzy sets on pseudo BCK-algebras and hyper BCK-algebras in to hyper pseudo BCK-algebras. For this purpose, we have considerd notion fuzzy sets on hyper pseudo BCK-algebras and defined some new fuzzy hyper pseudo BCK- ideals on hyper pseudo BCK-algebras. we defined fuzzy hyper pseudo BCK-ideals in such a way that their α-cuts (μ_α) would be hyper psedo BCK-ideals. We also were able to obtain the relationship between the fuzzy hyper pseudo BCK-ideals and show this relationship with their lattice diagram.

We note that if "$*$"="$\circ$" for all $x, y \in H$ then any fuzzy hyper pseudo BCK-ideal of types F_1, F_3, F_5 is a fuzzy hyper BCK-ideal and any fuzzy hyper pseudo BCK-ideal of types F_2, F_4, F_6 is a fuzzy weak hyper BCK-ideal in H. Also, If $x * y$, $x \circ y$ are singelton for all $x, y \in H$, then any fuzzy hyper pseudo BCK-ideal of types F_1, F_2 is a fuzzy pseudo BCK-ideal.

We hope that this results are helpful to futher studies in fuzzy set and fuzzy ideals.

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