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UNIQUENESS OF ALGEBROID FUNCTIONS CONCERNING NEVANLINNA'S FIVE-VALUE THEOREM ON ANNULI

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Abstract. In this paper, we prove a uniqueness theorem of derivative's of algebroid functions on annuli which improve and generalize the Navenlinna's five-value theorem for algebroid functions on annuli.

1. Introduction

The uniqueness theory of algebroid functions is an interesting problem in the Nevanlinna theory. The uniqueness problem of algebroid functions was first considered by Valiron, afterwards several scholars have got uniqueness theorems of algebroid functions in the complex plane $\mathbb{C}$ (see [3]-[25] and [14]-[48]). In 2005, Khrystiyanyn-Kondratyuk (see [21] and [22]) built the Nevanlinna Theory for meromorphic functions on annuli. Applying the Nevanlinna Theory for meromorphic functions on annuli, uniqueness questions of meromorphic functions sharing some values on annuli have been recently treated as well ([23]). Recently Tan-Zhang [13] built the fundamental theorems of algebroid functions on annuli. Combining these fundamental theorems and the notion of the weakly shared value, [11] first studied the uniqueness questions of algebroid functions on annuli. Thus we consider the uniqueness problem of algebroid functions in multiply connected domains. By Doubly connected mapping theorem [24] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}, 0 \leq r < R \leq +\infty$. We consider only two cases : $r = 0,\ R = +\infty$

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simultaneously and $0 \leq r < R \leq +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

2. Basic Notations and Definitions

We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and algebroid functions (see [7],[8] and [10]).

Let $A_v, A_{v-1}, ..., A_0$ be a group of analytic functions which have no common zeros and define on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$,

$$\psi(z, w) = A_v w^v + A_{v-1} w^{v-1} + ... + A_1 w + A_0(z) = 0. \tag{2.1}$$

Then irreducible equation (2.1) defines a ν-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$.

Let w be a ν-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, we use the following notations

$$m(r, w) = \frac{1}{\nu}\sum_{j=1}^{\nu} m(r, w_j) = \frac{1}{\nu}\sum_{j=1}^{\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|d\theta,$$

$$N_1(r, w) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{n_1(t, w)}{t}dt, \qquad N_2(r, w) = \frac{1}{\nu}\int_1^r \frac{n_2(t, w)}{t}dt,$$

$$\overline{N}_1\left(r, \frac{1}{w-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1\left(t, \frac{1}{w-a}\right)}{t}dt, \qquad \overline{N}_2\left(r, \frac{1}{w-a}\right) = \frac{1}{\nu}\int_1^r \frac{\overline{n}_2\left(t, \frac{1}{w-a}\right)}{t}dt,$$

$$m_0(r, w) = m(r, w) + m\left(\frac{1}{r}, w\right) - 2m(1, w), \quad N_0(r, w) = N_1(r, w) + N_2(r, w),$$

$$\overline{N}_0\left(r, \frac{1}{w-a}\right) = \overline{N}_1\left(r, \frac{1}{w-a}\right) + \overline{N}_2\left(r, \frac{1}{w-a}\right),$$

where $w_j((j = 1, 2, ..., \nu)$ is one valued branch of w, $n_1(t, w)$ is the counting function of poles of the function w in $\{z : t < |z| \leq 1\}$ and $n_2(t, w)$ is the counting function of poles of the function w in $\{z : 1 < |z| \leq t\}$ (both counting multiplicity).$\overline{n}_1\left(t, \frac{1}{w-a}\right)$ is the counting function of poles of the function $\frac{1}{w-a}$ in $\{z : t < |z| \leq 1\}$ and $\overline{n}_2\left(t, \frac{1}{w-a}\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ in $\{z : 1 < |z| \leq t\}$ (both ignoring multiplicity). $\overline{n}_1^{k)}\left(t, \frac{1}{w-a}\right)\left(\overline{n}_1^{(k}\left(t, \frac{1}{w-a}\right)\right)$ is the counting function of poles of the function $\frac{1}{w-a}$ with multiplicity $\leq k$ ($or \quad > k$) in $\{z : t < |z| \leq 1\}$, each point count only once; $\overline{n}_2^{k)}\left(t, \frac{1}{w-a}\right)$ $\left(\overline{n}_2^{(k}\left(t, \frac{1}{w-a}\right)\right)$ is the counting function of poles of the function $\frac{1}{w-a}$ with multiplicity $\leq k$ ($or \quad > k$) in $\{z : 1 < |z| \leq t\}$, each point count only once, respectively.

Definition 2.1. [**12**] Let w be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, the function

$$T_0(r, w) = m_0(r, w) + N_0(r, w), \quad 1 \leq r < R_0$$

is called Nevanlinna characteristic of w.

Definition 2.2. For $B \subset \mathbb{A}$ and $a \in \mathbb{C} \cup \infty$, we denote by $\overline{N}_0^B(r, \frac{1}{w-a})$ the reduced counting function of those zeros of $w - a$ on $\mathbb{A}$, which belong to the set B.

In 1930, Valiron [1] firstly began to study the uniqueness questions of algebroid functions (cf.[1]), and proved the following result:

Theorem 2.1. *Let $w = w(z)$ and $\hat{w} = \hat{w}(z)$ be two ν-valued algebroid functions, and let a_1, a_2, $\cdots$, $a_{4\nu}$, $a_{4\nu+1}$ be $4\nu + 1$ distinct values in the extended complex plane $\mathbb{C} \cup \{\infty\}$. If $w = w(z)$ and $\hat{w} = \hat{w}(z)$ share a_j CM for $1 \leq j \leq 4\nu + 1$, then $w = \hat{w}$.*

Later on, He[2] proved the following result that improved Theorem 2.1:

Theorem 2.2. *Let $w = w(z)$ and $\hat{w} = \hat{w}(z)$ be ν-valued and μ-valued algebroid functions respectively, where μ and ν are two positive integers satisfying $\mu \leq \nu$, and let a_1, a_2, $\cdots$, $a_{4\nu}$, $a_{4\nu+1}$ be $4\nu + 1$ distinct values in the extended complex plane $\mathbb{C} \cup \{\infty\}$. If $w = w(z)$ and $\hat{w} = \hat{w}(z)$ share a_j IM for $1 \leq j \leq 4\nu + 1$, then $w = \hat{w}$.*

Recently Tan-Zhang [13] built the fundamental theorems of algebroid functions on annuli. Combining these fundamental theorems and the notion of the weakly shared value, [13] first studied the uniqueness questions of algebroid functions on annuli, and proved the following results:

Theorem 2.3. *Let $w = w(z)$ and $\hat{w} = \hat{w}(z)$ be ν-valued and μ-valued algebroid functions on the annulus $A\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z_0| < R_0\right\}$ with $1 < R_0 \leq +\infty$,* *where ν and μ are positive integers satisfying $\mu \le \nu$, let $a_1,\ a_2,\ \cdots,\ a_p$ be p distinct finite complex values, and let $k_1,\ k_2,\ \cdots,\ k_p$ be p positive integers. If*

$$\overline{E}_{k_j)}(a_j, w) = \overline{E}_{k_j)}(a_j, \hat{w}), \quad \sum_{j=1}^{p} \frac{k_j}{k_j+1} - 2\nu \frac{2k+1}{k+1} \le 0$$

and

$$\frac{1}{k+1} \sum_{j=1}^{p} \Delta(a_j, w, \hat{w}) > 2\nu \frac{2k+1}{k+1} - \sum_{j=1}^{p} \frac{k_j}{k_j+1}$$

with $k = \max\limits_{1\le j\le p}\{k_j\}$, then $w = \hat{w}$.

Theorem 2.4. [11] *Let $w = w(z)$ and $\hat{w} = \hat{w}(z)$ be ν-valued and μ-valued algebroid functions on the annulus $A\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z_0| < R_0\right\}$ with $1 < R_0 \le +\infty$, where ν and μ are positive integers satisfying $\mu \le \nu$, let $a_1,\ a_2,\ \cdots,\ a_p$ be p distinct finite complex values, and let $k_1,\ k_2,\ \cdots,\ k_p$ be p positive integers such that $k_1 \ge k_2 \ge \cdots \ge k_p$, where p is a positive integer satisfying $p \ge 2\nu+1$. If $\overline{E}_{k_j)}(a_j, w) = \overline{E}_{k_j)}(a_j, \hat{w})$ for $1 \le j \le p$, then $w = \hat{w}$.*

3. Some Lemmas

Lemma 3.1. [21] *(Jensen theorem for meromorphic function on annuli) Let f be a meromorphic function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, then*

$$\begin{aligned} N_0\left(r, \frac{1}{f}\right) - N_0(r, f) &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(\frac{1}{r}e^{i\theta}\right)\right| d\theta \\ &\quad -\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta, \end{aligned}$$

where $1 \le r < R_0$.

Lemma 3.2. [11] *(The first fundamental theorem on annuli) Let w be ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, $a \in \mathbb{C}$*

$$m_0(r, a) + N_0(r, a) = T_0(r, w) + O(1).$$

Lemma 3.3. [11] *(The second fundamental theorem on annuli). Let W be ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le$*

$+\infty$), a_k ($k = 1, 2, .., p$) are p distinct complex numbers (finite or infinite), then we have

$$(3.1)\qquad (p-2v)T_0(r,w) \leq \sum_{k=1}^{p} N_0\left(r, \frac{1}{w-a_k}\right) - N_1(r,w) + S_0(r,w)$$

or

$$(3.2)\qquad (p-2v)T_0(r,w) \leq \sum_{k=1}^{p} \overline{N}_0\left(r, \frac{1}{w-a_k}\right) + S_0(r,w).$$

Lemma 3.4. [11] *Let w be ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ ($1 < R_0 \leq +\infty$), if the following conditions are satisfied*

$$\liminf_{r\to\infty} \frac{T_0(r,w)}{\log r} < \infty, \quad R_0 = +\infty,$$

$$\liminf_{r\to R_0^-} \frac{T_0(r,w)}{\log \frac{1}{(R_0-r)}} < \infty, \quad R_0 < +\infty,$$

then w is an algebraic function.

Remark 1. [11] *Let w be a ν-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$, where $1 < R_0 \leq +\infty$ and $\widehat{w}$ be a μ-valued algebroid functions which is determined by the following equation on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$, where $1 < R_0 \leq +\infty$,*

$$\varphi(z, \widehat{w}) = B_\mu \widehat{w}^\mu + B_{\mu-1}\widehat{w}^{\mu-1} + ... + B_1\widehat{w} + B_0(z) = 0.$$

Without loss of generality, let $\mu \leq \nu, \overline{n}_\Delta(r,a)$ denotes the counting function of the common values of $w = a$ and $\widehat{w} = a$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ ($1 < R_0 \leq +\infty$), ignoring multiplicity. And let

$$\overline{N}_\Delta(r,a) = \frac{\mu+\nu}{2\mu\nu}\int_{\frac{1}{r}}^{1} \frac{\overline{n}_{\Delta_1}(t,a)}{t}dt + \frac{\mu+\nu}{2\mu\nu}\int_{1}^{r} \frac{\overline{n}_{\Delta_2}(t,a)}{t}dt$$

$$\overline{N}_{12}(r,a) = \overline{N}_0\left(r, \frac{1}{w-a}\right) + \overline{N}_0\left(r, \frac{1}{\widehat{w}-a}\right) - 2\overline{N}_\Delta(r,a).$$

Let w be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$, where $1 < R_0 \leq +\infty$, and a be a complex number in the extended complex plane. Write $E(a,w) = \{z \in \mathbb{A} : w - a = 0\}$, where each zero with multiplicity m is counted m times. If we ignore the multiplicity, then the set is denoted by $\overline{E}(a,w)$. We use $\overline{E}_{k)}(a,w)$ to denote the set of zeros of $w - a$ with multiplicities not greater than k, in which each zero is counted only once.

In this paper, we say that two algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, share a function a if we have $w - a = 0$ if and only if $\widehat{w} - a = 0$. Now we consider the case that two algebroid function partially share small functions.

Definition 3.1. Let w be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ and a be a small function of w. We define

$$\overline{E}(a, w) = \{z | w - a = 0\}$$

in which each zero is counted only once.

Lemma 3.5. *Let w be ν-valued algebroid function which is determined by* (2.1) *on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ and $a_1, a_2, ..., a_q$ be $q(\geq 2\nu + 1)$ distinct complex numbers. If for a non-negative integer n, $E(0; w) \subseteq E(0, w^{(n)})$, then*

$$(q - 2\nu + o(1))T_0(r, w) \leq \sum_{j=1}^{q} \overline{N}_0\left(r, \frac{1}{w^{(n)} - a_j}\right).$$

Proof. By Nevanlinna's first fundamental theorem for algebroid functions on annuli, we have

$$\begin{aligned} T_0(r, w) &= T_0\left(r, \frac{1}{w}\right) + O(1) \\ &\leq N_0\left(r, \frac{1}{w}\right) + m_0\left(r, \frac{w^{(n)}}{w}\right) + m_0\left(r, \frac{1}{w^{(n)}}\right) + O(1) \\ &\leq N_0\left(r, \frac{1}{w}\right) + T_0(r, w^{(n)}) - N_0\left(r, \frac{1}{w^{(n)}}\right) + S_0(r, w). \qquad (3.3) \end{aligned}$$

By the Nevanlinna's second fundamental theorem for algebroid functions on annuli, we get

$$(q - 1)\, T_0(r, w^{(n)}) \leq \overline{N}_0(r, w^{(n)}) + \sum_{j=1}^{q-1} \overline{N}_0\left(r, \frac{1}{w^{(n)} - a_j}\right) + \overline{N}_0\left(r, \frac{1}{w^{(n)}}\right) + S_0(r, w).$$

Without loss of generality, we may assume that $a_q = 0$. Otherwise a suitable linear transformation is done. Then the above inequality reduces to

$$(q - 1)\, T_0(r, w^{(n)}) \leq \overline{N}_0(r, w^{(n)}) + \sum_{j=1}^{q} \overline{N}_0\left(r, \frac{1}{w^{(n)} - a_j}\right) + S_0(r, w). \qquad (3.4)$$

Using (3.4) in (3.3), we obtain

$$\begin{aligned}(q-1)T_0(r,w) &\leq (q-1)\,T_0\left(r,\frac{1}{w}\right)+\overline{N}_0(r,w^{(n)})+\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{w^{(n)}-a_j}\right)\\ &\quad -(q-1)N_0\left(r,\frac{1}{w^{(n)}}\right)+S_0(r,w).\end{aligned}$$

Thus

$$\begin{aligned}(q-1)T_0(r,w) &\leq (q-1)\,T_0\left(r,\frac{1}{w}\right)+\overline{N}_0(r,w)+\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{w^{(n)}-a_j}\right)\\ &\quad -(q-1)N_0\left(r,\frac{1}{w^{(n)}}\right)+S_0(r,w).\end{aligned} \tag{3.5}$$

Since $E(0,w)\subseteq E(0,w^{(n)})$, we have from (3.5)

$$(q-1)T_0(r,w)\leq \overline{N}_0(r,w)+\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{w^{(n)}-a_j}\right)+S_0(r,w).$$

Hence

$$(q-2\nu+o(1))T_0(r,w)\leq \sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{w^{(n)}-a_j}\right).$$

□

This completes the proof of the Lemma 3.5.

4. Main Results

By using the partly shared values and the notion of the weakly shared values and the fundamental theorems of algebroid functions on annuli, in this paper we have obtained some interesting and important new results concerning the uniqueness question of the n-order derivatives of algebroid functions on the annuli based upon some other assumptions.

Now we state and prove our main result in the following way

Theorem 4.1. *Let w_1 and w_2 be two ν-valued and μ-valued algebroid functions determined by* (2.1) *on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\leq+\infty)$, respectively and $\mu\leq\nu$,*

let a_j $(j = 1, 2, ..., q)$ *be* $q \geq 4\nu + 1$ *distinct complex numbers or* ∞. *Suppose that* $k_1 \geq k_2 \geq ... \geq k_q$ *are positive integers or* ∞ *and* $\delta_j(\geq 0)(j = 1, 2, ..., q)$ *are such that*

$$\frac{1}{k_1} + \left(1 + \frac{1}{k_m}\right) \sum_{j=2\nu}^{q} \frac{1}{1+k_j} + 1 + \delta < \frac{q-2\nu}{n+1}\left(1 + \frac{1}{k_1}\right).$$

for a non-negative integer n. *Let* $B_j = \overline{E}_{k_j}(a_j, w_1) \backslash \overline{E}_{k_j}(a_j, w_2)$ *for* $j = 1, 2\nu, ..., q$ *and* $E(0, w_i) \subseteq E(0, w_i^{(n)})$ *for* $i = 1, 2$. *If*

$$\overline{N}_0^{B_j}\left(r, \frac{1}{w_1^{(n)} - a_j}\right) \leq \delta_j T_0(r, w_1^{(n)})$$

and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{N}_0^{k_j)}(r, \frac{1}{w_1^{(n)}-a_j})}{\sum_{j=1}^{q} \overline{N}_0^{k_j)}(r, \frac{1}{w_2^{(n)}-a_j})}$$

$$> \frac{(n+1)k_1}{(p-2\nu)(1+k_1) - (n+1)(1+k_1)\sum_{j=2\nu}^{q} \frac{1}{1+k_j} - (n+1)\{(1+\delta)k_1 + 1\}},$$

then $w_1^{(n)} \equiv w_2^{(n)}$.

Proof. By Lemma 3.5, we have

$$(q - 2\nu + o(1))T_0(r, w_1) \leq \sum_{j=1}^{q} \overline{N}_0\left(r, \frac{1}{w_1^{(n)} - a_j}\right) \tag{4.1}$$

and

$$(q - 2\nu + o(1))T_0(r, w_2) \leq \sum_{j=1}^{q} \overline{N}_0\left(r, \frac{1}{w_2^{(n)} - a_j}\right). \tag{4.2}$$

From (4.1), we have

$$\begin{aligned}
&(q-2\nu+o(1))T_0(r,w_1)\\
\leq\ & \sum_{j=1}^{q}\left\{\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)+\overline{N}_0^{(k_j+1}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)\right\}\\
\leq\ & \sum_{j=1}^{q}\left\{\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)+\frac{1}{1+k_j}N_0^{(k_q+1}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)\right\}\\
\leq\ & \sum_{j=1}^{q}\left\{\frac{k_j}{1+k_j}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)+\frac{1}{1+k_j}N_0\left(r,\frac{1}{w_1^{(n)}-a_j}\right)\right\}\\
\leq\ & \sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)+\sum_{j=1}^{q}\frac{1}{1+k_j}T_0\left(r,w_1^{(n)}\right)\\
\leq\ & \sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)+(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}T_0\left(r,w_1^{(n)}\right).
\end{aligned}$$

Therefore

$$(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1))T_0(r,w_1)\leq\sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right).$$

Similarly from (4.2), we get

$$(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1))T_0(r,w_2)\leq\sum_{j=1}^{j}\frac{k_j}{1+k_j}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_2^{(n)}-a_j}\right).$$

Let $B_j=\overline{E}_{k_j}(a_j,w_1^{(n)})\backslash A_j$ for $j=1,2\nu,...,q$.

Now

$$\begin{aligned}
\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right) &= \sum_{j=1}^{q}\overline{N}_0^{A_j}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)+\sum_{j=1}^{q}\overline{N}_0^{B_j}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)\\
&\leq \delta T_0(r,w_1^{(n)})+N_0\left(r,\frac{1}{w_1^{(n)}-w_2^{(n)}}\right)\\
&\leq (1+\delta)(n+1)T_0(r,w_1)+\sum_{j=1}^{k-1}\overline{N}_{12}(r,a_j)+2\sum_{j=1}^{k-1}\overline{N}_{\Delta}(r,a_j). \qquad (4.3)
\end{aligned}$$

If $w_1^{(n)}\not\equiv w_2^{(n)}$, then we have

$$\sum\overline{n}_{\Delta}(r,a)\leq n_0\left(r,\frac{1}{R(\varphi,\psi)}\right),$$

$R(\varphi, \psi)$ denotes the resultant of $\varphi(z, w^{(n)})$ and $\psi(z, w^{(n)})$, it can be written as the following

$$R(\varphi,\psi) = [A_\nu]^\mu [B_\mu]^\nu \prod_{\substack{1\le j\le \nu \\ 1\le k\le \mu}} [w_j^{(n)} - \widehat{w}_j^{(n)}].$$

It can be written in the another form

$$R(\varphi,\psi) = \begin{vmatrix} A_\nu & A_{\nu-1} & \dots & \dots & A_0 & 0 & \dots & 0 \\ 0 & A_\nu & A_{\nu-1} & \dots & A_1 & A_0 & \dots & 0 \\ \vdots & \vdots & & & \vdots & & & \\ 0 & 0 & 0 & A_\nu & A_{\nu-1} & \dots & \dots & A_0 \\ B_\mu & B_{\mu-1} & \dots & \dots & B_0 & 0 & \dots & 0 \\ 0 & B_\mu & B_{\mu-1} & \dots & B_1 & B_0 & \dots & 0 \\ \vdots & \vdots & & & \vdots & & & \\ 0 & 0 & 0 & B_\mu & B_{\mu-1} & \dots & \dots & B_0 \end{vmatrix}$$

So we know that $R(\varphi, \psi)$ is a holomorphic function and using Jensen Theorem for meromorphic function on annuli, we have

$$\begin{aligned}
& N_0\left(r, \frac{1}{R(\varphi,\psi)}\right) \\
= \ & \frac{1}{2\pi}\int_0^{2\pi} \log|R[\psi(re^{i\theta}, w_1^{(n)}), \varphi(re^{i\theta}, w_2^{(n)})]|d\theta \\
+ \ & \frac{1}{2\pi}\int_0^{2\pi} \log\left|R\left[\psi\left(\frac{1}{r}e^{i\theta}, w_1^{(n)}\right), \varphi\left(\frac{1}{r}e^{i\theta}, w_2^{(n)}\right)\right]\right| d\theta \\
+ \ & 2\frac{1}{2\pi}\int_0^{2\pi} \log|R[\psi(e^{i\theta}, w_1^{(n)}), \varphi(e^{i\theta}, w_2^{(n)})]|d\theta \\
= \ & \frac{\mu}{2\pi}\int_0^{2\pi} \log|A_\nu(re^{i\theta})|d\theta + \frac{\nu}{2\pi}\int_0^{2\pi} \log|B_\mu(re^{i\theta})|d\theta \\
+ \ & \frac{1}{2\pi}\int_0^{2\pi} \log\left|\prod_{\substack{1\le j\le \nu \\ 1\le k\le \mu}} [w_j^{(n)}(re^{i\theta}) - \widehat{w}_j^{(n)}(re^{i\theta})]\right| d\theta \\
+ \ & \frac{\mu}{2\pi}\int_0^{2\pi} \log\left|A_\nu\left(\frac{1}{r}e^{i\theta}\right)\right| d\theta + \frac{\nu}{2\pi}\int_0^{2\pi} \log\left|B_\mu\left(\frac{1}{r}e^{i\theta}\right)\right| d\theta \\
+ \ & \frac{1}{2\pi}\int_0^{2\pi} \log\left|\prod_{\substack{1\le j\le \nu \\ 1\le k\le \mu}} \left[w_j^{(n)}\left(\frac{1}{r}e^{i\theta}\right) - \widehat{w}_j^{(n)}\left(\frac{1}{r}e^{i\theta}\right)\right]\right| d\theta - 2.\frac{\mu}{2\pi}\int_0^{2\pi} \log|A_\nu(e^{i\theta})|d\theta \\
- \ & 2\frac{\nu}{2\pi}\int_0^{2\pi} \log|B_\mu(e^{i\theta})|d\theta - 2\frac{1}{2\pi}\int_0^{2\pi} \log\left|\prod_{\substack{1\le j\le \nu \\ 1\le k\le \mu}} [w_j^{(n)}(e^{i\theta}) - \widehat{w}_j^{(n)}(e^{i\theta})]\right| d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu}{2\pi}\int_0^{2\pi}\log|A_\nu(re^{i\theta})|d\theta + \frac{\mu}{2\pi}\int_0^{2\pi}\log\left|A_\nu\left(\frac{1}{r}e^{i\theta}\right)\right|d\theta - 2\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_\nu(e^{i\theta})|d\theta \\
&+ \frac{\nu}{2\pi}\int_0^{2\pi}\log|B_\mu(re^{i\theta})|d\theta + \frac{\nu}{2\pi}\int_0^{2\pi}\log\left|B_\mu\left(\frac{1}{r}e^{i\theta}\right)\right|d\theta - 2\frac{\nu}{2\pi}\int_0^{2\pi}\log|B_\mu(e^{i\theta})|d\theta \\
&+ \frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}[w_j^{(n)}(re^{i\theta}) - \widehat{w}_j^{(n)}(re^{i\theta})]\right|d\theta \\
&+ \frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}\left[w_j^{(n)}\left(\frac{1}{r}e^{i\theta}\right) - \widehat{w}_j^{(n)}\left(\frac{1}{r}e^{i\theta}\right)\right]\right|d\theta \\
&- 2.\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}[w_j^{(n)}(e^{i\theta}) - \widehat{w}_j^{(n)}(e^{i\theta})]\right|d\theta \\
&\le \mu\left[m_0(r,A_\nu) - m_0\left(r,\frac{1}{A_\nu}\right)\right] + \nu\left[m_0(r,B_\mu) - m_0\left(r,\frac{1}{B_\mu}\right)\right] \\
&\quad +\mu\nu[m_0(r,w_1^{(n)}) + m_0(r,w_2^{(n)})] + O(1) \\
&= \mu\nu[T_0(r,w_1^{(n)}) + T_0(r,w_2^{(n)})] + O(1).
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\sum \overline{N}_\Delta(r,a_j) &\le \frac{2\mu\nu}{\mu+\nu}[T_0(r,w_1^{(n)}) + T_0(r,w_2^{(n)})] + O(1) \\
&\le (n+1)\nu[T_0(r,w_1) + T_0(r,w_2)] + O(1).
\end{aligned}
\tag{4.4}
$$

By the condition of Theorem 4.1, we know that the set of zeros of $w_1 - a_j$ and $w_2 - a_j$ in which each point counts only once, at the same time we get $\overline{N}_{12}(r,a_j) = 0$.

Therefore Therefore

$$
\begin{aligned}
\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{w_1^{(n)} - a_j}\right) &\le \sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{w_1^{(n)} - w_2^{(n)}}\right) \\
&\le (n+1)\nu[T_0(r,w_1) + T_0(r,\widehat{w_2})] + O(1).
\end{aligned}
\tag{4.5}
$$

From (4.3) and (4.5), we have

$$
\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{w_1^{(n)} - a_j}\right) \le (1+\delta)(n+1)T_0(r,w_1) + (n+1)T_0(r,w_2). \tag{4.6}
$$

Hence

$$\begin{aligned}&\left(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1)\right)\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)\\ \leq\;&(1+\delta)(n+1)\sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)+(n+1)\sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_2^{(n)}-a_j}\right).\end{aligned}$$

Since $1\geq\frac{k_1}{k_1+1}\geq\frac{k_2}{k_2+1}\geq...\geq\frac{k_q}{k_q+1}\geq\frac{1}{2}$, we get from the above inequality

$$\begin{aligned}&\left(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1)\right)\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)\\ \leq\;&(1+\delta)(n+1)\frac{k_1}{1+k_1}\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)+(n+1)\frac{k_1}{1+k_1}\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_2^{(n)}-a_j}\right).\end{aligned}$$

Since that implies

$$\begin{aligned}&\left(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}-(1+\delta)(n+1)\frac{k_1}{1+k_1}+o(1)\right)\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_1^{(n)}-a_j}\right)\\ \leq\;&(n+1)\frac{k_1}{1+k_1}\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{w_2^{(n)}-a_j}\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\liminf_{r\to\infty}\frac{\sum_{j=1}^{q}\overline{N}_0^{k_j}(r,\frac{1}{w_1-a_j})}{\sum_{j=1}^{q}\overline{N}_0^{k_j}(r,\frac{1}{w_2-a_j})}\\ \leq\;&\frac{(n+1)k_1}{(q-2\nu)(1+k_1)-(n+1)(1+k_1)\sum_{j=1}^{q}\frac{1}{1+k_j}-(n+1)\{(1+\delta)k_1}\\ \leq\;&\frac{(n+1)k_1}{(q-2\nu)(1+k_1)-(n+1)(1+k_1)\sum_{j=2\nu}^{q}\frac{1}{1+k_j}-(n+1)\{(1+\delta)k_1+1}.\end{aligned}\tag{4.7}$$

Which is a contradiction.

Thus, we have $w_1^{(n)}\not\equiv w_2^{(n)}$.

Therefore we complete the proof of Theorem 4.1. □

From Theorem 4.1, we can get the following consequences.

Corollary 4.1. Let $k_j = \infty$ for $j = 1, 2\nu, ..., q$ and

$$\gamma = \liminf_{r\to\infty} \frac{\overline{N}_0^{k_j)}\left(r, \frac{1}{w_1^{(n)}-a_j}\right)}{\overline{N}_0^{k_j)}\left(r, \frac{1}{w_2^{(n)}-a_j}\right)} > \frac{n+1}{q-(n+2\nu+1)}.$$

If $\overline{N}_0^{A_j}(r, \frac{1}{w_1^{(n)}-a_j}) \leq \delta_j T_0(r, w_1^{(n)})$ where $\delta(\geq 0)$ satisfy $0 \leq \delta_j < \frac{q-(n+2\nu+1)}{n+1} - \frac{1}{\gamma}$.
If we assume $E_{\infty)}(a_j, w_1^{(n)}) \subseteq E_{\infty)}(a_j, w_2^{(n)})$, then $A_j = \phi$ for $j = 1, 2\nu, ..., q$ and so we can choose $\delta = 0$.

Therefore Theorem 4.1 is an improvement of following theorem

Theorem 4.2. *Let $w_1(z)$ and $w_2(z)$ be two ν-valued and μ-valued algebroid functions determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, respectively and $\mu \leq \nu$, let a_j $(j = 1, 2, ..., q)$ be $q \geq 4\nu + 1$ distinct complex numbers or ∞. and for a non-negative integer n, $E_{\infty)}(a_j, w_1^{(n)}) \subseteq E_{\infty)}(a_j, w_2^{(n)})$ for $1 \leq j \leq q$,$E_{\infty)}(0, w_1) \subseteq E_{\infty)}(0, w_1^{(n)})$,$E_{\infty)}(0, w_2) \subseteq E_{\infty)}(0, w_2^{(n)})$ and*

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{N}_0^{k_j)}(r, \frac{1}{w_1^{(n)}-a_j})}{\sum_{j=1}^{q} \overline{N}_0^{k_j)}(r, \frac{1}{w_2^{(n)}-a_j})} > \frac{(n+1)}{q-(n+2\nu+1)},$$

then $w_1^{(n)} \equiv w_2^{(n)}$.

Corollary 4.2. Let $n = 0$, $k_j = \infty$ for $j = 1, 2\nu, ..., q$ and

$$\gamma = \liminf_{r\to\infty} \frac{\overline{N}_0^{k_j)}\left(r, \frac{1}{w_1-a_j}\right)}{\overline{N}_0^{k_j)}\left(r, \frac{1}{w_2-a_j}\right)} > \frac{1}{q-2\nu+1}$$

If $\overline{N}_0^{B_j}(r, \frac{1}{w_1-a_j}) \leq \delta_j T_0(r, w_1)$ where $\delta(\geq 0)$ satisfy $0 \leq \sum_{j=1}^{q} \delta_j < k - (2\nu+1) - \frac{1}{\gamma}$, then $w_1(z) \equiv w_2(z)$.

If we take $q = 4\nu+1$ and $\overline{E}(a_j, w_1) \subseteq \overline{E}(a_j, w_2)$, then $A_j = \phi$ for $j = 1, 2, ..., 4\nu+1$. Therefore, if we choose $\delta_j = 0$ for $j = 1, 2, ..., 4\nu+1$ and take any constant γ, such that $0 \leq 2\nu - \frac{1}{\gamma}$ in Corollary 2; we can get that $w_1(z) \equiv w_2(z)$. Especially, if $q = 4\nu+1$ and $\overline{E}(a_j, w_1) = \overline{E}(a_j, w_2)$, then $\gamma = 1$ and $\delta_j = 0$ for $j = 1, 2, ..., 4\nu+1$. We can obtain $w_1 \equiv w_2$.

Corollary 4.3. Let $w_1(z)$ and $w_2(z)$ be two ν-valued and μ-valued algebroid functions determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, respectively and $\mu \leq \nu$, let $a_j\,(j = 1, 2, ..., q)$ be $q \geq 5$ distinct complex numbers or ∞. Suppose that $k_1, k_2, ..., k_q$ are positive integers or ∞; with $k_1 \geq k_2 \geq ... \geq k_q$ if $\overline{E}_{k_j)}(a_j, w_1) \subseteq \overline{E}_{k_j)}(a_j, w_2)$ and :

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - \frac{k_1}{\gamma(k_1+1)} - 2\nu > 0,$$

where γ is stated as in Corollary 4.2; then $w_1 \equiv w_2$.

Corollary 4.4. Let $w_1(z)$ and $w_2(z)$ be two ν-valued and μ-valued algebroid functions determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, respectively and $\mu \leq \nu$, let $a_j\,(j = 1, 2, ..., q)$ be $q \geq 5$ distinct complex numbers in $\mathbb{C} \cup \infty$. Suppose that $k_1, k_2, ..., k_q$ are positive integers or ∞; with $k_1 \geq k_2 \geq ... \geq k_q$ if $\overline{E}_{k_j)}(a_j, w_1) = \overline{E}_{k_j)}(a_j, w_2)$ and :

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - \frac{k_1}{(k_1+1)} - 2\nu > 0,$$

then $w_1 \equiv w_2$.

Corollary 4.4 is an extension of Theorem 2.4 and also from Corollary 4.4 we obtain Nevanlinna's five-value theorem as follows

Theorem 4.3. *Let $w_1(z)$ and $w_2(z)$ be two ν-valued and μ-valued algebroid functions determined by* (2.1) *on the annulus* $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, *respectively and $\mu \leq \nu$, let $a_j\,(j = 1, 2, ..., 5)$ be* 5 *distinct complex numbers in* $\mathbb{C} \cup \infty$. *If $\overline{E}_{k_j)}(a_j, w_1) = \overline{E}_{k_j)}(a_j, w_2)$ for $j = 1, 2, ..., 5$, then then $w_1 \equiv w_2$.*

Corollary 4.5. Let w_1 and w_2 be two ν-valued and μ-valued algebroid functions determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, respectively and $\mu \leq \nu$, let $a_j\,(j = 1, 2, ..., q)$ be $q \geq 4\nu + 1$ distinct complex numbers or ∞. Suppose that $k_1, k_2, ..., k_q$ are positive integers or ∞; with $k_1 \geq k_2 \geq ... \geq k_q$ if $\overline{E}_{k_j)}(a_j, w_1) \subseteq$

$\overline{E}_{k_j)}(a_j, w_2)$ and :

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - 2\nu + \frac{(m-2\nu-\frac{1}{\gamma})k_m}{\gamma(k_m+1)} - 2\nu > 0,$$

where γ is stated as in Corollary 4.2; then $w_1 \equiv w_2$.

In Corollary 4.1 if $n = 0$ and $q = 4\nu + 1$ then we get the following theorem

Theorem 4.4. *Let $w_1(z)$ and $w_2(z)$ be two ν-valued and μ-valued algebroid functions determined by* (2.1) *on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, respectively and $\mu \leq \nu$ such that $E_{\infty)}(a_j, w_1) \subseteq E_{\infty)}(a_j, w_2)$ for $a_1, a_2, ..., a_5$ of $\mathbb{C} \cup \infty$. If*

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{4\nu+1} \overline{N}_0^{k_j)}(r, \frac{1}{w_1-a_j})}{\sum_{j=1}^{4\nu+1} \overline{N}_0^{k_j)}(r, \frac{1}{w_2-a_j})} > \frac{1}{2\nu},$$

then $w_1(z) \equiv w_2(z)$.

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