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COMPLEX LINEAR DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS OF ITERATED ORDER IN THE ANNULUS

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Abstract. In this paper, we study the growth properties of solutions of the linear differential equations

$$f^{(k)} + B_{k-1}(z) f^{(k-1)} + \cdots + B_1(z) f' + B_0(z) f = 0,$$

$$f^{(k)} + B_{k-1}(z) f^{(k-1)} + \cdots + B_1(z) f' + B_0(z) f = F,$$

where $B_{k-1}(z), ..., B_0(z)$ and $F(z)$ are analytic functions of iterated order in an annulus. We obtain some results concerning the estimates of the iterated order of solutions of the above equations.

1. Introduction and results

Throughout this article, we shall assume that the reader is familiar with the standard notations and fundamental results of the Nevanlinna value distribution theory of meromorphic functions in the complex plane and in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (see [4], [5], [11], [15], [18]).

Several authors have investigated the growth properties of solutions in the complex plane, in the unit disc and in a sector of the unit disc which are simple connected domains, by using the theory of value distribution of Nevanlinna [3, 6, 12, 16, 19]. It is well-known that Nevanlinna theory of meromorphic functions can be extended in a modified form to multiply-connected plane domains, in particular in the annulus [7, 8, 9, 10, 13, 14] which is a doubly-connected domain. In 2005, Khrystiyanyn and Kondratyuk [7, 8] gave an extension of the Nevanlinna value distribution theory

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for meromorphic functions in annuli. In their extension the main characteristics of meromorphic functions are one-parameter and possess the same properties as in the classical case of a simply connected domain. From the doubly-connected mapping theorem [1], we can get that each doubly-connected domain is conformally equivalent to the annulus $\{z : r < |z| < R,\ 0 \le r < R \le +\infty\}$. We consider only two cases: $r = 0$, $R = +\infty$ simultaneously and $0 \le r < R \le +\infty$. In the latter case, the homothety $z \longmapsto \frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\frac{1}{R_0} < |z| < R_0$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, every annulus is invariant with respect to the inversion $z \longmapsto \frac{1}{z}$ in two cases.

Before stating our main results, we give some notations and basic definitions of the theory of Nevanlinna of meromorphic functions in the complex plane and then in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R_0 \le +\infty$. Let f be a meromorphic function in the complex plane, we define

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\varphi}\right)\right| d\varphi,$$

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t}dt + n(0,f)\log r$$

and

$$T(r,f) = m(r,f) + N(r,f) \ \ (r > 0)$$

is the Nevanlinna characteristic function of f, where

$$\log^+ x = \max(0, \log x) = \begin{cases} \log x, \ x > 1, \\ 0, \ 0 \le x \le 1 \end{cases}$$

and $n(t,\ f)$ is the number of poles of f lying in $\{z : |z| \le t\}$, counted according to their multiplicity. Now, we give some basic notions of the Nevanlinna theory in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R_0 \le +\infty$. Set

$$N_1(r,f) = \int_{\frac{1}{r}}^1 \frac{n_1(t,f)}{t}dt, \quad N_2(r,f) = \int_1^r \frac{n_2(t,f)}{t}dt,$$

$$m_0(r,f) = m(r,f) + m\left(\frac{1}{r}, f\right) - 2m(1,f),$$

$$N_0(r,f) = N_1(r,f) + N_2(r,f),$$

where $n_1(t,f)$ and $n_2(t,f)$ are the counting functions of poles of f lying in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$ respectively, counted according to their multiplicity. The Nevanlinna characteristic of f in the annulus $\mathcal{A}$ is defined by

$$T_0(r,f) = m_0(r,f) + N_0(r,f).$$

Definition 1.1. ([17]) Let f be a nonconstant meromorphic function in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R_0 \leq +\infty$. The function f is called a transcendental or an admissible function in $\mathcal{A}$ provided that

$$\limsup_{r\to+\infty} \frac{T_0(r,f)}{\log r} = +\infty \text{ if } 1 < r < R_0 = +\infty$$

or

$$\limsup_{r\to R_0^-} \frac{T_0(r,f)}{\log \frac{1}{R_0 - r}} = +\infty \text{ if } 1 < r < R_0 < +\infty$$

respectively.

For all $r \in \mathbb{R}$, we define $\exp_1 r = \exp r = e^r$ and $\exp_{p+1} r = \exp(\exp_p r)$, $p \in \mathbb{N} = \{1, 2, 3, \dots\}$. Inductively, for all $r \in (0, +\infty)$ large enough, we define $\log_1 r = \log r$ and $\log_{p+1} r = \log(\log_p r), p \in \mathbb{N}$. We also denote $\exp_0 r = r = \log_0 r$, $\exp_{-1} r = \log_1 r$ and $\log_{-1} r = \exp_1 r$.

Definition 1.2. Let $p \geq 1$ be an integer and f be a nonconstant meromorphic function in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R_0 \leq +\infty$. The iterated $p-$order of f is defined as

$$\rho_{p,\mathcal{A}}(f) = \limsup_{r\to+\infty} \frac{\log_p T_0(r,f)}{\log r} \text{ if } 1 < r < R_0 = +\infty$$

or

$$\rho_{p,\mathcal{A}}(f) = \limsup_{r\to R_0^-} \frac{\log_p T_0(r,f)}{\log \frac{1}{R_0 - r}} \text{ if } 1 < r < R_0 < +\infty.$$

Remark 1.1 For $p = 1$, this notation is called order and for $p = 2$ hyper-order, see [17].

Definition 1.3 The finiteness degree of the order of a meromorphic function f is defined by

$$i_{\mathcal{A}}(f) := \begin{cases} 0, & \text{if } f \text{ is non admissible,} \\ \min\{j \in \mathbb{N} : \rho_{j,\mathcal{A}}(f) < \infty\}, & \text{if } f \text{ is admissible and } \rho_{j,\mathcal{A}}(f) < \infty \\ & \text{for some } j \in \mathbb{N}, \\ +\infty, & \text{if } \rho_{j,\mathcal{A}}(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

For $k \geq 2$, we consider the linear differential equations

$$f^{(k)} + B_{k-1}(z) f^{(k-1)} + \cdots + B_1(z) f' + B_0(z) f = 0, \tag{1.1}$$

$$f^{(k)} + B_{k-1}(z) f^{(k-1)} + \cdots + B_1(z) f' + B_0(z) f = F, \tag{1.2}$$

where $B_{k-1}(z), ..., B_0(z)$ and $F(z)$ are analytic in the annulus

$$\mathcal{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\} \ (1 < R_0 \leq +\infty).$$

Recently in [17], Wu and Xuan have studied the growth properties of solutions of higher order linear complex differential equations in $\mathcal{A}$ and obtained the following results.

Theorem A. ([17]) *Let $B_{k-1}(z), ..., B_1(z)$, $B_0(z)$ be analytic functions in the annulus $\mathcal{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\}$ $(1 < R_0 \leq +\infty)$ that satisfy*

$$\max\{\rho_{\mathcal{A}}(B_j) : j = 1, 2, ..., k-1\} < \rho_{\mathcal{A}}(B_0).$$

Then every solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{\mathcal{A}}(f) = +\infty$ and $\rho_{2,\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_0)$.

Theorem B. ([17]) *Let $B_{k-1}(z), ..., B_1(z)$, $B_0(z)$ be analytic functions in the annulus $\mathcal{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\}$ $(1 < R_0 \leq +\infty)$ that satisfy*

$$\max_{0 \leq j \leq k-1, j \neq l} \{\rho_{\mathcal{A}}(B_j)\} < \rho_{\mathcal{A}}(B_l).$$

Then every solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{\mathcal{A}}(f) \geq \rho_{\mathcal{A}}(B_l)$.

Remark 1.2 Hypothesis of Theorem B do not provide that a solution is an admissible in $\mathcal{A}$, so it is a priori assumed that f is an admissible.

In this paper, by using the concept of iterated order, we obtain some results which extend and improve Theorems A-B from usual order to iterated order for every non-trivial analytic solution of equations (1.1) and (1.2). We mainly obtain the following results.

Theorem 1.1 *Let $p \geq 1$ be an integer and $B_{k-1}(z), ..., B_1(z)$, $B_0(z)$ be analytic functions in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$ $(1 < R_0 \leq +\infty)$ such that*

$$\max\{\rho_{p,\mathcal{A}}(B_j) : j = 1, 2, ..., k-1\} < \rho_{p,\mathcal{A}}(B_0).$$

Then every solution $f \not\equiv 0$ of equation (1.1) satisfies $\rho_{p,\mathcal{A}}(f) = +\infty$ and $\rho_{p+1,\mathcal{A}}(f) \geq \rho_{p,\mathcal{A}}(B_0)$.

Remark 1.3 Setting $p = 1$ in Theorem 1.1, we obtain Theorem A.

Theorem 1.2 *Let $p \geq 2$ be an integer and $B_{k-1}(z), ..., B_1(z)$, $B_0(z)$ be analytic functions in the annulus $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$ $(1 < R_0 \leq +\infty)$. Suppose that there exist three positive real numbers α, β and μ with $0 \leq \beta < \alpha$, $\mu > 0$, such that we have*

$$T_0(r, B_0) \geq \exp_{p-1}\{\alpha r^{\mu}\} \tag{1.3}$$

and

$$T_0(r, B_j) \leq \exp_{p-1}\{\beta r^{\mu}\}, \ j = 1, \ldots, k-1 \tag{1.4}$$

if $1 < r < R_0 = +\infty$ as $|z| = r \to +\infty$ for $r \in E_r$ which satisfies $\int\limits_{E_r} \frac{dr}{r} = +\infty$, or

$$T_0(r, B_0) \geq \exp_{p-1}\left\{\frac{\alpha}{(R_0 - r)^{\mu}}\right\} \tag{1.5}$$

and

$$T_0(r, B_j) \leq \exp_{p-1}\left\{\frac{\beta}{(R_0 - r)^{\mu}}\right\} \quad (j = 1, \cdots, k-1) \tag{1.6}$$

if $1 < r < R_0 < +\infty$ *as* $|z| = r \to R_0^-$ *for* $r \in F_r$ *which satisfies* $\int\limits_{F_r} \frac{dr}{R_0 - r} = +\infty$. *Then every solution* $f \not\equiv 0$ *of equation* (1.1) *satisfies* $\rho_{p,\mathcal{A}}(f) = +\infty$ *and* $\rho_{p+1,\mathcal{A}}(f) \geq \mu$.

Remark 1.4 In [2], the Theorem 1.2 was obtained for $p = 1$ but under the condition $0 \leq (k-1)\beta < \alpha$ instead of $0 \leq \beta < \alpha$.

Theorem 1.3 *Let* $p \geq 1$ *be an integer, let* $B_{k-1}(z), ..., B_1(z)$, $B_0(z)$ *and* $F(z)$ *be analytic functions in the annulus* $\mathcal{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$ $(1 < R_0 \leq +\infty)$ *such that for some integer* s, $1 \leq s \leq k-1$, *we have* $\max\{\rho_{p,\mathcal{A}}(B_j)\ (j \neq s), \rho_{p,\mathcal{A}}(F)\} < \rho_{p,\mathcal{A}}(B_s)$. *Then every an admissible solution* f *of equation* (1.2) *satisfies* $\rho_{p,\mathcal{A}}(f) \geq \rho_{p,\mathcal{A}}(B_s)$.

Remark 1.5 Setting $p = 1$ and $F(z) \equiv 0$ in Theorem 1.3, we obtain Theorem B.

Theorem 1.4 *Let* $p \geq 1$ *be an integer, let* $B_{k-1}(z), ..., B_1(z)$, $B_0(z)$ *and* $F(z)$ *be analytic functions in the annulus* $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$ $(1 < R_0 \leq +\infty)$ *such that for some integer* s, $0 \leq s \leq k-1$, *we have* $\rho_{p,\mathcal{A}}(B_s) = \infty$ *and* $\max\{\rho_{p,\mathcal{A}}(B_j)\ (j \neq s), \rho_{p,\mathcal{A}}(F)\} < \infty$. *Then every an admissible solution* f *of equation* (1.2) *satisfies* $\rho_{p,\mathcal{A}}(f) = \infty$.

2. Some Auxiliary Lemmas

We need the following lemmas to prove our results.

Lemma 2.1 [8, 17] (The lemma of the logarithmic derivative). *Let* f *be a nonconstant meromorphic function in the annulus* $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, *where* $1 < r < R_0 \leq +\infty$ *and* $k \geq 1$ *be an integer. Then*

$$m_0\left(r, \frac{f^{(k)}}{f}\right) = \begin{cases} O(\log r), \ R_0 = +\infty \text{ and } \rho_{\mathcal{A}}(f) < +\infty, \\ O\left(\log \frac{1}{R_0 - r}\right), \ R_0 < +\infty \text{ and } \rho_{\mathcal{A}}(f) < +\infty, \\ O(\log r + \log T_0(r, f)), \ r \notin \Delta_r, \ R_0 = +\infty, \\ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right), \ r \notin \Delta_r', \ R_0 < +\infty, \end{cases}$$

where $\Delta_r \subset (1,+\infty)$ *and* $\Delta'_r \subset (1,R_0)$ *are sets with* $\int\limits_{\Delta_r} \frac{dr}{r} < +\infty$ *and* $\int\limits_{\Delta'_r} \frac{dr}{R_0-r} < +\infty$ *respectively.*

In the next, we give the generalized logarithmic derivative lemma.

Lemma 2.2 *Let* $p \geq 1$ *be an integer and* f *be a meromorphic function in the annulus* $\mathcal{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$ $(1 < R_0 \leq +\infty)$ *such that* $\rho_{p,\mathcal{A}}(f) = \rho < \infty$, *and let* $k \geq 1$ *be an integer. Then for any given* $\varepsilon > 0$,

$$m_0\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{r^{\rho+\varepsilon}\right\}\right), \quad \text{if } 1 < r < R_0 = +\infty$$

holds outside a set $\Delta_r \subset (1,+\infty)$ *with* $\int\limits_{\Delta_r} \frac{dr}{r} < +\infty$, *or*

$$m_0\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{R_0 - r}\right\}^{\rho+\varepsilon}\right), \text{ if } 1 < r < R_0 < +\infty$$

holds outside a set $\Delta'_r \subset (1,R_0)$ *with* $\int_{\Delta'_r} \frac{dr}{R_0-r} < +\infty$.

Proof. Case $R_0 = +\infty$. First for $k = 1$. Since $\rho_{p,\mathcal{A}}(f) = \rho < \infty$, then for any given $\varepsilon > 0$ and sufficiently large r, we have

$$T_0(r,f) \leq \exp_{p-1}\left\{r^{\rho+\varepsilon}\right\}. \tag{2.1}$$

By Lemma 2.1, we have

$$m_0\left(r, \frac{f'}{f}\right) = O\left(\log r + \log T_0(r,f)\right) \tag{2.2}$$

holds for all r outside a set Δ_r with $\int\limits_{\Delta_r} \frac{dr}{r} < +\infty$. Hence, by (2.1) and (2.2) we obtain

$$m_0\left(r, \frac{f'}{f}\right) = O\left(\exp_{p-2}\left\{r^{\rho+\varepsilon}\right\}\right), \ r \notin \Delta_r. \tag{2.3}$$

Next, we assume that we have

$$m_0\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{r^{\rho+\varepsilon}\right\}\right), \ r \notin \Delta_r \tag{2.4}$$

for a certain integer $k \geq 1$. Since $N_0\left(r, f^{(k)}\right) \leq (k+1) N_0(r,f)$, it holds that

$$T_0\left(r, f^{(k)}\right) = m_0\left(r, f^{(k)}\right) + N_0\left(r, f^{(k)}\right)$$

$$\leq m_0\left(r, \frac{f^{(k)}}{f}\right) + m_0(r,f) + (k+1) N_0(r,f)$$

$$\leq m_0\left(r, \frac{f^{(k)}}{f}\right) + (k+1)\,T_0\,(r,f) = O\left(\exp_{p-1}\left\{r^{\rho+\varepsilon}\right\}\right). \tag{2.5}$$

By (2.2) and (2.5), we again obtain

$$m_0\left(r, \frac{\left(f^{(k)}\right)'}{f^{(k)}}\right) = O\left(\log r + \log T_0\left(r, f^{(k)}\right)\right) = O\left(\exp_{p-2}\left\{r^{\rho+\varepsilon}\right\}\right),\ r \notin \Delta_r$$

and hence,

$$m_0\left(r, \frac{f^{(k+1)}}{f}\right) \leq m_0\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + m_0\left(r, \frac{f^{(k)}}{f}\right)$$
$$= O\left(\exp_{p-2}\left\{r^{\rho+\varepsilon}\right\}\right),\ r \notin \Delta_r.$$

Case $R_0 < +\infty$. First for $k = 1$. Since $\rho_{p,\mathcal{A}}(f) = \rho < \infty$, then for any given $\varepsilon > 0$ and $r \to R_0^-$, we have

$$T_0\,(r,f) \leq \exp_{p-1}\left\{\frac{1}{R_0 - r}\right\}^{\rho+\varepsilon}. \tag{2.6}$$

Again, by Lemma 2.1, we have

$$m_0\left(r, \frac{f'}{f}\right) = O\left(\log\frac{1}{R_0 - r} + \log T_0\,(r,f)\right) \tag{2.7}$$

holds for all r outside a set Δ_r' with $\int_{\Delta_r'} \frac{dr}{R_0 - r} < +\infty$. Hence, by (2.6) and (2.7) we obtain

$$m_0\left(r, \frac{f'}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{R_0 - r}\right\}^{\rho+\varepsilon}\right),\ r \notin \Delta_r'. \tag{2.8}$$

Next, we assume that we have

$$m_0\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{R_0 - r}\right\}^{\rho+\varepsilon}\right),\ r \notin \Delta_r' \tag{2.9}$$

for a certain integer $k \geq 1$. Since $N_0\left(r, f^{(k)}\right) \leq (k+1)\,N_0\,(r,f)$, we deduce

$$T_0\left(r, f^{(k)}\right) \leq m_0\left(r, \frac{f^{(k)}}{f}\right) + (k+1)\,T_0\,(r,f) = O\left(\exp_{p-1}\left\{\frac{1}{R_0 - r}\right\}^{\rho+\varepsilon}\right). \tag{2.10}$$

By (2.7) and (2.10), we again obtain

$$m_0\left(r, \frac{\left(f^{(k)}\right)'}{f^{(k)}}\right) = O\left(\log\frac{1}{R_0 - r} + \log T_0\left(r, f^{(k)}\right)\right)$$

$$= O\left(\exp_{p-2}\left\{\frac{1}{R_0 - r}\right\}^{\rho+\varepsilon}\right),\ r \notin \Delta_r'$$

and therefore

$$m_0\left(r,\frac{f^{(k+1)}}{f}\right)\leq m_0\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)+m_0\left(r,\frac{f^{(k)}}{f}\right)$$
$$=O\left(\exp_{p-2}\left\{\frac{1}{R_0-r}\right\}^{\rho+\varepsilon}\right),\ r\notin\Delta_r'.$$

□

Lemma 2.3 *Let f be a meromorphic function with finite iterated $p-$order $\rho_{p,\mathcal{A}}(f)<+\infty$. Then, for any set E_r of $(1,+\infty)$ with $\int\limits_{E_r}\frac{dr}{r}<+\infty$, there exists a sequence $\{r_n, r_n\notin E_r\}$ such that*

$$\lim_{r_n\to+\infty}\frac{\log_p T_0\left(r_n,f\right)}{\log r_n}=\rho_{p,\mathcal{A}}\left(f\right)\ \textit{ if }\ 1<r_n<R_0=+\infty,$$

or for any set $E_r^{/}$ of $(1,R_0)$ with $\int\limits_{E_r^{/}}\frac{dr}{R_0-r}<+\infty$, there exists a sequence $\{r_n^{/}, r_n^{/}\notin E_r^{/}\}$ such that

$$\lim_{r_n^{/}\to R_0^-}\frac{\log_p T_0\left(r_n^{/},f\right)}{\log\frac{1}{R_0-r_n^{/}}}=\rho_{p,\mathcal{A}}\left(f\right)\ \textit{ if }\ 1<r_n^{/}<R_0<+\infty.$$

Proof. Case $R_0=+\infty$. The definition of $\rho_{p,\mathcal{A}}\left(f\right)$ implies that there exists a sequence $\{s_n,n\geq1\}$, $s_n\longrightarrow+\infty$ such that

$$\lim_{s_n\to+\infty}\frac{\log_p T_0\left(s_n,f\right)}{\log s_n}=\rho_{p,\mathcal{A}}\left(f\right).$$

Setting $\int\limits_{E_r}\frac{dr}{r}=\delta<+\infty$. Then the interval $[s_n,\left(1+e^{\delta}\right)s_n]$ meets the complement of E_r since

$$\int\limits_{s_n}^{\left(1+e^{\delta}\right)s_n}\frac{dr}{r}=\log\left(1+e^{\delta}\right)>\delta$$

Therefore, there exists a point $r_n\in[s_n,\left(1+e^{\delta}\right)s_n]\backslash E_r$. For $r_n\in[s_n,\left(1+e^{\delta}\right)s_n]\backslash E_r$, we have

$$\frac{\log_p T_0\left(r_n,f\right)}{\log r_n}\geq\frac{\log_p T_0\left(s_n,f\right)}{\log\left(1+e^{\delta}\right)s_n}=\frac{\log_p T_0\left(s_n,f\right)}{\log\left(1+e^{\delta}\right)+\log s_n}.$$

Hence

$$\lim_{r_n\to+\infty}\frac{\log_p T_0\left(r_n,f\right)}{\log r_n}\geq\lim_{s_n\to+\infty}\frac{\log_p T_0\left(s_n,f\right)}{\left(1+\dfrac{\log\left(1+e^{\delta}\right)}{\log s_n}\right)\log s_n}=\rho_{p,\mathcal{A}}\left(f\right).$$

By

$$\lim_{r_n\to+\infty}\frac{\log_p T_0(r_n,f)}{\log r_n}\le\lim_{s_n\to+\infty}\frac{\log_p T_0\left(\left(1+e^{\delta}\right)s_n,f\right)}{\log s_n}$$

$$=\lim_{s_n\to+\infty}\left(\frac{\log_p T_0\left(\left(1+e^{\delta}\right)s_n,f\right)}{\log\left(1+e^{\delta}\right)s_n}\cdot\frac{\log\left(1+e^{\delta}\right)+\log s_n}{\log s_n}\right)=\rho_{p,\mathcal{A}}(f),$$

we deduce that

$$\lim_{r_n\to+\infty}\frac{\log_p T_0(r_n,f)}{\log r_n}=\rho_{p,\mathcal{A}}(f).$$

Case $R_0<+\infty$. The definition of $\rho_{p,\mathcal{A}}(f)$ implies that there exists a sequence $\{s'_n, n\ge 1\}$, $s'_n\longrightarrow R_0^-$ such that

$$\lim_{s'_n\to R_0^-}\frac{\log_p T_0(s'_n,f)}{\log\frac{1}{R_0-s'_n}}=\rho_{p,\mathcal{A}}(f).$$

Setting $\int\limits_{E'_r}\frac{dr}{R_0-r}=\log\delta'<+\infty$. Since

$$\int\limits_{s'_n}^{R_0-\frac{R_0-s'_n}{\delta'+1}}\frac{dr}{R_0-r}=\log(1+\delta')>\log\delta',$$

then there exists a point $r'_n\in[s'_n,R_0-\frac{R_0-s'_n}{\delta'+1}]\backslash E'_r$. For $r'_n\in[s'_n,R_0-\frac{R_0-s'_n}{\delta'+1}]\backslash E'_r$, we have

$$\frac{\log_p T_0\left(r'_n,f\right)}{\log\frac{1}{R_0-r'_n}}\ge\frac{\log_p T_0(s'_n,f)}{\log\frac{\delta'+1}{R_0-s'_n}}=\frac{\log_p T_0(s'_n,f)}{\log(1+\delta')+\log\frac{1}{R_0-s'_n}}.$$

Hence

$$\lim_{r'_n\to R_0^-}\frac{\log_p T_0\left(r'_n,f\right)}{\log\frac{1}{R_0-r'_n}}\ge\lim_{s'_n\to R_0^-}\frac{\log_p T_0(s'_n,f)}{\left(1+\frac{\log(1+\delta')}{\log\frac{1}{R_0-s'_n}}\right)\log\frac{1}{R_0-s'_n}}=\rho_{p,\mathcal{A}}(f).$$

By

$$\lim_{r'_n\to R_0^-}\frac{\log_p T_0\left(r'_n,f\right)}{\log\frac{1}{R_0-r'_n}}\le\lim_{s'_n\to R_0^-}\frac{\log_p T_0\left(R_0-\frac{R_0-s'_n}{\delta'+1},f\right)}{\log\frac{1}{R_0-s'_n}}$$

$$=\lim_{s'_n\to R_0^-}\left(\frac{\log_p T_0\left(R_0-\frac{R_0-s'_n}{\delta'+1},f\right)}{\log\frac{\delta'+1}{R_0-s'_n}}\cdot\frac{\log(\delta'+1)+\log\frac{1}{R_0-s'_n}}{\log\frac{1}{R_0-s'_n}}\right)=\rho_{p,\mathcal{A}}(f),$$

we obtain

$$\lim_{r'_n\to R_0^-}\frac{\log_p T_0\left(r'_n,f\right)}{\log\frac{1}{R_0-r'_n}}=\rho_{p,\mathcal{A}}(f).$$

□

3. Proof of Theorem 1.1

Proof. Let $f \not\equiv 0$ be a solution of (1.1). Set $b = \max\{\rho_{p,\mathcal{A}}(B_j) : j = 1, 2, ..., k-1\} < \rho_{p,\mathcal{A}}(B_0) = a$. We divide through equation (1.1) by f to get

$$-B_0(z) = \frac{f^{(k)}(z)}{f(z)} + \sum_{j=1}^{k-1} B_j(z) \frac{f^{(j)}(z)}{f(z)}. \tag{3.1}$$

By (3.1) and Lemma 2.1, it follows that

$$m_0(r, B_0) \le \sum_{j=1}^{k-1} m_0(r, B_j) + \sum_{j=1}^{k} m_0\left(r, \frac{f^{(j)}}{f}\right) + O(1)$$

$$\le \sum_{j=1}^{k-1} m_0(r, B_j) + \begin{cases} O(\log r + \log T_0(r, f)), & R_0 = +\infty, \ r \notin \Delta_r, \\ O\left(\log \frac{1}{R_0 - r} + \log T_0(r, f)\right), & R_0 < +\infty, \ r \notin \Delta_r', \end{cases} \tag{3.2}$$

where Δ_r and Δ_r' are sets with $\int_{\Delta_r} \frac{dr}{r} < +\infty$ and $\int_{\Delta_r'} \frac{dr}{R_0 - r} < +\infty$ respectively.

Case $R_0 = +\infty$. Since $\rho_{p,\mathcal{A}}(B_0) = a$ and $N_0(r, B_0) \equiv 0$, then by the definition of the characteristic function and Lemma 2.3, there exists a sequence $\{r_n, \ r_n \notin \Delta_r\}$ such that

$$\lim_{r_n \to +\infty} \frac{\log_p T_0(r_n, B_0)}{\log r_n} = \lim_{r_n \to +\infty} \frac{\log_p m_0(r_n, B_0)}{\log r_n} = a.$$

Then, for any given ε $(0 < \varepsilon < (a-b)/2)$, we have

$$m_0(r_n, B_0) \ge \exp_{p-1}\left\{r_n^{a-\varepsilon}\right\} \tag{3.3}$$

and for $j = 1, 2, ..., k-1$, we have

$$m_0(r_n, B_j) \le \exp_{p-1}\left\{r_n^{b+\varepsilon}\right\}. \tag{3.4}$$

By substituting (3.3) and (3.4) into (3.2), we conclude for $r_n \notin \Delta_r$ sufficiently large

$$\exp_{p-1}\left\{r_n^{a-\varepsilon}\right\} \le (k-1)\exp_{p-1}\left\{r_n^{b+\varepsilon}\right\} + O(\log r_n + \log T_0(r_n, f)). \tag{3.5}$$

Noting that $a - \varepsilon > b + \varepsilon$, by (3.5) we obtain

$$(1 - o(1))\exp_{p-1}\left\{r_n^{a-\varepsilon}\right\} \le O(\log r_n + \log T_0(r_n, f))$$

which leads to $\rho_{p,\mathcal{A}}(f) = +\infty$ and $\rho_{p+1,\mathcal{A}}(f) \ge a = \rho_{p,\mathcal{A}}(B_0)$.

Case $R_0 < +\infty$. Since $\rho_{p,\mathcal{A}}(B_0) = a$ and $N_0(r, B_0) \equiv 0$, then by the definition of the characteristic function and Lemma 2.3, there exists a sequence $\{r_n^/,\ r_n^/ \notin \Delta_r'\}$ such that

$$\lim_{r_n^/ \to R_0^-} \frac{\log_p m_0(r_n^/, B_0)}{\log \frac{1}{R_0 - r_n^/}} = a.$$

Then, for any given ε $(0 < \varepsilon < (a-b)/2)$, we have

$$m_0(r_n^/, B_0) \geq \exp_{p-1}\left\{ \left(\frac{1}{R_0 - r_n^/} \right)^{a-\varepsilon} \right\} \tag{3.6}$$

and for $j = 1, 2, ..., k-1$, we have

$$m_0(r_n^/, B_j) \leq \exp_{p-1}\left\{ \left(\frac{1}{R_0 - r_n^/} \right)^{b+\varepsilon} \right\}. \tag{3.7}$$

By substituting (3.6) and (3.7) into (3.2), we conclude for $r_n^/ \to R_0^-$, $r_n^/ \notin \Delta_r'$

$$\exp_{p-1}\left\{ \left(\frac{1}{R_0 - r_n^/} \right)^{a-\varepsilon} \right\} \leq (k-1)\exp_{p-1}\left\{ \left(\frac{1}{R_0 - r_n^/} \right)^{b+\varepsilon} \right\}$$

$$+O\left(\log \frac{1}{R_0 - r_n^/} + \log T_0(r_n^/, f) \right). \tag{3.8}$$

Since $a - \varepsilon > b + \varepsilon$, then by (3.8) we obtain

$$(1 - o(1))\exp_{p-1}\left\{ \left(\frac{1}{R_0 - r_n^/} \right)^{a-\varepsilon} \right\} \leq O\left(\log \frac{1}{R_0 - r_n^/} + \log T_0(r_n^/, f) \right)$$

which leads to $\rho_{p,\mathcal{A}}(f) = +\infty$ and $\rho_{p+1,\mathcal{A}}(f) \geq a = \rho_{p,\mathcal{A}}(B_0)$. □

4. Proof of Theorem 1.2

Proof. Case $R_0 = +\infty$. Let $f \not\equiv 0$ be a solution of (1.1). By substituting (1.3) and (1.4) into (3.2), we conclude for $r \in E_r \backslash \Delta_r$ sufficiently large

$$\exp_{p-1}\{\alpha r^\mu\} \leq (k-1)\exp_{p-1}\{\beta r^\mu\} + O(\log r + \log T_0(r, f)). \tag{4.1}$$

Noting that $p \geq 2$ and $\alpha > \beta \geq 0$, by (4.1) we obtain

$$(1 - o(1))\exp_{p-1}\{\alpha r^\mu\} \leq O(\log r + \log T_0(r, f))$$

which leads to $\rho_{p,\mathcal{A}}(f) = +\infty$ and $\rho_{p+1,\mathcal{A}}(f) \geq \mu$.

Case $R_0 < +\infty$. Let $f \not\equiv 0$ be a solution of (1.1). By substituting (1.5) and (1.6) into (3.2), we conclude for $r \in F_r \backslash \Delta'_r$, $r \to R_0^-$

$$\exp_{p-1}\left\{\frac{\alpha}{(R_0-r)^{\mu}}\right\} \leq (k-1)\exp_{p-1}\left\{\frac{\beta}{(R_0-r)^{\mu}}\right\}$$

$$+O(\log\frac{1}{R_0-r}+\log T_0(r,f)). \tag{4.2}$$

Since $p \geq 2$ and $\alpha > \beta \geq 0$, then by (4.2) we obtain

$$(1-o(1))\exp_{p-1}\left\{\frac{\alpha}{(R_0-r)^{\mu}}\right\} \leq O(\log\frac{1}{R_0-r}+\log T_0(r,f))$$

which leads to $\rho_{p,\mathcal{A}}(f) = +\infty$ and $\rho_{p+1,\mathcal{A}}(f) \geq \mu$. □

5. Proof of Theorem 1.3

Proof. Case $R_0 = +\infty$. Set $d = \max\{\rho_{p,\mathcal{A}}(B_j)\,(j \neq s), \rho_{p,\mathcal{A}}(F)\} < \rho_{p,\mathcal{A}}(B_s) = c$. If $\rho_{p,\mathcal{A}}(f) = \infty$, then the result is trivial. Suppose that f is an admissible solution of (1.2) with $\rho = \rho_{p,\mathcal{A}}(f) < \infty$. It follows from (1.2) that

$$B_s(z) = \frac{F(z)}{f^{(s)}} - \frac{f^{(k)}}{f^{(s)}} - B_{k-1}(z)\frac{f^{(k-1)}}{f^{(s)}} - \cdots - B_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}}$$

$$-B_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}} - \cdots - B_1(z)\frac{f'}{f^{(s)}} - B_0(z)\frac{f}{f^{(s)}}. \tag{5.1}$$

Since $N_0\left(r, f^{(j+1)}\right) = 0$, it holds for $j = 0, ..., k-1$ that

$$T_0\left(r, f^{(j+1)}\right) = m_0\left(r, f^{(j+1)}\right) \leq m_0\left(r, \frac{f^{(j+1)}}{f}\right) + m_0(r,f)$$

$$= T_0(r,f) + m_0\left(r, \frac{f^{(j+1)}}{f}\right). \tag{5.2}$$

By using (5.2), we can obtain from (5.1) that

$$T_0(r,B_s) \leq T_0(r,F) + M\cdot T_0(r,f) + \sum_{j\neq s} T_0(r,B_j)$$

$$+\sum_{j=0}^{k-1} m_0\left(r, \frac{f^{(j+1)}}{f}\right) + O(1), \tag{5.3}$$

where $M > 0$ is a constant. Applying Lemma 2.2, we have

$$m_0\left(r, \frac{f^{(j+1)}}{f}\right) = O\left(\exp_{p-2}\left\{r^{\rho+\varepsilon}\right\}\right) \ (j = 0, ..., k-1) \tag{5.4}$$

holds for all r outside a set $\Delta_r \subset (1, +\infty)$ with $\int\limits_{\Delta_r} \frac{dr}{r} < +\infty$. By substituting (5.4) into (5.3), we obtain

$$T_0(r, B_s) \leq T_0(r, F) + MT_0(r, f) + \sum_{j \neq s} T_0(r, B_j)$$

$$+O\left(\exp_{p-2}\left\{r^{\rho+\varepsilon}\right\}\right), \; r \notin \Delta_r. \tag{5.5}$$

Since $\rho_{p,\mathcal{A}}(B_s) = c$, then by Lemma 2.3, there exists a sequence $\{r_n, \, r_n \notin \Delta_r\}$ such that

$$\lim_{r_n \to +\infty} \frac{\log_p T_0(r_n, B_s)}{\log r_n} = c.$$

Then, for any given ε $(0 < \varepsilon < (c-d)/2)$ and sufficiently large $r_n \notin \Delta_r$, we have

$$T_0(r_n, B_s) \geq \exp_{p-1}\left\{r_n^{c-\varepsilon}\right\} \tag{5.6}$$

and

$$T_0(r_n, F) \leq \exp_{p-1}\left\{r_n^{d+\varepsilon}\right\}, \quad T_0(r_n, B_j) \leq \exp_{p-1}\left\{r_n^{d+\varepsilon}\right\} \quad (j \neq s). \tag{5.7}$$

By substituting (5.6) and (5.7) into (5.5), we conclude for $r_n \notin \Delta_r$ sufficiently large

$$\exp_{p-1}\left\{r_n^{c-\varepsilon}\right\} \leq k \exp_{p-1}\left\{r_n^{d+\varepsilon}\right\} + MT_0(r_n, f) + O\left(\exp_{p-2}\left\{r_n^{\rho+\varepsilon}\right\}\right). \tag{5.8}$$

Noting that $c - \varepsilon > d + \varepsilon$, it follows from (5.8) that for $r_n \notin \Delta_r$ sufficiently large

$$(1 - o(1)) \exp_{p-1}\left\{r_n^{c-\varepsilon}\right\} \leq MT_0(r_n, f) + O\left(\exp_{p-2}\left\{r_n^{\rho+\varepsilon}\right\}\right). \tag{5.9}$$

Therefore, by (5.9) we obtain

$$\limsup_{r_n \mapsto +\infty} \frac{\log_p T_0(r_n, f)}{\log r_n} \geq c - \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, we get $\rho_{p,\mathcal{A}}(f) \geq \rho_{p,\mathcal{A}}(B_s) = c$.

Case $R_0 < +\infty$. Set $d = \max\{\rho_{p,\mathcal{A}}(B_j)(j \neq s), \rho_{p,\mathcal{A}}(F)\} < \rho_{p,\mathcal{A}}(B_s) = c$. If $\rho_{p,\mathcal{A}}(f) = \infty$, then the result is trivial. Suppose that f is an admissible solution of (1.2) with $\rho = \rho_{p,\mathcal{A}}(f) < \infty$. Applying Lemma 2.2, we have

$$m_0\left(r, \frac{f^{(j+1)}}{f}\right) = O\left(\exp_{p-2}\left\{\frac{1}{R_0 - r}\right\}^{\rho+\varepsilon}\right) \quad (j = 0, ..., k-1) \tag{5.10}$$

outside a set $\Delta_r' \subset (1, R_0)$ with $\int_{\Delta_r'} \frac{dr}{R_0-r} < +\infty$. By (5.10), we can obtain from (5.3) that

$$T_0(r, B_s) \leq T_0(r, F) + MT_0(r, f) + \sum_{j\neq s} T_0(r, B_j)$$

$$+O\left(\exp_{p-2}\left\{\frac{1}{R_0-r}\right\}^{\rho+\varepsilon}\right) \ (r \notin \Delta_r'), \tag{5.11}$$

where $M > 0$ is a constant. Since $\rho_{p,\mathcal{A}}(B_s) = c$, then by Lemma 2.3, there exists a sequence $\{r_n^/, r_n^/ \notin \Delta_r'\}$ such that

$$\lim_{r_n^/ \mapsto R_0^-} \frac{\log_p T_0(r_n^/, B_s)}{\log \frac{1}{R_0-r_n^/}} = c.$$

Then, for any given ε $(0 < \varepsilon < (c-d)/2)$ and $r_n^/ \to R_0^-$, $r_n^/ \notin \Delta_r'$, we have

$$T_0(r_n^/, B_s) \geq \exp_{p-1}\left\{\left(\frac{1}{R_0-r_n^/}\right)^{c-\varepsilon}\right\} \tag{5.12}$$

and

$$\begin{aligned} T_0\left(r_n^/, F\right) &\leq \exp_{p-1}\left\{\left(\frac{1}{R_0-r_n^/}\right)^{d+\varepsilon}\right\}, \\ T_0(r_n^/, B_j) &\leq \exp_{p-1}\left\{\left(\frac{1}{R_0-r_n^/}\right)^{d+\varepsilon}\right\} \ (j \neq s). \end{aligned} \tag{5.13}$$

By substituting (5.12) and (5.13) into (5.11), we conclude for $r_n^/ \to R_0^-$, $r_n^/ \notin \Delta_r'$

$$\exp_{p-1}\left\{\left(\frac{1}{R_0-r_n^/}\right)^{c-\varepsilon}\right\} \leq k\exp_{p-1}\left\{\left(\frac{1}{R_0-r_n^/}\right)^{d+\varepsilon}\right\}$$

$$+MT_0\left(r_n^/, f\right) + O\left(\exp_{p-2}\left\{\frac{1}{R_0-r_n^/}\right\}^{\rho+\varepsilon}\right). \tag{5.14}$$

Noting that $c-\varepsilon > d+\varepsilon$, it follows from (5.14) that for $r_n^/ \to R_0^-$, $r_n^/ \notin \Delta_r'$

$$(1-o(1))\exp_{p-1}\left\{\left(\frac{1}{R_0-r_n^/}\right)^{c-\varepsilon}\right\} \leq MT_0(r_n^/, f)$$

$$+O\left(\exp_{p-2}\left\{\frac{1}{R_0-r_n^/}\right\}^{\rho+\varepsilon}\right). \tag{5.15}$$

Therefore, by (5.15) we obtain

$$\lim_{r_n^{/}\mapsto R_0^-}\frac{\log_p T_0\left(r_n^{/},f\right)}{\log\frac{1}{R_0-r_n^{/}}}\geq c-\varepsilon$$

and since $\varepsilon>0$ is arbitrary, we get $\rho_{p,\mathcal{A}}(f)\geq\rho_{p,\mathcal{A}}(B_s)=c$. This proves Theorem 1.3. □

6. Proof of Theorem 1.4

Proof. Case $R_0=+\infty$. Contrary to our assertion, we assume that f is an admissible solution of (1.2) with $\rho=\rho_{p,\mathcal{A}}(f)<\infty$. For any given $\varepsilon>0$ and sufficiently large r, we have

$$T_0(r,f)\leq\exp_{p-1}\left\{r^{\rho+\varepsilon}\right\}. \tag{6.1}$$

Set $\max\{\rho_{p,\mathcal{A}}(B_j)\,(j\neq s),\rho_{p,\mathcal{A}}(F)\}=\eta<+\infty$. Then, for the above $\varepsilon>0$ and sufficiently large r, we have

$$T_0(r,B_j)\leq\exp_{p-1}\left\{r^{\eta+\varepsilon}\right\}\ \ (j\neq s),\ T_0(r,F)\leq\exp_{p-1}\left\{r^{\eta+\varepsilon}\right\}. \tag{6.2}$$

Thus, by substituting (5.4), (6.1) and (6.2) into (5.3), we get for any given $\varepsilon>0$ and sufficiently large $r\notin\Delta_r$

$$T_0(r,B_s)\leq k\exp_{p-1}\left\{r^{\eta+\varepsilon}\right\}+M\exp_{p-1}\left\{r^{\rho+\varepsilon}\right\}$$

$$+O\left(\exp_{p-2}\left\{r^{\rho+\varepsilon}\right\}\right). \tag{6.3}$$

Therefore

$$\rho_{p,\mathcal{A}}(B_s)\leq\max\{\eta+\varepsilon,\rho+\varepsilon\}<\infty.$$

This contradicts the fact that $\rho_{p,\mathcal{A}}(B_s)=\infty$. Hence, every an admissible solution f of (1.2) satisfies $\rho_{p,\mathcal{A}}(f)=\infty$.

Case $R_0<+\infty$. We suppose the contrary. Let f be an admissible solution of (1.2) with $\rho=\rho_{p,\mathcal{A}}(f)<\infty$. For any given $\varepsilon>0$ and $r\to R_0^-$, we have

$$T_0(r,f)\leq\exp_{p-1}\left\{\left(\frac{1}{R_0-r}\right)^{\rho+\varepsilon}\right\}. \tag{6.4}$$

Set $\max\{\rho_{p,\mathcal{A}}(B_j)(j\neq s),\rho_{p,\mathcal{A}}(F)\}=\eta<+\infty$. Then for the above $\varepsilon>0$ and $r\to R_0^-$, we have

(6.5)

$$T_0(r,B_j)\leq \exp_{p-1}\left\{\left(\frac{1}{R_0-r}\right)^{\eta+\varepsilon}\right\},\quad (j\neq s),\; T_0(r,F)\leq \exp_{p-1}\left\{\left(\frac{1}{R_0-r}\right)^{\eta+\varepsilon}\right\}.$$

Thus, by substituting (5.10), (6.4) and (6.5) into (5.3), we get for any given $\varepsilon>0$ and $r\to R_0^-$, $r\notin\Delta'_r$

$$T_0(r,B_s)\leq k\exp_{p-1}\left\{\left(\frac{1}{R_0-r}\right)^{\eta+\varepsilon}\right\}+M\exp_{p-1}\left\{\left(\frac{1}{R_0-r}\right)^{\rho+\varepsilon}\right\}$$

$$+O\left(\exp_{p-2}\left\{\left(\frac{1}{R_0-r}\right)^{\rho+\varepsilon}\right\}\right). \tag{6.6}$$

Therefore

$$\rho_{p,\mathcal{A}}(B_s)\leq\max\{\eta+\varepsilon,\rho+\varepsilon\}<\infty.$$

This contradicts the fact that $\rho_{p,\mathcal{A}}(B_s)=\infty$. Thus, every an admissible solution f of (1.2) satisfies $\rho_{p,\mathcal{A}}(f)=\infty$. This proves Theorem 1.4. □

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