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ON THE STRUCTURE OF CHARACTERISTIC SUBGROUP LATTICES OF FINITE ABELIAN p-GROUPS

AFIF HUMAM[(1)] AND PUDJI ASTUTI[(2)]

ABSTRACT. This paper gives explicit descriptions of characteristic and fully invariant subgroups of a finite abelian p-group in term of its cyclic decomposition. The results are then utilized to identify the lattice of characteristic subgroups is self-dual.

1. INTRODUCTION

Let G be a finite abelian p-group with p a prime number and $S \subseteq G$ be a subgroup of G. S is called *fully invariant* if $f(S) \subseteq S$ for all f endomorphisms of G and S is called *characteristic* if $f(S) \subseteq S$ for all f automorphisms of G. The set of all fully invariant subgroups of G, denoted by $FI(G)$, and the set of all characteristic subgroups of G, denoted by $Char(G)$, both form lattices with respect to intersection and product operations such that $FI(G)$ is a sublattice of $Char(G)$.

Kerby and Rode [5] addressed the question when two finite abelian groups have isomorphic lattices of characteristic subgroups. By utilizing an explicit description of the structure of the groups, the problem can be reduced to both groups of being primary. If p is an odd prime number, then every characteristic subgroup of a finite abelian p-group is fully invariant. A complete answer of the question for the class of the groups having odd order was obtained. The case of 2-groups is more complicated as a result of the present of irregular characteristic subgroups.

Kerby and Rode [5] reviewed that any characteristic subgroup of a finite abelian p-group with $p \neq 2$ is regular and canonical and so, according to Theorem 2.1, it is

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fully invariant. Hence, for $p \neq 2$ we obtain $Char(G) = FI(G)$. The result of the lattice isomorphism in [5] is, in fact, concerning lattices of fully invariant subgroups. In this paper we would like to investigate the structure of lattice $Char(G)$ when $p = 2$, particularly when the lattice $Char(G)$ contains a characteristic subgroup that is not fully invariant. We will show that the $Char(G)$ is self-dual.

In spirit to be able to complete the above investigations, in this paper we will explain an explicit description of the structure of finite abelian 2-groups. Then we will utilize it to characterize the lattice of characteristic subgroups. An explicit description concerning characteristic subgroup lattices of a number abelian p-groups of rank two was recently obtained by Sarita and Jakhar [**?**].

The organization of the paper is as follows. Section 2 will review some results regarding the lattice of fully invariant subgroups. The main result of this paper is explained in Section 3. Finally, this paper is closed with conclusions.

2. Lattice of fully invariant subgroups

Let G be a finite abelian p-group, with addition operation, for some prime element p. It is a well known fact that G can be decomposed as

$$G = Z_{p^{\beta_1}}^{\alpha_1} \oplus Z_{p^{\beta_2}}^{\alpha_2} \oplus \dots \oplus Z_{p^{\beta_n}}^{\alpha_n} \tag{2.1}$$

where Z_{p^λ} is a cyclic group of order p^λ, $1 \leq \beta_1 < \beta_2 < \cdots < \beta_n$ and $\alpha_1, \alpha_2, \dots, \alpha_n \geq 1$. Chew et al. exploited such cyclic decomposition to deduce their results concerning finite abelian p-groups [**?**]. We follow the idea to deduce our results.

Related to the decomposition (2.1), we define the exponent tuple of G as

$$\lambda(G) = (\lambda_1, \lambda_2, \dots, \lambda_m) = (\beta_1, \dots, \beta_1, \beta_2, \dots, \beta_n, \dots, \beta_n),$$

i.e. $\lambda_j = \beta_\nu$ for $\sum_{\ell=0}^{\nu-1} \alpha_\ell < j \leq \sum_{\ell=1}^{\nu} \alpha_\ell$, where $\alpha_0 = 0$ and $m = \sum_{\ell=1}^{n} \alpha_\ell$. The integer λ_j is called a simple exponent if $\lambda_j \neq \lambda_i$ for all i with $j \neq i$. A tuple of elements of G, denoted by $(t_1, \dots, t_m)$, is called a generator tuple of G, if

$$G = \langle t_1 \rangle \oplus \cdots \oplus \langle t_m \rangle = \oplus_{i=1}^{m} \langle t_i \rangle \tag{2.2}$$

with order $o(t_i) = p^{\lambda_i}$ for all $i = 1, \dots, m$, and $\lambda_1 \leq \cdots \leq \lambda_m$.

In context of module theory, an abelian group is a module over the ring of integers which is a principal ideal domain. In this context we found the following fact [7].

Theorem 2.1. *Let G be a finite abelian p-group with a cyclic decomposition (2.2), $o(t_i) = p^{\lambda_i}$, $1 \leq \lambda_1 \leq \cdots \leq \lambda_m$. A subgroup $S \subseteq G$ is fully invariant if and only if*

$$S = \oplus_{i=1}^{m} \langle p^{\lambda_i - a_i} t_i \rangle \tag{2.3}$$

for some non-negative integers a_i satisfying the following conditions

$$0 \leq a_i \leq \lambda_i, \quad 0 \leq a_i - a_{i-1} \leq \lambda_i - \lambda_{i-1}, \tag{2.4}$$

for $i = 1, 2, \ldots, m$, and $\lambda_0 = 0$.

Subgroup of the form (2.3) is called regular. Conditions (2.4) for the tuple of integers $(a_1, \ldots, a_m)$ is called canonical [5].

Let $\Lambda(G)$ be a set of tuple of integers as follows

$$\Lambda(G) = \{(a_1, \ldots, a_m) \ : \ 0 \leq a_i \leq \lambda_i, \ i = 1, \ldots, m\}.$$

For any two tuple of integers $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \in \Lambda(G)$ we define

$a \leq b$ if $a_i \leq b_i$ for all $i \in \{1, \ldots, m\}$;

$a \wedge b = (\min\{a_1, b_1\}, \ldots, \min\{a_m, b_m\})$

$a \vee b = (\max\{a_1, b_1\}, \ldots, \max\{a_m, b_m\})$.

Then, the set $\Lambda(G)$, equipped with the above partially order and operations, forms a finite lattice. Referring to [2] the sublattice

$$\mathcal{C}(G) = \{(a_1, \ldots, a_m) \in \Lambda(G) \ : 0 \leq a_i - a_{i-1} \leq \lambda_i - \lambda_{i-1}, i = 2, \ldots, m\},$$

which consists of all canonical tuples, is distributive and self-dual with an anti-isomorphism

$$(a_1, \ldots, a_m) \longmapsto (\lambda_1 - a_1, \ldots, \lambda_m - a_m).$$

Then referring to Theorem 2.1 above and Theorem 2.3 in [5] we can conclude that the lattices $\mathcal{C}(G)$ and $IF(G)$ are isomorphic. Particularly, the mapping $\theta(a) = \oplus_{i=1}^{m} \langle p^{\lambda_i - a_i} t_i \rangle$ for any $a = (a_1, \ldots, a_m) \in \mathcal{C}(G)$ is a lattice isomorphism. As a result, the lattice $IF(G)$ is self-dual. Particularly the mapping D, defined as

$$\text{for every } S \in IF(G), \quad D(S) = \theta(\lambda(G) - a) \text{ when } S = \theta(a), \tag{2.5}$$

is an anti isomorphism.

3. Mean Results

Shoda [8] showed that any finite abelian 2-group G has an irregular characteristic subgroup if and only if G has at least two simple exponents which are not consecutive. In this section we restrict our discussion to finite abelian 2-groups having irregular characteristic subgroups. Before we explain about the structure of the lattice of characteristic subgroups, the following are some properties of any characteristic subgroups that some of them are adapted from [1].

Consider a cyclic decomposition of a finite abelian 2-group G (2.2) with order $o(t_i) = 2^{\lambda_i}$ for $i = 1, \dots, m$. Let $x \in G$. A non-negative r is called the height of x, denoted by $h(x) = r$ if r is the largest integers such that $x = 2^r y$ for some $y \in G$ [3]. If $(t_1, \dots, t_m)$ is a tuple of generators which corresponds to a cyclic decomposition of G and r is a non-negative integer, then $h(2^r t_i) = r$. Any automorphism will preserve order and height.

Furthermore, we have the following interesting result.

Lemma 3.1. *Let λ_i be a simple exponent and f be an automorphism of G. Then the map of t_i by f can be written as*

$$f(t_i) = x + y + z$$

for some $x \in \oplus_{j=1}^{i-1}\langle t_j\rangle$, $y \in \langle t_i\rangle$, and $z \in \oplus_{j=i+1}^{m}\langle t_j\rangle$, satisfying the conditions $h(y) = 0$ and $o(z) = 2^r$ with $r \leq \lambda_i$.

Proof. Let

$$f(t_i) = x + y + z$$

for some $x \in \oplus_{j=1}^{i-1}\langle t_j\rangle$, $y \in \langle t_i\rangle$, $z \in \oplus_{j=i+1}^{m}\langle t_j\rangle$. From $o(f(t_i)) = o(t_i) = 2^{\lambda_i}$, we obtain $o(z) = 2^r$, with $r \leq \lambda_i$. Hence $z = 2^{\lambda_{i+1}-r}z_1$ for some $z_1 \in \oplus_{j=i+1}^{m}\langle t_j\rangle$. Suppose $h(y) = \nu > 0$. Then $y = 2^\nu y_1$ for some $y_1 \in \langle t_i\rangle$. Then,

$$f(2^{\lambda_i-1}t_i) = 2^{\lambda_i-1}f(t_i) = 2^{\lambda_i-1}(x + 2^\nu y_1 + 2^{\lambda_{i+1}-r}z_1) = 2^{\lambda_i-1+\lambda_{i+1}-r}z_1.$$

Hence $h(f(2^{\lambda_i-1}t_i)) \geq \lambda_i - 1 + \lambda_{i+1} - r > \lambda_i - 1$ since $\lambda_{i+1} - r > 0$. This is a contradiction to the fact that f preserves height which implies $h(f(2^{\lambda_i-1}t_i)) = \lambda_i - 1$ since $h((2^{\lambda_i-1}t_i)) = \lambda_i - 1$. Thus $h(y) = 0$.

□

For any i, we define $\pi_i : G \to \langle t_i \rangle$ the projection on the subgroup $\langle t_i \rangle$ with the kernel $\oplus_{j=1, j\neq i}^{m} \langle t_j \rangle$.

Theorem 3.1. *Let* $S \in Char(G)$, $S \cap \langle t_i \rangle = \langle 2^{\lambda_i - a_i} t_i \rangle$, $\pi_i(S) = \langle 2^{\lambda_i - b_i} t_i \rangle$ *for some non-negative integers* a_i, b_i, $i \in \{1, \ldots, m\}$. *Then*

(i) $a = (a_1, \ldots, a_m)$ *is canonical and* $\theta(a)$ *is the largest fully invariant subgroup contained in* S.

(ii) $b = (b_1, \ldots, b_m)$ *is canonical and* $\theta(b)$ *is the smallest fully invariant subgroup containing* S.

(iii) *If* $a_i \neq b_i$ *then* λ_i *is a simple exponent and* $b_i = a_i + 1$.

Proof. Here we will show the proof for parts (i) and (iii). Part (ii) can be done similarly.

(i) Let $i \in \{1, \ldots, m-1\}$. Consider the automorphism of G, denoted by f, defined as $f(t_j) = t_j$ for $j \neq i$ and $f(t_i) = t_i + 2^{\lambda_{i+1} - \lambda_i} t_{i+1}$. Then

$$f(2^{\lambda_i - a_i} t_i) = 2^{\lambda_i - a_i}(t_i + 2^{\lambda_{i+1} - \lambda_i} t_{i+1}) = 2^{\lambda_i - a_i} t_i + 2^{\lambda_{i+1} - a_i} t_{i+1}.$$

Since $2^{\lambda_i - a_i} t_i \in S$ then $2^{\lambda_{i+1} - a_i} t_{i+1} \in S$. Hence $\lambda_{i+1} - a_i \geq \lambda_{i+1} - a_{i+1}$ which implies $a_i \leq a_{i+1}$.

Let $i \in \{2, \ldots, m\}$ and define an automorphism of G, denoted by g, as $g(t_j) = t_j$ if $j \neq i$ and $g(t_i) = t_i + t_{i-1}$. We obtain that

$$g(2^{\lambda_i - a_i} t_i) = 2^{\lambda_i - a_i}(t_i + t_{i-1}).$$

The fact $2^{\lambda_i - a_i} t_i \in S$ implies $2^{\lambda_i - a_i} t_{i-1} \in S$. Hence $\lambda_i - a_i \geq \lambda_{i-1} - a_{i-1}$. Thus $(a_1, \ldots, a_m) \in \mathcal{C}(G)$.

(iii) Let $i \in \{1, \ldots, m\}$ and $x \in S$ such that $\pi_i(x) = 2^{\lambda_i - b_i} t_i$. Using the automorphism of G defined as $f(t_j) = t_j$ for $j \neq i$ and $f(t_i) = t_i + 2t_i$, we have

$f(x) = x + 2^{\lambda_i - b_i + 1} t_i$. Hence $2^{\lambda_i - b_i + 1} t_i \in S$ which that implies $\lambda_i - b_i + 1 \geq \lambda_i - a_i$. As a result $b_i \leq a_i + 1$. It is trivial that $a_i \leq b_i$. Thus point (iii) is proven. □

We can give the following corollary as a direct consequence of Thoerem 3.1.g

Corollary 3.1. *Let $S \in Char(G)$. Then $S \in FI(G)$ if and only if for any $i \in \{1, \ldots, m\}$, $S \cap \langle t_i \rangle = \pi_i(S)$.*

For any characteristic subgroup $S \in Char(G)$, let $\underline{S}$ and $\overline{S}$ denote respectively the largest fully invariant subgroup contained in S and the smallest fully invariant subgroup contains S. From Theorem 3.1 we obtained that the quotient group $\overline{S}/\underline{S}$ is an elementary 2-group. Meanwhile let the interval $[\underline{S}, \overline{S}]$ denote the lattice subgroups of G in between $\underline{S}$ and $\overline{S}$. It is well known that the lattice subgroups $[\underline{S}, \overline{S}]$ and the lattice subgroups of the quotient group $\overline{S}/\underline{S}$ are isomorphic. Moreover, we also have the following lemma.

Lemma 3.2. *Let $S \in Char(G)$ but $S \notin IF(G)$. Then $[\underline{S}, \overline{S}] \subseteq Char(G)$.*

Proof. : Let $\underline{S} = \oplus_{i=1}^{m} \langle 2^{\lambda_i - a_i} t_i \rangle$, $\overline{S} = \oplus_{i=1}^{m} \langle 2^{\lambda_i - b_i} t_i \rangle$, and $I(S) = \{i \ : \ b_i = a_i + 1\}$. We will show that for every $\eta, \mu \in I(S)$ with $\eta < \mu$ we have $b_\eta < b_\mu$ and $\lambda_\eta - b_\eta < \lambda_\mu - b_\mu$. Consider the automorphism $f_1 : G \to G$ that maps $f(t_i) = t_i$ for $i \neq \eta$ and $f_1(t_\eta) = t_\eta + 2^{\lambda_\mu - \lambda_\eta} t_\mu$. Since $\pi_\eta(S) = \langle 2^{\lambda_\eta - b_\eta} t_\eta \rangle$, let $x \in S$ such that $\pi_\eta(x) = 2^{\lambda_\eta - b_\eta} t_\eta$. We obtain

$$f_1(x) = x + 2^{\lambda_\eta - b_\eta} 2^{\lambda_\mu - \lambda_\eta} t_\mu = x + 2^{\lambda_\mu - b_\eta} t_\mu.$$

Hence $2^{\lambda_\mu - b_\eta} t_\mu \in S$. As a result $\lambda_\mu - b_\eta > \lambda_\mu - b_\mu$ or $b_\eta < b_\eta$. Using similar approach for the automorphism $f_2 : G \to G$ that maps $f_2(t_i) = t_i$ for all $i \neq \mu$ and $f_2(t_\mu) = t_\eta + t_\mu$, we finally can obtain the second inequality $\lambda_\eta - b_\eta < \lambda_\mu - b_\mu$. As a result, for every $i \in I(S)$ we have $(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_m) \in \mathcal{C}(G)$.

Write $w_k = 2^{\lambda_k - b_k} t_k$ for $k = 1, 2, \ldots, m$. Then, $w_k \in \overline{S}$ and $2w_k \in \underline{S}$. Let $i \in I(S)$ and g be an arbitrary automorphism on G. By Lemma 3.1, we have

$$g(t_i) = \sum_{i > j \in I(S)} \alpha_j t_j + (2\alpha_i + 1) t_i + \sum_{i < j \in I(S)} \alpha_j 2^{\lambda_j - \lambda_i} t_j + \sum_{j \notin I(S)} \alpha_j t_j$$

for some $\alpha_j \in \mathbb{Z}$, $j = 1, 2, \ldots, m$. Clearly, $s = \sum_{j \notin I(S)} \alpha_j t_j \in \underline{S}$. Then,

$$g(w_i) = w_i + \sum_{i > j \in I(S)} \alpha_j 2^{(\lambda_i - b_i) - (\lambda_j - b_j)} w_j + \alpha_i (2w_i) + \sum_{i < j \in I(S)} \alpha_j 2^{\lambda_j - b_i} w_j + 2^{\lambda_i - b_i} s.$$

Therefore, $g(w_i) - w_i \in \underline{S}$. In addition, for every $x \in \overline{S}$ and for every g an automorphism of G we obtain $g(x) \in \underline{S} + \langle x \rangle$. Thus, every subgroup in $[\underline{S}, \overline{S}]$ is characteristic. □

Consider again the characteristic subgroup $S \in Char$ in Lemma 3.2 with $\underline{S} = \theta(a)$ and $\overline{S} = \theta(b)$. Suppose $I(S) = \{i_1, \ldots, i_k\}$ with $i_j < i_{j+1}$. We obtain the quotient group $\overline{S}/\underline{S} = \oplus_{j \in I} \langle 2^{\lambda_j - b_j} t_j + \underline{S} \rangle$ is elementary 2-group. Hence it is isomorphic to $\mathbb{Z}_2^k$ where $\mathbb{Z}_2 = \{0, 1\}$ is the Galois field of two elements with an isomorphism $\Omega_{a,I(S)}$ that maps

$$\Omega_{a,I(S)}(2^{\lambda_{i_j} - b_{i_j}} t_{i_j} + \underline{S}) = e_j = (0, \ldots, 1, \ldots, 0)^t.$$

The group $\mathbb{Z}_2^k$ can be viewed as k-dimensional vector space over the Galois field $\mathbb{Z}_2$. It is well known that the lattice of subspaces of a finite dimensional vector space is self-dual by using its dual space of linear functionals. Particularly, the mapping $A_k : U \longmapsto ann(U)$, for any U is a subgroup of $\mathbb{Z}_2^k$, where

$$ann(U) = \{(w_1, \ldots, w_k)^t \in \mathbb{Z}_2^k \ : \ \sum_{i=1}^{k} u_i w_i = 0, \text{for all } (u_1, \ldots, u_k)^t \in U\},$$

is a lattice anti isomorphism on the lattice of subgroups of $\mathbb{Z}_2^k$,

$$SG(\mathbb{Z}_2^k) = \{U \subseteq \mathbb{Z}_2^k \ : \ U \text{ subgroup}\}.$$

Hence the mapping

$$\Omega^{-1}_{(\lambda(G)-b),I(S)} A_k \Omega_{a,I(S)} : [\underline{S}, \overline{S}] \to [D(\overline{S}), D(\underline{S})]$$

is an anti isomorphism.

Hence we have the following theorem.

Theorem 3.2. *Let G be a finite abelian 2-group with a cyclic decomposition (2.2) and $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \in \mathcal{C}(G)$. If for every $i \in \{1, 2, , \ldots, m\}$ with $b_i \neq a_i$ implies λ_i is a simple exponent and $b_i = a_i + 1$ then there exists an anti-isomorphism $D_{a,b}$ from $[\theta(a), \theta(b)]$ onto $[\theta(\lambda(G) - b), \theta(\lambda(G) - a)]$ such that if $S \in [\theta(a), \theta(b)]$*

is fully invariant then $D_{a,b}(S)$ is fully invariant and if $S \in [\theta(a), \theta(b)]$ is irregular characteristic then $D_{a,b}(S)$ is also irregular characteristic.

Now combining Theorem 3.2 and the anti isomorphism (2.5) we obtain the following theorem.

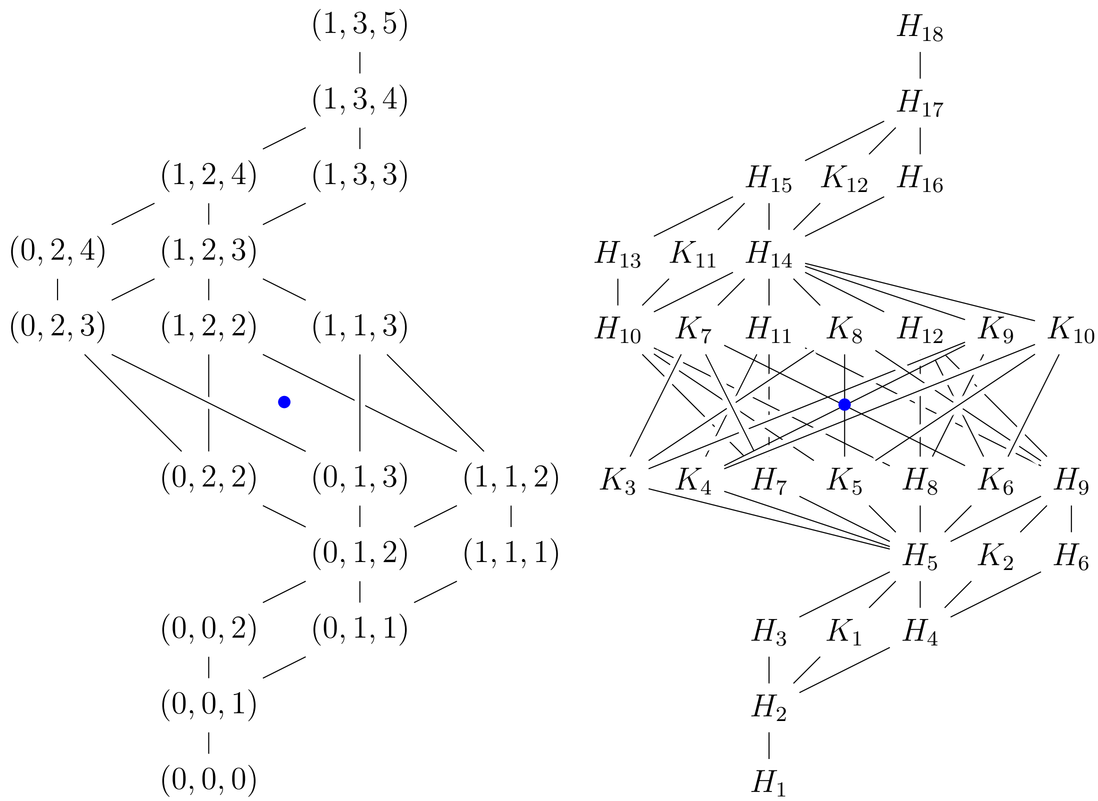
Theorem 3.3. *Let G be a finite abelian 2-group with a cyclic decomposition (2.2). The anti isomorphism $D : IF(G) \to IF(G)$ defined in (2.5) can be extended to become an anti isomorphism $\overline{D} : Char(G) \to Char(G)$ with $\overline{D}(S) = D_{a,b}(S)$ if $S \in Char(G)$ is irregular with $\underline{S} = \theta(a)$ and $\overline{S} = \theta(b)$.*

Mingueza et al. [6] also showed an anti isomorphism on characteristic subspaces lattice as the restriction of an anti isomorphism on lattice of subspaces to the characteristic subspaces lattice.

We close this section with the following example.

Example 3.1. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_{32} = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ be generated by three elements x, y, z with orders $o(x) = 2, o(y) = 2^3, o(z) = 2^5$. The exponents of generators of $G, 1, 3, 5,$ are simple and are not consecutive, hence we can expect G has an irregular characteristic subgroup. The lattices $\mathcal{C}(G)$ and $Char(G)$ are shown as follows:*

H_1	$\langle 0 \rangle$	H_7	$\langle 2y \rangle + \langle 8z \rangle$	H_{13}	$\langle 2y \rangle + \langle 2z \rangle$
H_2	$\langle 16z \rangle$	H_8	$\langle 4y \rangle + \langle 4z \rangle$	H_{14}	$\langle x \rangle + \langle 2y \rangle + \langle 4z \rangle$
H_3	$\langle 8z \rangle$	H_9	$\langle x \rangle + \langle 4y \rangle + \langle 8z \rangle$	H_{15}	$\langle x \rangle + \langle 2y \rangle + \langle 2z \rangle$
H_4	$\langle 4y \rangle + \langle 16z \rangle$	H_{10}	$\langle 2y \rangle + \langle 4z \rangle$	H_{16}	$\langle x \rangle + \langle y \rangle + \langle 4z \rangle$
H_5	$\langle 4y \rangle + \langle 8z \rangle$	H_{11}	$\langle x \rangle + \langle 2y \rangle + \langle 8z \rangle$	H_{17}	$\langle x \rangle + \langle y \rangle + \langle 2z \rangle$
H_6	$\langle x \rangle + \langle 4y \rangle + \langle 16z \rangle$	H_{12}	$\langle x \rangle + \langle 4y \rangle + \langle 4z \rangle$	H_{18}	$\langle x \rangle + \langle y \rangle + \langle z \rangle$
K_1	$H_2 + \langle 4y + 8z \rangle$	K_5	$H_5 + \langle 2y + 4z \rangle$	K_9	$H_8 + \langle x + 2y \rangle$
K_2	$H_4 + \langle x + 8z \rangle$	K_6	$H_5 + \langle x + 4z \rangle$	K_{10}	$H_5 + \langle x + 4z \rangle + \langle 2y + 4z \rangle$
K_3	$H_5 + \langle x + 2y + 4z \rangle$	K_7	$H_7 + \langle x + 4z \rangle$	K_{11}	$H_{10} + \langle x + 2z \rangle$
K_4	$H_5 + \langle x + 2y \rangle$	K_8	$H_9 + \langle 2y + 4z \rangle$	K_{12}	$H_{14} + \langle y + 2z \rangle$



The anti isomorphism D defined in (2.5) satisfies $D(H_i) = H_{17-i}$ for $i = 1, 2, \ldots, 16$ and the extension $\overline{D}$ satisfies $\overline{D}(K_j) = K_{13-j}$ for $j = 1, 2, \ldots, 12$. The self-dual property of the lattices $\mathcal{C}(G)$ and $Char(G)$ are also shown by their pictures, each of them is symmetric with respect to its center (the blue dot).

4. Conclusions

According to facts above we may read that results of Kerby and Rode [5] addressed the question: when do two finite abelian groups of odd order have isomorphic lattices of fully invariant subgroups? In connection with this view we are quite certain that the approach in [5] can be extended to obtain a complete answer of the question for any two finite abelian groups, including 2-groups. With this finding and Theorem 3.3, we expect the original considered question, that is when do two finite abelian 2-groups have isomorphic lattices of characteristic subgroups?, can be investigated through the following open question: given G, H two finite abelian 2-groups having both lattices $Char(G)$ and $Char(H)$ are isomorphic, are both the sublattices $FI(G)$ and $FI(H)$ isomorphic too?

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References

[1] P. Astuti and H.K. Wimmer, Characteristic subspaces and hyperinvariant frames, *Linear Algebra and its Applications*, **482** (2015), 21–46.

[2] P.A. Fillmore, D.A. Herrero, and W.E. Longstaff, The hyperinvariant subspace of a linear transformation, *Linear Algebra and its Applications*, **17** (1977), 125–132.

[3] L. Fuchs, *Infinite Abelian Groups*, vol.I, Academic Press, 1973.

[4] I.M Isaacs, *Algebra: A Graduate Course*, American Mathematical Society, 2009.

[5] B.L. Kerby and E. Rode, Characteristic subgroups of finite abelian groups, *Communications in Algebra,* **39(4)** (2011), 1315–1343.

[6] D. Mingueza, M.E. Montoro, and A. Roca, The characteristic subspace lattice of a linear transformation, *Linear Algebra and its Applications*, **506** (2016), 329–341.

[7] K. Saleh, P. Astuti and I. Muchtadi-Alamsyah, On the structure of finitely generated primary modules, *JP Journal of Algebra Number Theory and Applications,* **38(5)** (2016), 519-533.

[8] K.Shoda, Über die characteristischen Untergruppen einer endlichen Abelschen Gruppe, *Mathematische Zeitschrift*, **31** (1930), 611-624.

[9] Sarita, M. Jakhar, Characteristic subgroups of a finite abelian p-group, *Global and Stochastic Analysis,* **8(2)** (2021), 243-253.

[10] C. Y. Chew, A. Y. M. Chin, and C. S. Lim, The number of subgroups of finite abelian p-groups of rank 4 and higher, *Communications in Algebra*, **48(4)** (2020) DOI: 10.1080/00927872.2019.1691571, 2019.

(1) Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, 40132, West Java, Indonesia

Email address: afif.humam@math.itb.ac.id

(2) Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, 40132, West Java, Indonesia

Email address: pudji@math.itb.ac.id