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# CONGRUENCES FOR 5-REGULAR PARTITIONS WITH ODD PARTS OVERLINED

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Abstract. Let $\overline{a}_5(n)$ denote the number of 5-regular partitions of $n$ with the first occurrence of an odd number may be overlined. In this paper, we establish many infinite families of congruences modulo powers of 2 for $\overline{a}_5(n)$. For example, for all $n \geq 0$ and $\beta \geq 0$,

$$\overline{a}_5\left(16 \cdot 5^{2\beta+1}n + \frac{k_1 \cdot 5^{2\beta} - 1}{3}\right) \equiv 0 \pmod{16},$$

where $k_1 \in \{142, 238\}$.

## 1. Introduction

A partition of a positive integer $n$ is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. For positive integer $\ell > 1$, a partition is an $\ell$-regular partition of $n$ if none of the parts are divisible by $\ell$. Let $b_\ell(n)$ denote the number of $\ell$-regular partitions of $n$ with $b_\ell(0) = 1$ and the generating function for $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1},$$

where

$$f_\ell := (q^\ell; q^\ell)_\infty = \prod_{k=1}^{\infty}(1 - q^{k\ell})$$

and $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$, for any complex numbers $a$ and $q$ with $|q| < 1$.

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Arithmetic properties of $\ell$-regular partition functions have been studied by a number of mathematicians. We can see [3, 6, 14].

For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}.$$

By using the Jacobi's triple product identity [2, Entry 19, p. 35], the function $f(a,b)$ can be written as

$$f(a,b) := (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.1}$$

The most important special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}, \tag{1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1 \tag{1.4}$$

and

$$\chi(q) = (-q;q^2)_\infty. \tag{1.5}$$

An overpartition of a non-negative integer $n$ is a non-increasing sequence of natural numbers whose sum is $n$ where the first occurrence of parts of each size may be overlined. For example, the overpartitions of 5 are

$$\begin{gathered} 5,\ \overline{5},\ 4+1, \overline{4}+1,\ 4+\overline{1},\ \overline{4}+\overline{1},\ 3+2,\ \overline{3}+2,\ 3+\overline{2},\ \overline{3}+\overline{2},\ 3+1+1,\ \overline{3}+1+1, \\ 3+\overline{1}+1,\ \overline{3}+\overline{1}+1,\ 2+2+1,\ \overline{2}+2+1,\ 2+2+\overline{1},\ \overline{2}+2+\overline{1},\ 2+1+1+1, \\ \overline{2}+1+1+1,\ 2+\overline{1}+1+1,\ \overline{2}+\overline{1}+1+1,\ 1+1+1+1+1,\ \overline{1}+1+1+1+1. \end{gathered}$$

Corteel and Lovejoy [5] obtained the following generating function for $\overline{p}(n)$, the number of overpartitions of $n$ with $\overline{p}(0) = 1$.

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + \cdots. \tag{1.6}$$

In [5], the authors extensively studied on overpartition function $\overline{p}(n)$ as a means of better understanding and interpreting various $q$-series identities. Later, Hirschhorn and Sellers [9] proved a number of arithmetic relations satisfied by $\overline{p}(n)$ and also obtained many Ramanujan-type congruences modulo powers of 2 for $\overline{p}(n)$. For example, for all $n \geq 0$,

$$\overline{p}(9n+3) \equiv 0 \pmod{8}$$

and

$$\overline{p}(9n+4) \equiv 0 \pmod{8}.$$

For more details about $\overline{p}(n)$, one can see [1, 11, 12, 16, 19, 20].

Hirschhorn and Sellers [10] considered the partition function $\overline{p_o}(n)$, the number of overpartitions of $n$ into odd parts. The generating function for $\overline{p_o}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p_o}(n) q^n = \prod_{n=1}^{\infty} \frac{1+q^{2n+1}}{1-q^{2n-1}} = 1 + 2q + 2q^2 + 4q^3 + 6q^4 + \cdots . \tag{1.7}$$

They proved a number of arithmetic results including several Ramanujan-type congruences satisfied by $\overline{p_o}(n)$ and some easily-stated characterizations of $\overline{p_o}(n)$ modulo small powers of 2. For example, for all $n \geq 1$,

$$\overline{p_o}(n) \equiv \begin{cases} 2 \pmod{4} & \textit{if } n \textit{ is square or } n \textit{ is twice a square}, \\ 0 \pmod{4} & \textit{otherwise}. \end{cases} \tag{1.8}$$

Later, Chen [4] proved an identity of $\overline{p_o}(n)$ and established many explicit Ramanujan-type congruences for $\overline{p_o}(n)$ modulo 32 and 64. For example, let $t \geq 0$ be an integer and $p \equiv 1 \pmod{8}$ be a prime, then for all non-negative integers $n$ with $n \not\equiv -\frac{7}{8} \pmod{p}$,

$$\overline{p_o}(16p^{2t+1}n + 16\lambda_{p,t} + 14) \equiv 0 \pmod{32}$$

and

$$\overline{p_o}(16p^{4t+3}n + 16\delta_{p,t} + 14) \equiv 0 \pmod{64},$$

where $\lambda_{p,t} = \dfrac{7(p^{2t+1}-1)}{8}$ and $\delta_{p,t} = \dfrac{7(p^{4t+3}-1)}{8}$.

For more details about $\overline{p_o}(n)$, one can see [18].

In [13], the authors defined $\overline{a}_{4,5}(n)$, the number of $(4,5)$-regular partitions of $n$ with the first occurrence of an odd number may be overlined. Also, they established many infinite families of congruences modulo powers of 2 for $\overline{a}_{4,5}(n)$.

By the motivation of the above work, in this paper, we define $\overline{a}_5(n)$, the number of 5-regular partitions of $n$ with the first occurrence of an odd number may be overlined. The generating function for $\overline{a}_5(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{a}_5\,(n)\,q^n = \frac{(-q;q^2)_\infty\,(q^5;q^5)_\infty}{(q;q)_\infty\,(-q^5;q^{10})_\infty}. \tag{1.9}$$

For example, there are 14 partitions for $\overline{a}_5(5)$, namely

$$4+1, 4+\overline{1},\ 3+2,\ \overline{3}+2,\ 3+1+1,\ \overline{3}+1+1,\ 3+\overline{1}+1,\ \overline{3}+\overline{1}+1,\ 2+2+1,$$
$$2+2+\overline{1},\ 2+1+1+1,\ 2+\overline{1}+1+1,\ 1+1+1+1+1,\ \overline{1}+1+1+1+1.$$

Also, we establish many infinite families of congruences modulo powers of 2 for $\overline{a}_5(n)$. For example, for all $n \geq 0$ and $\beta \geq 0$,

$$\overline{a}_5\left(16\cdot 5^{2\beta+1}n + \frac{k_1\cdot 5^{2\beta}-1}{3}\right) \equiv 0 \pmod{16},$$

where $k_1 \in \{142, 238\}$.

## 2. Preliminary results

In this section, we collect some identities which are useful in proving our main results.

**Lemma 2.1.** *The following 2-dissections hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.1}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2.2}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.3}$$

The identity (2.1) is the 2-dissection of $\phi(q)$ [7, 1.9.4]. The equation (2.2) obtained from (2.1) by replacing $q$ by $-q$. The identity (2.3) is the 2-dissection of $\phi(q)^2$ [7, 1.10.1]. Also, one can see [2, p.40].

**Lemma 2.2.** *The following 2-dissections hold:*

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2} \tag{2.4}$$

*and*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{2.5}$$

The equation (2.4) was proved by Hirschhorn and Sellers [8]; see also [17]. Replacing $q$ by $-q$ in (2.4) and using the fact that $(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}$, we obtain (2.5).

**Lemma 2.3.** *We have*

$$\frac{1}{f_1^3 f_5} = \frac{f_4^4}{f_2^7 f_{10}} - 2q\frac{f_4^6 f_{20}^2}{f_2^9 f_{10}^3} + 5q\frac{f_4^3 f_{20}}{f_2^8} + 2q^2\frac{f_4^9 f_{40}^2}{f_2^{10} f_8^2 f_{10}^2 f_{20}}, \tag{2.6}$$

$$f_1^3 f_5 = \frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3 + 2q^2\frac{f_4^6 f_{10} f_{40}^2}{f_2 f_8^2 f_{20}^2} \tag{2.7}$$

*and*

$$f_1 f_5^3 = f_2^3 f_{10} - q\frac{f_2^2 f_{10}^2 f_{20}}{f_4} + 2q^2 f_4 f_{20}^3 - 2q^3\frac{f_4^4 f_{10} f_{40}^2}{f_2 f_8^2}. \tag{2.8}$$

The equations (2.6) and (2.7) obtained from (4.25) and (4.26) in [15] respectively. The equation (2.8) obtained from (4.13) in [15].

**Lemma 2.4.** [7, p. 85, 8.1.1] *We have the following* 5*-dissection formula*

$$f_1 = f_{25}(a(q^5) - q - q^2/a(q^5)), \tag{2.9}$$

*where*

$$a := a(q) := \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty}. \tag{2.10}$$

**Lemma 2.5.** *For any positive integers $k$ and $m$, we have*

$$f_k^{2m} \equiv f_{2k}^m \pmod 2, \tag{2.11}$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod 4 \tag{2.12}$$

*and*

$$f_k^{8m} \equiv f_{2k}^{4m} \pmod 8. \tag{2.13}$$

## 3. Congruences for $\overline{a}_5(n)$

In this section, we prove many infinite families of congruences modulo powers of 2 for $\overline{a}_5(n)$.

**Theorem 3.1.** Let $k_1 \in \{142, 238\}$ and $k_2 \in \{86, 134\}$, then for all $n \geq 0$ and $\beta \geq 0$, we have

$$\overline{a}_5\left(16n+7\right) \equiv 0 \pmod{16}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16 \cdot 5^{2\beta} n + \frac{46 \cdot 5^{2\beta}-1}{3}\right) q^n \equiv 8 f_1^3 f_{20} \pmod{16}, \tag{3.2}$$

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16 \cdot 5^{2\beta+1} n + \frac{38 \cdot 5^{2\beta+1}-1}{3}\right) q^n \equiv 8 f_4 f_5^3 \pmod{16}, \tag{3.3}$$

$$\overline{a}_5\left(16 \cdot 5^{2\beta+1} n + \frac{k_1 \cdot 5^{2\beta}-1}{3}\right) \equiv 0 \pmod{16}, \tag{3.4}$$

$$\overline{a}_5\left(16 \cdot 5^{2\beta+2} n + \frac{k_2 \cdot 5^{2\beta+1}-1}{3}\right) \equiv 0 \pmod{16}. \tag{3.5}$$

*Proof.* From the equation (1.9), we see that

$$\sum_{n=0}^{\infty} \overline{a}_5\left(n\right) q^n = \frac{f_2^2 f_5^2 f_{20}}{f_1^2 f_4 f_{10}^2}. \tag{3.6}$$

Employing (2.5) in (3.6) and then collecting the coefficients of $q^{2n+1}$ from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{a}_5\left(2n+1\right) q^n = 2\frac{f_2^2 f_{10}^2}{f_1^3 f_5}. \tag{3.7}$$

Using (2.3) and (2.4) in (3.7) and then collecting the even and odd terms from both sides, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5\left(4n+1\right) q^n = 2\frac{f_2^{13} f_{10}^3}{f_1^{11} f_4^3 f_5 f_{20}} - 8q\frac{f_2^4 f_4^3 f_{20}}{f_1^8} \tag{3.8}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_5\left(4n+3\right) q^n = 8\frac{f_2 f_4^5 f_{10}^3}{f_1^7 f_5 f_{20}} - 2\frac{f_2^{16} f_{20}}{f_1^{12} f_4^5}. \tag{3.9}$$

Invoking (2.11) and (2.13) in (3.9), we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_5(4n+3)q^n \equiv 8f_1^3 f_2^6 f_5 + 14\frac{f_2^4 f_{20}}{f_1^4 f_4} \pmod{16}. \tag{3.10}$$

Substituting (2.3) and (2.7) in (3.10), we get

$$\sum_{n=0}^{\infty} \overline{a}_5(8n+3)q^n \equiv 8\frac{f_1^8 f_2 f_5^2}{f_{10}} + 14\frac{f_2^{13} f_{10}}{f_1^{10} f_4^4} \pmod{16} \tag{3.11}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_5(8n+7)q^n \equiv 8f_1^7 f_5^3 + 8\frac{f_2 f_4^4 f_{10}}{f_1^6} \pmod{16}. \tag{3.12}$$

The equation (3.12) reduces to

$$\sum_{n=0}^{\infty} \overline{a}_5(8n+7)q^n \equiv 8f_1 f_2^3 f_5^3 + 8f_4^3 f_{10} \pmod{16}. \tag{3.13}$$

Substituting (2.8) in (3.13), we have

$$\sum_{n=0}^{\infty} \overline{a}_5(16n+7)q^n \equiv 8f_1^6 f_5 + 8f_2^3 f_5 \pmod{16} \tag{3.14}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_5(16n+15)q^n \equiv 8\frac{f_1^5 f_5^2 f_{10}}{f_2} \pmod{16}. \tag{3.15}$$

From the equation (3.14), we arrive at (3.1).

The equation (3.15) becomes

$$\sum_{n=0}^{\infty} \overline{a}_5(16n+15)q^n \equiv 8f_1^3 f_{20} \pmod{16}, \tag{3.16}$$

which is $\beta = 0$ case of (3.2). Suppose that the congruence (3.2) is true for $\beta \geq 0$, we have

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16 \cdot 5^{2\beta} n + \frac{46 \cdot 5^{2\beta} - 1}{3}\right) q^n \equiv 8f_1^3 f_{20} \pmod{16}. \tag{3.17}$$

Employing (2.9) in (3.17) and then collecting the coefficients of $q^{5n+3}$ from both sides, we get

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16 \cdot 5^{2\beta+1} n + \frac{38 \cdot 5^{2\beta+1} - 1}{3}\right) q^n \equiv 8f_4 f_5^3 \pmod{16}. \tag{3.18}$$

Again, using (2.9) in (3.18) and then comparing the terms involving $q^{5n+4}$ on both sides, we have

$$\sum_{n=0}^{\infty} \overline{a}_5 \left(16 \cdot 5^{2\beta+2} n + \frac{46 \cdot 5^{2\beta+2} - 1}{3}\right) q^n \equiv 8 f_1^3 f_{20} \pmod{16}, \tag{3.19}$$

which implies that the congruence (3.2) is true for $\beta + 1$. Hence, by mathematical induction, the congruence (3.2) holds for all integer $\beta \geq 0$.

Employing (2.9) in (3.2) and then collecting the coefficients of $q^{5n+3}$ from both sides of the resultant equation, we obtain (3.3).

Employing (2.9) in (3.2) and then comparing the coefficients of $q^{5n+i}$ for $i = 2, 4$ on both sides of the resultant equation, we get (3.4).

Substituting (2.9) in (3.3) and then collecting the terms involving $q^{5n+i}$ for $i = 1, 2$ from both sides, we arrive at (3.5). □

**Theorem 3.2.** If $n$ can not be represented as a sum of twenty times a pentagonal number and once a triangular number, then

$$\overline{a}_5 (16n + 15) \equiv 0 \pmod{16}. \tag{3.20}$$

*Proof.* The equation (3.15) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_5 (16n + 15) q^n \equiv 8 \frac{f_2^2 f_{20}}{f_1} \pmod{16}. \tag{3.21}$$

In view of (1.3) and (3.21), we have

$$\sum_{n=0}^{\infty} \overline{a}_5 (16n + 15) q^n \equiv 8 f_{20} \psi(q) \pmod{16}. \tag{3.22}$$

Combining (1.3), (1.4) and (3.22), we have

$$\sum_{n=0}^{\infty} \overline{a}_5 (16n + 15) q^n \equiv 8 \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} q^{10k(3k-1) + \frac{n(n+1)}{2}} \pmod{16}. \tag{3.23}$$

The result (3.20) follows from (3.23). □

**Theorem 3.3.** For all $n \geq 0$ and $\beta \geq 0$, we have

$$\overline{a}_5 \left(16 \cdot 5^{2\beta+2} n + \frac{34 \cdot 5^{2\beta+2} - 1}{3}\right) \equiv 3^{\beta+1} \cdot \overline{a}_5 (16n + 11) \pmod{16}, \tag{3.24}$$

$$\overline{a}_5 (16(5n + i) + 11) \equiv 0 \pmod{16}, \tag{3.25}$$

$$\overline{a}_5\left(80(5n+j)+43\right) \equiv 0 \pmod{16}, \tag{3.26}$$

where $i = 1, 3$ and $j = 0, 1$.

*Proof.* The equation (3.11) implies

$$\sum_{n=0}^{\infty} \overline{a}_5\left(8n+3\right) q^n \equiv 8f_2^5 + 14\frac{f_2 f_{10}}{f_1^2} \pmod{16}. \tag{3.27}$$

Substituting (2.1) in (3.27), we have

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16n+3\right) q^n \equiv 8f_1^5 + 14\frac{f_4^5 f_5}{f_1^4 f_8^2} \pmod{16} \tag{3.28}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16n+11\right) q^n \equiv 12\frac{f_2^2 f_5 f_8^2}{f_1^4 f_4} \pmod{16}. \tag{3.29}$$

The equation (3.29) becomes

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16n+11\right) q^n \equiv 12f_4^3 f_5 \pmod{16}. \tag{3.30}$$

Substituting (2.9) in (3.30) and then comparing the coefficients of $q^{5n+2}$ on both sides, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{a}_5\left(80n+43\right) q^n &\equiv 12q^2 f_1 f_{20}^3 \\ &\equiv 12q^2 f_{20}^3 f_{25}(a(q^5) - q - q^2/a(q^5)) \pmod{16}, \end{aligned} \tag{3.31}$$

which implies

$$\sum_{n=0}^{\infty} \overline{a}_5\left(400n+283\right) q^n \equiv 4f_4^3 f_5 \pmod{16}. \tag{3.32}$$

In view of the congruences (3.30) and (3.32), we see that

$$\overline{a}_5\left(400n+283\right) \equiv 3 \cdot \overline{a}_5\left(16n+11\right) \pmod{16}. \tag{3.33}$$

By induction on $\beta$, we arrive at (3.24).

Employing (2.9) in (3.30) and then extracting the terms involving $q^{5n+i}$ for $i = 1, 3$ from both sides of the resultant equation, we obtain (3.25).

The equation (3.26) can be obtained by collecting the coefficients of $q^{5n}$ and $q^{5n+1}$ from both sides of the equation (3.31). □

**Theorem 3.4.** For all $n \geq 0$, we have

$$\overline{a}_5\left(16(5n+i)+3\right) \equiv 0 \pmod{4}, \quad \text{where } i = 1, 2, 3, 4, \tag{3.34}$$

$$\overline{a}_5\left(80n+3\right) \equiv \begin{cases} 2 & \pmod{4} \quad \text{if n is a pentagonal number,} \\ 0 & \pmod{4} \quad \text{otherwise.} \end{cases} \tag{3.35}$$

*Proof.* The equation (3.28) becomes

$$\sum_{n=0}^{\infty} \overline{a}_5\left(16n+3\right) q^n \equiv 2f_5 \pmod{4}. \tag{3.36}$$

Extracting the coefficients of $q^{5n+i}$ for $i = 1, 2, 3, 4$ from both sides of the above equation, we get (3.34).
The equation (3.36) implies

$$\sum_{n=0}^{\infty} \overline{a}_5\left(80n+3\right) q^n \equiv 2f_1 \pmod{4}. \tag{3.37}$$

The result (3.35) obtained from the equations (1.4) and (3.37). □

**Theorem 3.5.** Let $k_3 \in \{568, 952\}$ and $k_4 \in \{344, 536\}$, then for all $n \geq 0$ and $\beta \geq 0$, we have

$$\overline{a}_5\left(64n+29\right) \equiv 0 \pmod{16}, \tag{3.38}$$

$$\sum_{n=0}^{\infty} \overline{a}_5\left(64 \cdot 5^{2\beta} n + \frac{184 \cdot 5^{2\beta} - 1}{3}\right) q^n \equiv 8f_1^3 f_{20} \pmod{16}, \tag{3.39}$$

$$\sum_{n=0}^{\infty} \overline{a}_5\left(64 \cdot 5^{2\beta+1} n + \frac{152 \cdot 5^{2\beta+1} - 1}{3}\right) q^n \equiv 8f_4 f_5^3 \pmod{16}, \tag{3.40}$$

$$\overline{a}_5\left(64 \cdot 5^{2\beta+1} n + \frac{k_3 \cdot 5^{2\beta} - 1}{3}\right) \equiv 0 \pmod{16}, \tag{3.41}$$

$$\overline{a}_5\left(64 \cdot 5^{2\beta+2} n + \frac{k_4 \cdot 5^{2\beta+1} - 1}{3}\right) \equiv 0 \pmod{16}. \tag{3.42}$$

*Proof.* Invoking (2.11) and (2.13) in (3.8), we find that

$$\sum_{n=0}^{\infty} \overline{a}_5\,(4n+1)\,q^n \equiv 2\frac{f_2 f_4 f_{10}^3}{f_1^3 f_5 f_{20}} + 8q f_4^3 f_{20} \pmod{16}. \tag{3.43}$$

Substituting (2.3) and (2.4) in (3.43), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5\,(8n+1)\,q^n \equiv 2\frac{f_2^2 f_4 f_{10}^2}{f_1^4 f_{20}} + 8q f_1 f_4^3 f_5^3 \pmod{16} \tag{3.44}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_5\,(8n+5)\,q^n \equiv 14\frac{f_2^5 f_5 f_{20}}{f_1^5 f_4 f_{10}} + 8 f_2^8 + 8 f_2^3 f_{10} \pmod{16}. \tag{3.45}$$

Using (2.3) and (2.5) in (3.45), we get

$$\sum_{n=0}^{\infty} \overline{a}_5\,(16n+5)\,q^n \equiv 14\frac{f_2 f_4 f_{10}^3}{f_1^3 f_5 f_{20}} + 8 f_2^4 + 8 f_1^3 f_5 + 8q f_4^3 f_{20} \pmod{16} \tag{3.46}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_5\,(16n+13)\,q^n \equiv 8 f_1^3 f_4^3 f_5 + 14\frac{f_2^4 f_{20}}{f_1^4 f_4} \pmod{16}. \tag{3.47}$$

Employing (2.3) and (2.7) in (3.47), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5\,(32n+13)\,q^n \equiv 8 f_2^5 + 14\frac{f_2 f_{10}}{f_1^2} \pmod{16} \tag{3.48}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_5\,(32n+29)\,q^n \equiv 8 f_1 f_2^3 f_5^3 + 8 f_4^3 f_{10} \pmod{16}. \tag{3.49}$$

Using (2.8) in (3.49), we get

$$\sum_{n=0}^{\infty} \overline{a}_5\,(64n+29)\,q^n \equiv 8 f_1^6 f_5 + 8 f_2^3 f_5 \pmod{16} \tag{3.50}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_5\,(64n+61)\,q^n \equiv 8 f_1^3 f_{20} \pmod{16}. \tag{3.51}$$

From the equation (3.50), we arrive at (3.38).
The equation (3.51) is $\beta = 0$ case of (3.39). The rest of the proofs of the congruences (3.39)-(3.42) are similar to the proofs of the congruences (3.2)-(3.5). So, we omit the details. □

**Theorem 3.6.** If $n$ can not be represented as a sum of twenty times a pentagonal number and once a triangular number, then

$$\overline{a}_5\left(64n+61\right)\equiv 0 \pmod{16}. \tag{3.52}$$

*Proof.* The equation (3.51) can be written as

$$\sum_{n=0}^{\infty}\overline{a}_5\left(64n+61\right)q^n\equiv 8\frac{f_2^2f_{20}}{f_1} \pmod{16}. \tag{3.53}$$

In view of (1.3) and (3.53), we have

$$\sum_{n=0}^{\infty}\overline{a}_5\left(64n+61\right)q^n\equiv 8f_{20}\psi(q) \pmod{16}. \tag{3.54}$$

Combining (1.3), (1.4) and (3.54), we have

$$\sum_{n=0}^{\infty}\overline{a}_5\left(64n+61\right)q^n\equiv 8\sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty}q^{10k(3k-1)+\frac{n(n+1)}{2}} \pmod{16}. \tag{3.55}$$

The result (3.52) follows from (3.55). □

**Theorem 3.7.** For all $n\geq 0$ and $\beta\geq 0$, we have

$$\overline{a}_5\left(64\cdot 5^{2\beta+2}n+\frac{136\cdot 5^{2\beta+2}-1}{3}\right)\equiv 3^{\beta+1}\cdot\overline{a}_5\left(64n+45\right) \pmod{16}, \tag{3.56}$$

$$\overline{a}_5\left(64(5n+i)+45\right)\equiv 0 \pmod{16}, \tag{3.57}$$

$$\overline{a}_5\left(320(5n+j)+173\right)\equiv 0 \pmod{16}, \tag{3.58}$$

where $i=1,3$ and $j=0,1$.

*Proof.* Employing (2.1) in (3.48), we obtain

$$\sum_{n=0}^{\infty}\overline{a}_5\left(64n+13\right)q^n\equiv 8f_1^5+14\frac{f_4^5f_5}{f_1^4f_8^2} \pmod{16} \tag{3.59}$$

and

$$\sum_{n=0}^{\infty}\overline{a}_5\left(64n+45\right)q^n\equiv 12f_4^3f_5 \pmod{16}. \tag{3.60}$$

Substituting (2.9) in (3.60) and then comparing the coefficients of $q^{5n+2}$ on both sides, we get

$$\sum_{n=0}^{\infty} \overline{a}_5\,(320n+173)\,q^n \equiv 12q^2 f_1 f_{20}^3$$

$$\equiv 12q^2 f_{20}^3 f_{25}(a(q^5) - q - q^2/a(q^5)) \pmod{16}, \tag{3.61}$$

which implies

$$\sum_{n=0}^{\infty} \overline{a}_5\,(1600n+1133)\,q^n \equiv 4f_4^3 f_5 \pmod{16}. \tag{3.62}$$

In view of the congruences (3.60) and (3.62), we see that

$$\overline{a}_5\,(1600n+1133) \equiv 3\cdot\overline{a}_5\,(64n+45) \pmod{16}. \tag{3.63}$$

By induction on $\beta$, we arrive at (3.56).

Employing (2.9) in (3.60) and then extracting the terms involving $q^{5n+i}$ for $i = 1,3$ from both sides of the resultant equation, we obtain (3.57).

The equation (3.58) can be obtained by collecting the coefficients of $q^{5n}$ and $q^{5n+1}$ from both sides of the equation (3.61). □

**Theorem 3.8.** For all $n \geq 0$, we have

$$\overline{a}_5\,(64(5n+i)+13) \equiv 0 \pmod{4}, \quad \text{where } i = 1,2,3,4, \tag{3.64}$$

$$\overline{a}_5\,(320n+13) \equiv \begin{cases} 2 & \pmod{4} \quad \text{if n is a pentagonal number,} \\ 0 & \pmod{4} \quad \text{otherwise.} \end{cases} \tag{3.65}$$

*Proof.* The equation (3.59) becomes

$$\sum_{n=0}^{\infty} \overline{a}_5\,(64n+13)\,q^n \equiv 2f_5 \pmod{4}. \tag{3.66}$$

Extracting the coefficients of $q^{5n+i}$ for $i = 1,2,3,4$ from both sides of the above equation, we get (3.64).

The equation (3.66) implies

$$\sum_{n=0}^{\infty} \overline{a}_5\,(320n+13)\,q^n \equiv 2f_1 \pmod{4}. \tag{3.67}$$

The result (3.65) obtained from the equations (1.4) and (3.67). □

**Theorem 3.9.** Let $k_5 \in \{352, 448\}$ and $k_6 \in \{224, 416\}$, then for all $n \geq 0$ and $\alpha, \beta \geq 0$, we have

(3.68)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(32 \cdot 2^{2\alpha} \cdot 5^{2\beta} n + \frac{64 \cdot 2^{2\alpha} \cdot 5^{2\beta} - 1}{3}\right) q^n \equiv 3^{\alpha+\beta}(12 f_2^3 f_{10} + 14 f_1 f_5^3) \pmod{16},$$

(3.69)
$$\overline{a}_5\left(32 \cdot 2^{2\alpha} \cdot 5^{2\beta+1} n + \frac{k_5 \cdot 2^{2\alpha} \cdot 5^{2\beta} - 1}{3}\right) \equiv 0 \pmod{16},$$

(3.70)
$$\overline{a}_5\left(32 \cdot 2^{2\alpha} \cdot 5^{2\beta+2} n + \frac{k_6 \cdot 2^{2\alpha} \cdot 5^{2\beta+1} - 1}{3}\right) \equiv 0 \pmod{16}.$$

*Proof.* Using (2.6) and (2.7) in (3.46), we have

(3.71)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(32n + 5\right) q^n \equiv 14 \frac{f_2^5 f_5^2}{f_1^6 f_{10}} + 12 q f_1^3 f_5 f_{10}^2 \pmod{16}$$

and

(3.72)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(32n + 21\right) q^n \equiv 12 f_2^3 f_{10} + 14 f_1 f_5^3 \pmod{16}.$$

The equation (3.72) is $\alpha = \beta = 0$ case of (3.68). Suppose that the congruence (3.68) is true for $\alpha \geq 0$ with $\beta = 0$, we have

(3.73)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(32 \cdot 2^{2\alpha} n + \frac{64 \cdot 2^{2\alpha} - 1}{3}\right) q^n \equiv 3^{\alpha}(12 f_2^3 f_{10} + 14 f_1 f_5^3) \pmod{16}.$$

Using (2.8) in (3.73) and then comparing the coefficients of $q^{2n}$ on both sides of the resultant equation, we get

(3.74)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(32 \cdot 2^{2\alpha+1} n + \frac{64 \cdot 2^{2\alpha} - 1}{3}\right) q^n \equiv 3^{\alpha}(10 f_1^3 f_5 + 12 q f_2 f_{10}^3) \pmod{16}.$$

Substituting (2.7) in (3.74) and then collecting the coefficients of $q^{2n+1}$ from both sides of the resultant equation, we arrive at

(3.75)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(32 \cdot 2^{2\alpha+2} n + \frac{64 \cdot 2^{2\alpha+2} - 1}{3}\right) q^n \equiv 3^{\alpha}(4 f_2^3 f_{10} + 10 q f_1 f_5^3) \pmod{16},$$

which implies that the congruence (3.68) is true for $\alpha+1$ with $\beta = 0$. So, by induction, the congruence (3.68) holds for all integer $\alpha \geq 0$ with $\beta = 0$. Suppose that the congruence (3.68) is true for $\alpha, \beta \geq 0$, we have

(3.76)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(32\cdot 2^{2\alpha}\cdot 5^{2\beta}n+\frac{64\cdot 2^{2\alpha}\cdot 5^{2\beta}-1}{3}\right)q^n \equiv 3^{\alpha+\beta}(12f_2^3f_{10}+14f_1f_5^3) \pmod{16}.$$

Employing (2.9) in (3.76) and then extracting the terms involving $q^{5n+1}$ from both sides, we obtain

(3.77)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(32\cdot 2^{2\alpha}\cdot 5^{2\beta+1}n+\frac{32\cdot 2^{2\alpha}\cdot 5^{2\beta+1}-1}{3}\right)q^n \equiv 3^{\alpha+\beta}(2f_1^3f_5+12qf_2f_{10}^3) \pmod{16}.$$

Again, using (2.9) in (3.77) and then collecting the coefficients of $q^{5n+3}$ from both sides of the resultant equation, we arrive at

(3.78)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(32\cdot 2^{2\alpha}\cdot 5^{2\beta+2}n+\frac{64\cdot 2^{2\alpha}\cdot 5^{2\beta+2}-1}{3}\right)q^n \equiv 3^{\alpha+\beta}(4f_2^3f_{10}+10f_1f_5^3) \pmod{16},$$

which implies that the congruence (3.68) is true for $\beta+1$. Hence, by mathematical induction, the congruence (3.68) holds for all integers $\alpha, \beta \geq 0$.
Using (2.9) in (3.68) and then collecting the coefficients of $q^{5n+i}$ for $i = 3, 4$ from both sides of the resultant equation, we get (3.69).
Using (2.9) in (3.77) and then comparing the coefficients of $q^{5n+i}$ for $i = 2, 4$ on both sides of the resultant equation, we obtain (3.70). □

**Theorem 3.10.** Let $k_7 \in \{172, 268\}$ and $k_8 \in \{284, 476\}$, then for all $n \geq 0$ and $\beta \geq 0$, we have

(3.79)
$$\overline{a}_5(32n+9) \equiv 0 \pmod{16},$$

(3.80)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(32\cdot 5^{2\beta}n+\frac{76\cdot 5^{2\beta}-1}{3}\right)q^n \equiv 8f_4f_5^3 \pmod{16},$$

(3.81)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(32\cdot 5^{2\beta+1}n+\frac{92\cdot 5^{2\beta+1}-1}{3}\right)q^n \equiv 8f_1^3f_{20} \pmod{16},$$

(3.82)
$$\overline{a}_5\left(32\cdot 5^{2\beta+1}n+\frac{k_7\cdot 5^{2\beta}-1}{3}\right) \equiv 0 \pmod{16},$$

$$\overline{a}_5\left(32\cdot 5^{2\beta+2}n+\frac{k_8\cdot 5^{2\beta+1}-1}{3}\right)\equiv 0 \pmod{16}. \tag{3.83}$$

*Proof.* Employing (2.3) and (2.8) in (3.44), we get

$$\sum_{n=0}^{\infty}\overline{a}_5\left(16n+1\right)q^n\equiv 2\frac{f_2^3f_5^2}{f_1^4f_{10}}+8qf_2^3f_{20} \pmod{16} \tag{3.84}$$

and

$$\sum_{n=0}^{\infty}\overline{a}_5\left(16n+9\right)q^n\equiv 8f_2^7+8f_1^3f_2^3f_5 \pmod{16}. \tag{3.85}$$

Substituting (2.7) in (3.85), we obtain

$$\sum_{n=0}^{\infty}\overline{a}_5\left(32n+9\right)q^n\equiv 8f_1^7+8\frac{f_1^5f_2f_5^2}{f_{10}} \pmod{16} \tag{3.86}$$

and

$$\sum_{n=0}^{\infty}\overline{a}_5\left(32n+25\right)q^n\equiv 8f_4f_5^3 \pmod{16}. \tag{3.87}$$

From the equation (3.86), we arrive at (3.79).

The equation (3.87) is $\beta=0$ case of (3.80). Suppose that the congruence (3.80) is true for $\beta\geq 0$ and employing (2.9) in (3.80), we have

$$\sum_{n=0}^{\infty}\overline{a}_5\left(32\cdot 5^{2\beta+1}n+\frac{92\cdot 5^{2\beta+1}-1}{3}\right)q^n\equiv 8f_1^3f_{20} \pmod{16}. \tag{3.88}$$

Again, using (2.9) in (3.88) and then collecting the coefficients of $q^{5n+3}$ from both sides, we get

$$\sum_{n=0}^{\infty}\overline{a}_5\left(32\cdot 5^{2\beta+2}n+\frac{76\cdot 5^{2\beta+2}-1}{3}\right)q^n\equiv 8f_4f_5^3 \pmod{16}, \tag{3.89}$$

which implies that the congruence (3.80) is true for $\beta+1$. So, by induction, the congruence (3.80) holds for all integer $\beta\geq 0$.

Employing (2.9) in (3.80) and then extracting the terms involving $q^{5n+4}$ from both sides of the resultant equation, we arrive at (3.81).

From the equation (3.80) along with (2.9), we obtain (3.82).

Using (2.9) in (3.81) and then comparing the coefficients of $q^{5n+2}$ and $q^{5n+4}$ on both sides of the resultant equation, we arrive at (3.83). □

**Theorem 3.11.** If $n$ can not be represented as a sum of four times a pentagonal number and five times a triangular number, then

$$\overline{a}_5\,(32n+25) \equiv 0 \pmod{16}. \tag{3.90}$$

*Proof.* The equation (3.87) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_5\,(32n+25)\,q^n \equiv 8\frac{f_4 f_{10}^2}{f_5} \pmod{16}. \tag{3.91}$$

In view of (1.3) and (3.91), we have

$$\sum_{n=0}^{\infty} \overline{a}_5\,(32n+25)\,q^n \equiv 8 f_4 \psi(q^5) \pmod{16}. \tag{3.92}$$

Combining (1.3), (1.4) and (3.92), we have

$$\sum_{n=0}^{\infty} \overline{a}_5\,(32n+25)\,q^n \equiv 8 \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} q^{2k(3k-1)+\frac{5n(n+1)}{2}} \pmod{16}. \tag{3.93}$$

The result (3.90) follows from (3.93). □

**Theorem 3.12.** For all $n \geq 0$ and $\beta \geq 0$, we have

$$\overline{a}_5\left(32 \cdot 5^{2\beta+2} n + \frac{52 \cdot 5^{2\beta+2} - 1}{3}\right) \equiv 3^{\beta+1} \cdot \overline{a}_5\,(32n+17) \pmod{16}, \tag{3.94}$$

$$\overline{a}_5\,(160(5n+i)+113) \equiv 0 \pmod{16}, \tag{3.95}$$

where $i = 1, 3$.

*Proof.* Using (2.2) and (2.3) in (3.84), we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_5\,(32n+1)\,q^n \equiv 2\frac{f_2^2 f_{20}^5}{f_1^3 f_{10}^2 f_{40}^2} \pmod{16} \tag{3.96}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_5\,(32n+17)\,q^n \equiv 8 f_1^{13} + 8 f_1^3 f_{10} + 12 q^2 f_1 f_{20}^3 \pmod{16}. \tag{3.97}$$

Employing (2.9) in (3.97) and then comparing the coefficients of $q^{5n+3}$ on both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{a}_5\,(160n+113)\,q^n \equiv 4 f_4^3 f_5 + 8 f_2 f_5^3 + 8 q^2 f_5^{13} \pmod{16}. \tag{3.98}$$

Again, using (2.9) in (3.98) and then collecting the coefficients of $q^{5n+2}$ from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_5\,(800n+433)\,q^n \equiv 8f_1^{13}+8f_1^3 f_{10}+4q^2 f_1 f_{20}^3 \pmod{16}. \tag{3.99}$$

In view of the congruences (3.97) and (3.99), we see that

$$\overline{a}_5\,(800n+433) \equiv 3\cdot \overline{a}_5\,(32n+17) \pmod{16}. \tag{3.100}$$

By induction on $\beta$, we arrive at (3.94).

Substituting (2.9) in (3.98) and then extracting the coefficients of $q^{5n+i}$ for $i=1,3$ from both sides of the resultant equation, we obtain (3.95). □

**Theorem 3.13.** For all $n \geq 0$, we have

$$\overline{a}_5\,(32n+1) \equiv \begin{cases} 2 \pmod 4 & \text{if n is a pentagonal number,} \\ 0 \pmod 4 & \text{otherwise.} \end{cases} \tag{3.101}$$

*Proof.* From the equation (3.96), we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_5\,(32n+1)\,q^n \equiv 2f_1 \pmod 4. \tag{3.102}$$

The result (3.101) obtained from the equations (1.4) and (3.102). □

**Theorem 3.14.** Let $k_9 \in \{688, 1072\}$ and $k_{10} \in \{1136, 1904\}$, then for all $n \geq 0$ and $\beta \geq 0$, we have

$$\sum_{n=0}^{\infty} \overline{a}_5 \left(128\cdot 5^{2\beta} n + \frac{304\cdot 5^{2\beta}-1}{3}\right) q^n \equiv 8f_4 f_5^3 \pmod{16}, \tag{3.103}$$

$$\sum_{n=0}^{\infty} \overline{a}_5 \left(128\cdot 5^{2\beta+1} n + \frac{368\cdot 5^{2\beta+1}-1}{3}\right) q^n \equiv 8f_1^3 f_{20} \pmod{16}, \tag{3.104}$$

$$\overline{a}_5 \left(128\cdot 5^{2\beta+1} n + \frac{k_9\cdot 5^{2\beta}-1}{3}\right) \equiv 0 \pmod{16}, \tag{3.105}$$

$$\overline{a}_5 \left(128\cdot 5^{2\beta+2} n + \frac{k_{10}\cdot 5^{2\beta+1}-1}{3}\right) \equiv 0 \pmod{16}. \tag{3.106}$$

*Proof.* Employing (2.3), (2.5) and (2.7) in (3.71) and then comparing the coefficients of $q^{2n+1}$ on both sides of the resultant equation, we obtain

(3.107)
$$\sum_{n=0}^{\infty} \overline{a}_5\left(64n+37\right)q^n \equiv 8\frac{f_1^3f_4^4f_5}{f_{10}} + 8f_4f_{10} + 8qf_1^3f_2^5f_5f_{20} + 8qf_1^3f_5f_{10}^3 \pmod{16}.$$

Substituting (2.7) in (3.107) and then collecting the coefficients of $q^{2n+1}$ from both sides of the resultant equation, we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_5\left(128n+101\right)q^n \equiv 8f_4f_5^3 \pmod{16}, \tag{3.108}$$

which is $\beta = 0$ case of (3.103). The rest of the proofs of the congruences (3.103)-(3.106) are similar to the proofs of the congruences (3.80)-(3.83). So, we omit the details. □

**Theorem 3.15.** If $n$ can not be represented as a sum of four times a pentagonal number and five times a triangular number, then

$$\overline{a}_5\left(128n+101\right) \equiv 0 \pmod{16}. \tag{3.109}$$

*Proof.* The equation (3.108) can be written as

$$\sum_{n=0}^{\infty} \overline{a}_5\left(128n+101\right)q^n \equiv 8\frac{f_4f_{10}^2}{f_5} \pmod{16}. \tag{3.110}$$

In view of (1.3) and (3.110), we have

$$\sum_{n=0}^{\infty} \overline{a}_5\left(128n+101\right)q^n \equiv 8f_4\psi(q^5) \pmod{16}. \tag{3.111}$$

Combining (1.3), (1.4) and (3.111), we have

$$\sum_{n=0}^{\infty} \overline{a}_5\left(128n+101\right)q^n \equiv 8\sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty} q^{2k(3k-1)+\frac{5n(n+1)}{2}} \pmod{16}. \tag{3.112}$$

The result (3.109) follows from (3.112). □

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