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WEIGHTED ESTIMATES FOR MULTILINEAR STRONGLY SINGULAR CALDERÓN-ZYGMUND OPERATORS WITH MULTIPLE WEIGHTS

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Abstract. In this paper, the authors establish the weighted boundedness properties for the multilinear strongly singular Calderón-Zygmund operators, their multilinear commutators and multilinear iterated commutators through sharp maximal estimates, respectively. The weight involved is the multiple weight.

1. Introduction

The multilinear Calderón-Zygmund theory comes from the work of Coifman and Meyer in [3, 4, 5]. Since then, many scholars have done a lot of research on this topic from different perspectives.

Grafakos-Martell in [7] studied the weighted boundedness properties of the m-linear Calderón-Zygmund operator T.

$$T : L^{p_1}(\omega_1)\times\cdots\times L^{p_m}(\omega_m) \to L^p(v),\ where\ v=\prod_{j=1}^{m}\omega_j^{\frac{p}{p_j}}\ and\ \omega_j\in A_{p_j},\ j=1,\cdots,m.$$

Grafakos-Torres in [10] posed a question: is there a multiple weight theory? If it exists, it will better match the multilinear operator. Based on this idea, Lerner-Ombrosi-Pérez-Torres-Trujillo in [13] answered this question. They gave a kind of multiple weights, which are defined as follows.

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Definition 1.1. Suppose $1 \leq p_1, \cdots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$. Given $\vec{\omega} = (\omega_1, \cdots, \omega_m)$, let $v_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{\frac{p}{p_j}}$. We say $\vec{\omega} \in A_{\vec{p}}$ if

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q v_{\vec{\omega}}\Big)^{1/p} \prod_{j=1}^m \Big(\frac{1}{|Q|}\int_Q \omega_j^{1-p_j'}\Big)^{1/p_j'} < \infty.$$

When $p_j = 1$, $(\frac{1}{|Q|}\int_Q \omega_j^{1-p_j'})^{1/p_j'}$ is understood as $(\inf_Q \omega_j)^{-1}$.

And the authors in [13] gave boundedness properties of the m-linear Calderón-Zygmund operator T and its multilinear commutator $T_{\vec{b}}$ with multiple weight.

$$T \textit{ and } T_{\vec{b}} : L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m) \to L^p(v_{\vec{\omega}}), \textit{ where } v_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{\frac{p}{p_j}} \textit{ and } \vec{\omega} \in A_{\vec{p}}.$$

One can refer to [2, 8, 9, 11, 12, 14, 19, 21, 22, 23, 24] for other research results on this topic.

Remark 1. *The condition $\vec{\omega} \in A_{\vec{p}}$ does not guarantee that $\omega_j \in L^1_{loc}$, so it does not guarantee $\omega_j \in A_{p_j}$ for any j. On the other hand, when $\omega_j \in A_{p_j}$ $(j = 1, \cdots, m)$, we know by the properties of A_p weight that $v_{\vec{\omega}} \in A_{\max\{p_1,\cdots,p_m\}}$ and $A_{mp} \subset A_{\max\{p_1,\cdots,p_m\}}$. By Lemma 3.3 in Section 3, we can't guarantee $\vec{\omega} \in A_{\vec{p}}$.*

In [15], Lin introduced the multilinear strongly singular Calderón-Zygmund operator defined as follows.

Definition 1.2. T is an m-linear operator with kernel K. The following is its integral expression.

$$T(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) dy_1 \cdots dy_m,$$

where $f_j \in C_c^\infty(\mathbb{R}^n)(j = 1, \ldots, m)$ and $x \notin \cap_{j=1}^m \mathrm{supp} f_j$.

$K(y_0, y_1, \ldots, y_m)$ is a function defined away from the diagonal of $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. For some $\varepsilon > 0$ and $0 < \alpha \leq 1$, it satisfies

$$|K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \leq \frac{C|x - x'|^\varepsilon}{(|x - y_1| + \cdots + |x - y_m|)^{mn+\varepsilon/\alpha}}, \tag{1.1}$$

whenever $|x - x'|^\alpha \leq \frac{1}{2} \max_{1\leq j\leq m} |x - y_j|$.

T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ to $L^{r,\infty}$ and from $L^{l_1} \times \cdots \times L^{l_m}$ to $L^{q,\infty}$, where $r_1, \cdots, r_m, l_1, \cdots, l_m$ satisfy $1 \leq r_1, \ldots, r_m < \infty$ with $1/r = 1/r_1 + \cdots + 1/r_m$,

$1 \leq l_1, \ldots, l_m < \infty$ with $1/l = 1/l_1 + \cdots + 1/l_m$ and $0 < l/q \leq \alpha$. Then T is called an *m-linear strongly singular Calderón-Zygmund operator.*

It should be pointed out that the kernel of the multilinear strongly singular Calderón-Zygmund operator do not need any size condition and is more singular near the diagonal than the standard case. And it can be seen from the special case $\alpha = 1$ that the standard multilinear Calderón-Zygmund operator is included in the multilinear strongly singular Calderón-Zygmund operator.

Lin in [15], Lin-Lu-Lu in [17], Lin-Han in [16] and Lin-Yan in [20] obtained the boundedness of multilinear strongly singular Calderón-Zygmund operators, their multilinear commutators and multilinear iterated commutators on product of weighted Lebesgue spaces and product of weighted Morrey spaces, respectively. Other related results of this kind of operators can be seen in [1] and so on. The weights in the above papers are the Muckenhoupt weights. Because in [13] the multiple weight that match the multilinear operators was given, inspired by this, we also want to consider the corresponding results of multilinear strongly singular Calderón-Zygmund operators with the multiple weight of [13].

Therefore, in this paper we will study the weighted boundedness properties for multilinear strongly singular Calderón-Zygmund operators and multilinear commutators with multiple weight.

In order to establish our main results, we need some notations as follows.

Definition 1.3. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls B containing x. Denote by $M_s(f) = [M(|f|^s)]^{1/s}$, where $0 < s < \infty$.

Definition 1.4. Suppose $\vec{f} = (f_1, \cdots, f_m)$. The maximal operator $\mathcal{M}$ is define by

$$\mathcal{M}(\vec{f})(x) = \sup_{B \ni x} \prod_{i=1}^{m} \frac{1}{|B|} \int_B |f_i(y_i)| dy_i.$$

Denote by $\mathcal{M}_s(\vec{f}) = \sup_{B \ni x} \prod_{i=1}^{m} \left(\frac{1}{|B|} \int_B |f_i(y_i)|^s dy_i \right)^{\frac{1}{s}}$, where $0 < s < \infty$.

Obviously, we have $\mathcal{M}(\vec{f})(x) \leq C\mathcal{M}_s(\vec{f})$ and $\mathcal{M}(\vec{f})(x) \leq \prod_{i=1}^m M(f_i)$, where $1 \leq s < \infty$.

Definition 1.5. The sharp maximal function is defined by

$$M^\sharp(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - f_B| dy \sim \sup_{B \ni x} \inf_{a \in \mathbb{C}} \frac{1}{|B|} \int_B |f(y) - a| dx,$$

where $f_B = \frac{1}{|B|} \int_B |f(x)| dx$. Denote by $M_s^\sharp(f) = [M^\sharp(|f|^s)]^{1/s}$, where $0 < s < \infty$.

Definition 1.6. A non-negative measurable function ω belongs to Muckenhoupt class A_p with $1 < p < \infty$ if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where $1/p + 1/p' = 1$.

When $p = 1$, we say that ω belongs to A_1, if there exists a constant $C > 0$ such that for any cube Q,

$$\frac{1}{|Q|} \int_Q \omega(y) dy \leq C\omega(x), \quad a.e.\, x \in Q.$$

Denote by $A_\infty = \bigcup_{p \geq 1} A_p$.

Definition 1.7. Suppose T is an m-linear operator and $\vec{b} = (b_1, \cdots, b_m)$ are locally integrable functions. The m-linear commutator of T with $\vec{b}$ is defined by

$$T_{\vec{b}}(f_1, \cdots, f_m) = \sum_{j=1}^m T_{\vec{b}}^j(\vec{f}),$$

where

$$T_{\vec{b}}^j(\vec{f}) = b_j T(f_1, \cdots, f_m) - T(f_1, \cdots, f_{j-1}, b_j f_j, f_{j+1}, \cdots, f_m).$$

The m-linear iterated commutator of T with $\vec{b}$ is defined by

$$T_{\Pi\vec{b}}(f_1, \cdots, f_m) = [b_1, [b_2, \cdots, [b_{m-1}, [b_m, T]_m]_{m-1} \cdots]_2]_1(\vec{f}).$$

If T is an m-linear operator with K as the kernel, it can be denoted as

$$T_{\Pi\vec{b}}(f_1, \cdots, f_m)(x) = \int_{(R^n)^m} \prod_{j=1}^m (b_j(x) - b_j(y_j)) K(x, y_1, \cdots, y_m) \times f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m.$$

The notation $\vec{b} \in BMO^m$ is defined by $b_j \in BMO$ for $j = 1, \cdots, m$. Denote by $\|\vec{b}\|_{BMO^m} = \max_{1 \le j \le m} \|b_j\|_{BMO}$.

Definition 1.8. Suppose $j, m \in \mathbb{N}_+$, $1 \le j \le m$, and C_j^m is the totality of the finite subset $\psi = \{\psi(1), \cdots, \psi(j)\}$ composed of j different elements in $\{1, \cdots, m\}$. If $k < l$, then $\psi(k) < \psi(l)$. For any $\psi \in C_j^m$, let $\psi' = \{1, \cdots, m\} \setminus \psi$ be the complementary sequence of ψ. In particular, $C_0^m = \varnothing$. For any m-fold $\vec{b}$ and $\psi \in C_j^m$, then j-fold $\vec{b}_\psi = (\vec{b}_{\psi(1)}, \cdots, \vec{b}_{\psi(j)})$ is a finite subset of $\vec{b} = (b_1, \cdots, b_m)$.

Suppose T is an m-linear operator, $\psi \in C_j^m$ and $\vec{b}_\psi = (\vec{b}_{\psi(1)}, \cdots, \vec{b}_{\psi(j)})$. The m-linear iterated commutator of T with $\vec{b}_\psi$ is defined by

$$T_{\Pi \vec{b}_\psi}(f_1, \cdots, f_m) = [b_{\psi(1)}, [b_{\psi(2)}, \cdots, [b_{\psi(j-1)}, [b_{\psi(j)}, T]_{\psi(j)}]_{\psi(j-1)} \cdots]_{\psi(2)}]_{\psi(1)}(\vec{f}).$$

It can also be expressed as

$$T_{\Pi \vec{b}_\psi}(f_1, \cdots, f_m) = \int_{(R^n)^m} \prod_{i=1}^{j} (b_{\psi(i)}(x) - b_{\psi(i)}(y_{\psi(i)})) K(x, y_1, \cdots, y_m) \times f_1(y_1) \cdots f_m(y_m) d\vec{y}.$$

Obviously, when $\psi = \{1, 2, \cdots, m\}$, $T_{\Pi \vec{b}_\psi}$ is equal to $T_{\Pi \vec{b}}$, and when $\psi = \{j\}$, $T_{\Pi \vec{b}_\psi}$ is just the $T^j_{b_j}$.

Definition 1.9. Denote by $\vec{f^s} = (f_1^s, \cdots, f_m^s)$ for $0 < s < \infty$.

This paper will be organized as follows. The weigheted boundedness properties with multiple weights of multilinear strongly singular Calderón-Zygmund operators, their multilinear commutators and multilinear iterated commutators through sharp maximal estimates are obtained as main results in Section 2. We will provide some necessary lemmas to prove the main results in Section 3. Finally, the proof details of the main results will be given in Section 4.

2. Main results

Firstly, we establish the sharp maximal estimate of the multilinear strongly singular Calderón-Zygmund operator.

Theorem 2.1. *Suppose T is an m-linear strongly singular Calderón-Zygmund operator and $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, r_j and l_j $(j = 1, \ldots, m)$ here are given by Definition 1.2. If $0 < \delta < 1/m$, then there is a constant $C > 0$ such that for all bounded measurable functions with compact support m-tuples $\vec{f} = (f_1, \ldots, f_m)$, we have*

$$M_\delta^\sharp(T(\vec{f}))(x) \leq C\mathcal{M}_{s_0}(\vec{f})(x).$$

Then the weighted estimates of the multilinear strongly singular Calderón-Zygmund operator can be obtained.

Theorem 2.2. *Suppose T is an m-linear strongly singular Calderón-Zygmund operator and $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, r_j and l_j $(j = 1, \ldots, m)$ here are given by Definition 1.2. Then for any $s_0 < p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{\omega} = (\omega_1, \cdots, \omega_m)$ satisfying $\vec{\omega} \in A_{\frac{\vec{p}}{s_0}}$ and $v_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{\frac{p}{p_j}}$, we have*

$$\|T(\vec{f})\|_{L^p(v_{\vec{\omega}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)}.$$

By establishing the sharp maximal estimate of the multilinear commutator, we can obtain its boundedness on product of weighted Lebesgue spaces with mutiple weight.

Theorem 2.3. *Suppose T is an m-linear strongly singular Calderón-Zygmund operator and $0 < l/q < \alpha$ in Definition 1.2. Let $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, r_j and l_j $(j = 1, \ldots, m)$ here are given by Definition 1.2. If $s_0 < s < \infty$, $0 < \delta < 1/m$, $\delta < t < \infty$ and $\vec{b} \in BMO^m$, then there is a constant $C > 0$ such that for all bounded measurable functions with compact support m-tuples $\vec{f} = (f_1, \ldots, f_m)$, we have*

$$M_\delta^\sharp(T_{\vec{b}}(\vec{f}))(x) \leq C\|\vec{b}\|_{BMO^m}\Big(M_t(T(\vec{f}))(x) + \mathcal{M}_s(\vec{f})(x)\Big).$$

Theorem 2.4. *Suppose T is an m-linear strongly singular Calderón-Zygmund operator and $0 < l/q < \alpha$ in Definition 1.2. Let $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, r_j and l_j $(j = 1, \ldots, m)$ here are given by Definition 1.2. If $\vec{b} \in BMO^m$, then for any $s_0 < p_1, \cdots, p_m < \infty$, with $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{\omega} = (\omega_1, \cdots, \omega_m)$ satisfying $\vec{\omega} \in A_{\frac{\vec{p}}{s_0}}$ and $v_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{\frac{p}{p_j}}$,*

$$\|T_{\vec{b}}(\vec{f})\|_{L^p(v_{\vec{\omega}})} \leq C\|\vec{b}\|_{BMO^m} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)}.$$

The sharp maximal estimate of the multilinear iterated commutator can also be established as follows.

Theorem 2.5. *Suppose T is an m-linear strongly singular Calderón-Zygmund operator and $0 < l/q < \alpha$ in Definition 1.2. Let $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, r_j and l_j $(j = 1, \ldots, m)$ here are given by Definition 1.2. If $s_0 < s < \infty$, $0 < \delta < 1/m$, $\delta < \varepsilon < \infty$ and $\vec{b} \in BMO^m$, then there is a constant $C > 0$ such that for all bounded measurable functions with compact support m-tuples $\vec{f} = (f_1, \ldots, f_m)$, we have*

$$\begin{aligned} M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))(x) \;\leq\; & C\prod_{j=1}^m \|b_j\|_{BMO}\Big(\mathcal{M}_s(\vec{f})(x) + M_\varepsilon(T(\vec{f}))(x)\Big) \\ & + C\sum_{j=1}^{m-1}\sum_{\psi\in C_j^m}\prod_{i=1}^{j}\|b_{\psi(i)}\|_{BMO} M_\varepsilon(T_{\Pi\vec{b}_{\psi'}}(\vec{f}))(x). \end{aligned}$$

Then the boundedness of the multilinear iterated commutator on product of weighted Lebesgue spaces with mutiple weight can be deduced.

Theorem 2.6. *Suppose T is an m-linear strongly singular Calderón-Zygmund operator and $0 < l/q < \alpha$ in Definition 1.2. Let $s_0 = \max\{r_1, \ldots, r_m, l_1, \ldots, l_m\}$, r_j and l_j $(j = 1, \ldots, m)$ here are given by Definition 1.2. If $\vec{b} \in BMO^m$, then for any $s_0 < p_1, \cdots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{\omega} = (\omega_1, \cdots, \omega_m)$ satisfying $\vec{\omega} \in A_{\frac{\vec{p}}{s_0}}$ and $v_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{\frac{p}{p_j}}$,*

$$\|T_{\Pi\vec{b}}(\vec{f})\|_{L^p(v_{\vec{\omega}})} \leq C\prod_{j=1}^m \|b_j\|_{BMO}\prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)}.$$

3. Necessary lemmas

Lemma 3.1. (see [6, 13]). *Suppose $0 < p < r < \infty$, then there is a positive constant $C = C_{p,r}$ such that for any measurable function f*

$$|Q|^{-1/p}\|f\|_{L^p(Q)} \leq C|Q|^{-1/r}\|f\|_{L^{r,\infty}(Q)}.$$

Lemma 3.2. (see [13]). *Suppose $0 < p, \delta < \infty$ and $\omega \in A_\infty$. Then there exists a constant $C > 0$ depending only on the A_∞ constant of ω such that for every function f,*

$$\int_{\mathbb{R}^n}[M_\delta(f)(x)]^p\omega(x)dx \leq C\int_{\mathbb{R}^n}[M_\delta^\sharp(f)(x)]^p\omega(x)dx.$$

Lemma 3.3. (see [13]). *Suppose* $\vec{\omega} = (\omega_1, \cdots, \omega_m)$ *and* $1 \le p_1, \cdots, p_m < \infty$. *Then* $\vec{\omega} \in A_{\vec{p}}$ *if and only if*

$$\begin{cases} \omega_j^{1-p'_j} \in A_{mp'_j}, \qquad j = 1, \cdots, m, \\ \qquad v_{\vec{\omega}} \in A_{mp}, \end{cases}$$

where the condition $\omega_j^{1-p'_j} \in A_{mp'_j}$ *in the case* $p_j = 1$ *is replaced by* $\omega_j^{1/m} \in A_1$.

Lemma 3.4. (see [13]). *Suppose* $1 < p_1, \cdots, p_m < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. *Then* $\vec{\omega}$ *satisfies the* $A_{\vec{p}}$ *condition if and only if for every* $\vec{f}$ *it satisfies*

$$\|\mathcal{M}(\vec{f})\|_{L^p(v_{\vec{\omega}})} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\omega_j)}.$$

Lemma 3.5. (see [18]). *Suppose* f *is a function in* BMO. *Suppose* $1 \le p < \infty$, $x \in \mathbb{R}^n$, *and* $r_1, r_2 > 0$. *Then there is a constant* $C > 0$ *independent of* f, x, r_1 *and* r_2 *such that*

$$\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |f(y) - f_{B(x,r_2)}|^p dy\right)^{1/p} \le C\left(1 + \left|\ln \frac{r_1}{r_2}\right|\right)\|f\|_{BMO}.$$

Lemma 3.6. *If* $\varepsilon > 0$, *then*

$$\ln x \le \frac{1}{\varepsilon} x^{\varepsilon}$$

for all $x \ge 1$.

4. Proof of main results

Proof of Theorem 2.1.

Suppose $f_j (j = 1, \cdots, m)$ are bounded measurable functions with compact support. Then for any ball $B = B(x_0, r_B)$ containing x with $r_B > 0$, we're going to consider two cases, respectively.

Case 1: $r_B \ge 1$.

Write

$$f_j = f_j\chi_{2B} + f_j\chi_{(2B)^c} := f_j^0 + f_j^{\infty}, \quad j = 1, \cdots, m.$$

Then

$$\prod_{j=1}^{m} f_j(y_j) \;=\; \prod_{j=1}^{m}\left(f_j^0(y_j) + f_j^{\infty}(y_j)\right) = \sum_{\alpha_1,\cdots,\alpha_m \in \{0,\infty\}} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m)$$

$$= \prod_{j=1}^{m} f_j^0(y_j) + {\sum}' f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m) + \prod_{j=1}^{m} f_j^{\infty}(y_j),$$

where each term of $\sum'$ contains at least one $\alpha_j \neq 0$ and one $\alpha_j \neq \infty$. Write then

$$T(\vec{f})(z) = T(f_1^0, \cdots, f_m^0)(z) + {\sum}' T(f_1^{\alpha_1}, \cdots, f_m^{\alpha_m})(z) + T(f_1^{\infty}, \cdots, f_m^{\infty})(z).$$

Take a $c_0 = \sum' T(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(x_0) + T(f_1^{\infty}, \cdots, f_m^{\infty})(x_0)$, then

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B \big||T(\vec{f})(z)|^\delta - |c_0|^\delta\big| dz\right)^{1/\delta} \le \left(\frac{1}{|B|}\int_B |T(\vec{f})(z) - c_0|^\delta dz\right)^{1/\delta} \\
\le\ & C\left(\frac{1}{|B|}\int_B |T(f_1^0, \cdots, f_m^0)|^\delta dz\right)^{1/\delta} \\
&+C\left(\frac{1}{|B|}\int_B |{\sum}' T(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(z) - {\sum}' T(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(x_0)|^\delta dz\right)^{1/\delta} \\
&+C\left(\frac{1}{|B|}\int_B |T(f_1^{\infty}, \cdots, f_m^{\infty})(z) - T(f_1^{\infty}, \cdots, f_m^{\infty})(x_0)|^\delta dz\right)^{1/\delta} := \sum_{j=1}^{3} I_j.
\end{aligned}$$

Notice that $0 < \delta < r < \infty$ and the r here is given in Definition 1.2. By Lemma 3.1 and Hölder's inequality, we have

$$\begin{aligned}
I_1 \ \le\ & C|B|^{-1/\delta}\|T(f_1^0, \cdots, f_m^0)\|_{L^\delta(B)} \\
\le\ & C|B|^{-1/r}\|T(f_1^0, \cdots, f_m^0)\|_{L^{r,\infty}(B)} \\
\le\ & C\left(\frac{1}{|2B|}\int_{2B} |f_1(y_1)|^{r_1} dy_1\right)^{\frac{1}{r_1}} \times \cdots \times \left(\frac{1}{|2B|}\int_{2B} |f_m(y_m)|^{r_m} dy_m\right)^{\frac{1}{r_m}} \\
\le\ & C\mathcal{M}_{s_0}(\vec{f})(x).
\end{aligned}$$

We consider when $\alpha_{j_1} = \cdots = \alpha_{j_l} = \infty$ for some $\{j_1, \cdots, j_l\} \subset \{1, \cdots, m\}$ and $1 < l < m$. For $z \in B$ and $y_{j_1} \in (2B)^c$, $\cdots$, $y_{j_l} \in (2B)^c$, there are $|z - x_0|^\alpha \le r_B^\alpha \le r_B \le \frac{1}{2}|y_{j_\eta} - x_0|$, $\eta = 1, \cdots, l$. By Hölder's inequality, the condition (1.1), we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |T(f_1^{\alpha_1}, \cdots, f_m^{\alpha_m})(z) - T(f_1^{\alpha_1}, \cdots, f_m^{\alpha_m})(x_0)|^\delta dz\right)^{1/\delta} \\
\le\ & C\frac{1}{|B|}\int_B \int_{(\mathbb{R}^n)^m} \frac{|z - x_0|^\varepsilon}{(|y_1 - x_0| + \cdots + |y_m - x_0|)^{mn+\varepsilon/\alpha}} \prod_{j=1}^{m} |f_j^{\alpha_j}(y_j)| d\vec{y} dz \\
\le\ & Cr_B^\varepsilon \prod_{j\in\{1,\cdots,m\}\setminus\{j_1,\cdots,j_l\}} \int_{2B} |f_j(y_j)| dy_j
\end{aligned}$$

$$\begin{aligned}
&\times\sum_{k=1}^{\infty}\int_{(2^{k+1}B)^l\setminus(2^kB)^l}\frac{\prod_{j\in\{j_1,\cdots,j_l\}}|f_j(y_j)|dy_j}{(|x_0-y_{j_1}|+\cdots+|x_0-y_{j_l}|)^{mn+\varepsilon/\alpha}}\\
\le\ & Cr_B^{\varepsilon}\prod_{j\in\{1,\cdots,m\}\setminus\{j_1,\cdots,j_l\}}\int_{2B}|f_j(y_j)|dy_j\sum_{k=1}^{\infty}(2^kr_B)^{-(mn+\frac{\varepsilon}{\alpha})}\\
&\times\int_{(2^{k+1}B)^l}\prod_{j\in\{j_1,\cdots,j_l\}}|f_j(y_j)|dy_j\\
\le\ & Cr_B^{\varepsilon}\sum_{k=1}^{\infty}(2^kr_B)^{-\frac{\varepsilon}{\alpha}}\prod_{j=1}^{m}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_j(y_j)|dy_j\\
\le\ & Cr_B^{\varepsilon-\frac{\varepsilon}{\alpha}}\sum_{k=1}^{\infty}2^{-\frac{k\varepsilon}{\alpha}}\mathcal{M}(\vec{f})(x)\\
\le\ & C\mathcal{M}_{s_0}(\vec{f})(x).
\end{aligned}$$

From what has been discussed above, we can get that

$$I_2\le C\mathcal{M}_{s_0}(\vec{f})(x).$$

For $z\in B$ and $y_1,\cdots,y_m\in(2B)^c$, there are $|z-x_0|^{\alpha}\le\frac{1}{2}|y_j-x_0|$, $j=1,\cdots,m$. By Hölder's inequality and the condition (1.1), we have

$$\begin{aligned}
I_3\ \le\ & C\frac{1}{|B|}\int_B|T(f_1^{\infty},\cdots,f_m^{\infty})(z)-T(f_1^{\infty},\cdots,f_m^{\infty})(x_0)|dz\\
\le\ & Cr_B^{\varepsilon}\sum_{k=1}^{\infty}\int_{(2^{k+1}B)^m\setminus(2^kB)^m}\frac{\prod_{j=1}^m|f_j(y_j)|dy_j}{(|x_0-y_1|+\cdots+|x_0-y_m|)^{mn+\varepsilon/\alpha}}\\
\le\ & Cr_B^{\varepsilon}\sum_{k=1}^{\infty}(2^kr_B)^{-\frac{\varepsilon}{\alpha}}\prod_{j=1}^{m}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_j(y_j)|dy_j\\
\le\ & Cr_B^{\varepsilon-\frac{\varepsilon}{\alpha}}\sum_{k=1}^{\infty}2^{-\frac{k\varepsilon}{\alpha}}\mathcal{M}(\vec{f})(x)\\
\le\ & C\mathcal{M}_{s_0}(\vec{f})(x).
\end{aligned}$$

Case 2: $0<r_B<1$.

Denote by $\tilde{B}=B(x_0,r_B^{\alpha})$. Write

$$f_j=f_j\chi_{2\tilde{B}}+f_j\chi_{(2\tilde{B})^c}:=\tilde{f}_j^0+\tilde{f}_j^{\infty},\quad j=1,\cdots,m.$$

Then

$$\prod_{j=1}^{m} f_j(y_j) = \prod_{j=1}^{m} \tilde{f}_j^0(y_j) + \sum{}' f_1^{\tilde{\alpha}_1}(y_1)\cdots f_m^{\tilde{\alpha}_m}(y_m) + \prod_{j=1}^{m} f_j^{\tilde{\infty}}(y_j).$$

Take a $\tilde{c_0} = \sum' T(f_1^{\tilde{\alpha}_1}\cdots f_m^{\tilde{\alpha}_m})(x_0) + T(f_1^{\tilde{\infty}},\cdots,f_m^{\tilde{\infty}})(x_0)$, then

$$\begin{aligned}
&\Big(\frac{1}{|B|}\int_B \big||T(\vec{f})(z)|^\delta - |\tilde{c_0}|^\delta\big| dz\Big)^{1/\delta} \le \Big(\frac{1}{|B|}\int_B |T(\vec{f})(z) - \tilde{c_0}|^\delta dz\Big)^{1/\delta} \\
\le\ & C\Big(\frac{1}{|B|}\int_B |T(\tilde{f}_1^0,\cdots,\tilde{f}_m^0)(z)|^\delta dz\Big)^{1/\delta} \\
&+C\Big(\frac{1}{|B|}\int_B |\sum{}' T(f_1^{\tilde{\alpha}_1}\cdots f_m^{\tilde{\alpha}_m})(z) - \sum{}' T(f_1^{\tilde{\alpha}_1}\cdots f_m^{\tilde{\alpha}_m})(x_0)|^\delta dz\Big)^{1/\delta} \\
&+C\Big(\frac{1}{|B|}\int_B |T(f_1^{\tilde{\infty}},\cdots,f_m^{\tilde{\infty}})(z) - T(f_1^{\tilde{\infty}},\cdots,f_m^{\tilde{\infty}})(x_0)|^\delta dz\Big)^{1/\delta} := \sum_{j=1}^{3} \tilde{I}_j.
\end{aligned}$$

Notice that $0<\delta<q<\infty$, $0<l/q\le\alpha$ and the r, q here are given in Definition 1.2. By Lemma 3.1 and Hölder's inequality, we have

$$\begin{aligned}
\tilde{I}_1 \ \le\ & C|B|^{-1/\delta}\|T(\tilde{f}_1^0,\cdots,\tilde{f}_m^0)\|_{L^\delta(B)} \\
\le\ & C|B|^{-1/q}\|T(\tilde{f}_1^0,\cdots,\tilde{f}_m^0)\|_{L^{q,\infty}(B)} \\
\le\ & C|B|^{-1/q}|\tilde{B}|^{1/l}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |f_1(y_1)|^{l_1} dy_1\Big)^{\frac{1}{l_1}} \times\cdots\times \Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} |f_m(y_m)|^{l_m} dy_m\Big)^{\frac{1}{l_m}} \\
\le\ & C\mathcal{M}_{s_0}(\vec{f})(x).
\end{aligned}$$

We consider when $\alpha_{j_1} = \cdots = \alpha_{j_l} = \infty$ for some $\{j_1,\cdots,j_l\}\subset\{1,\cdots,m\}$ and $1<l<m$. For $z\in B$ and $y_{j_1}\in(2\tilde{B})^c$, $\cdots$, $y_{j_l}\in(2\tilde{B})^c$, there are $|z-x_0|^\alpha \le r_B^\alpha \le \frac{1}{2}|y_{j_\eta} - x_0|$, $\eta = 1,\cdots,l$. By Hölder's inequality and the condition (1.1), we have

$$\begin{aligned}
&\Big(\frac{1}{|B|}\int_B |T(f_1^{\tilde{\alpha}_1},\cdots,f_m^{\tilde{\alpha}_m})(z) - T(f_1^{\tilde{\alpha}_1},\cdots,f_m^{\tilde{\alpha}_m})(x_0)|^\delta dz\Big)^{1/\delta} \\
\le\ & C\frac{1}{|B|}\int_B\int_{(\mathbb{R}^n)^m} \frac{|z-x_0|^\varepsilon}{(|y_1-x_0|+\cdots+|y_m-x_0|)^{mn+\varepsilon/\alpha}} \prod_{j=1}^{m} |f_j^{\tilde{\alpha}_j}(y_j)| d\vec{y} dz \\
\le\ & Cr_B^\varepsilon \prod_{j\in\{1,\cdots,m\}\setminus\{j_1,\cdots,j_l\}} \int_{2\tilde{B}} |f_j(y_j)| dy_j \\
&\times\sum_{k=1}^{\infty}\int_{(2^{k+1}\tilde{B})^l\setminus(2^k\tilde{B})^l} \frac{\prod_{j\in\{j_1,\cdots,j_l\}}|f_j(y_j)|dy_j}{(|x_0-y_{j_1}|+\cdots+|x_0-y_{j_l}|)^{mn+\varepsilon/\alpha}}
\end{aligned}$$

$$\begin{aligned}
&\le Cr_B^{\varepsilon}\prod_{j\in\{1,\cdots,m\}\setminus\{j_1,\cdots,j_l\}}\int_{2\tilde{B}}|f_j(y_j)|dy_j\\
&\quad\times\sum_{k=1}^{\infty}(2^k r_B^{\alpha})^{-(mn+\frac{\varepsilon}{\alpha})}\int_{(2^{k+1}\tilde{B})^l}\prod_{j\in\{j_1,\cdots,j_l\}}|f_j(y_j)|dy_j\\
&\le Cr_B^{\varepsilon}\sum_{k=1}^{\infty}(2^k r_B^{\alpha})^{-\frac{\varepsilon}{\alpha}}\prod_{j=1}^{m}\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_j(y_j)|dy_j\\
&\le C\sum_{k=1}^{\infty}2^{-\frac{k\varepsilon}{\alpha}}\mathcal{M}(\vec{f})(x)\\
&\le C\mathcal{M}_{s_0}(\vec{f})(x).
\end{aligned}$$

From what has been discussed above, we can get that

$$\tilde{I}_2 \le C\mathcal{M}_{s_0}(\vec{f})(x).$$

For $z\in B$ and $y_1,\cdots,y_m\in(2\tilde{B})^c$, there are $|z-x_0|^{\alpha}\le\frac{1}{2}|y_j-x_0|$, $j=1,\cdots,m$. By Hölder's inequality and the condition (1.1), we have

$$\begin{aligned}
\tilde{I}_3 &\le C\frac{1}{|B|}\int_B|T(\tilde{f_1^{\infty}},\cdots,\tilde{f_m^{\infty}})(z)-T(\tilde{f_1^{\infty}},\cdots,\tilde{f_m^{\infty}})(x_0)|dz\\
&\le Cr_B^{\varepsilon}\sum_{k=1}^{\infty}\int_{(2^{k+1}\tilde{B})^m\setminus(2^k\tilde{B})^m}\frac{\prod_{j=1}^m|f_j(y_j)|dy_j}{(|x_0-y_1|+\cdots+|x_0-y_m|)^{mn+\varepsilon/\alpha}}\\
&\le Cr_B^{\varepsilon}\sum_{k=1}^{\infty}(2^k r_B^{\alpha})^{-\frac{\varepsilon}{\alpha}}\prod_{j=1}^{m}\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_j(y_j)|dy_j\\
&\le C\sum_{k=1}^{\infty}2^{-\frac{k\varepsilon}{\alpha}}\mathcal{M}(\vec{f})(x)\\
&\le C\mathcal{M}_{s_0}(\vec{f})(x).
\end{aligned}$$

Thus, according to the estimates in both cases, there is

$$M_{\delta}^{\sharp}(T(\vec{f}))(x)\sim\sup_{B\ni x}\inf_{a\in\mathbb{C}}\left(\frac{1}{|B|}\int_B\big||T(\vec{f})(z)|^{\delta}-a\big|dz\right)^{1/\delta}\le C\mathcal{M}_{s_0}(\vec{f})(x),$$

which completes the proof of the theorem. □

Proof of Theorem 2.2.

By Lemma 3.3, we can get that $v_{\vec{\omega}}\in A_{\frac{mp}{s_0}}\subset A_{\infty}$.

Take a δ such that $0<\delta<1/m$, then by Lemma 3.2, Theorem 2.1 and Lemma 3.4, we have

$$\begin{aligned}
&\|T(\vec{f})\|_{L^p(v_{\vec{\omega}})} \leq \|M_\delta(T(\vec{f}))\|_{L^p(v_{\vec{\omega}})} \leq C\|M_\delta^\sharp(T(\vec{f}))\|_{L^p(v_{\vec{\omega}})}\\
&\leq C\Big\|\mathcal{M}_{s_0}(\vec{f})\Big\|_{L^p(v_{\vec{\omega}})} = C\Big\|\mathcal{M}(\vec{f}^{s_0})\Big\|^{\frac{1}{s_0}}_{L^{\frac{p}{s_0}}(v_{\vec{\omega}})}\\
&\leq C\prod_{j=1}^m \||f_j|^{s_0}\|^{\frac{1}{s_0}}_{L^{\frac{p_j}{s_0}}(\omega_j)} = C\prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)},
\end{aligned}$$

which completes the proof of the theorem. □

Proof of Theorem 2.3.

Suppose $f_j (j=1,\cdots,m)$ are bounded measurable functions with compact support. Then for any ball $B=B(x_0,r_B)$ containing x with $r_B>0$, we're going to consider two cases, respectively.

Case 1: $r_B\geq 1$.

Write

$$f_j = f_j\chi_{2B} + f_j\chi_{(2B)^c} := f_j^0 + f_j^\infty, \quad j=1,\cdots,m,$$

then

$$\prod_{j=1}^m f_j(y_j) = \prod_{j=1}^m f_j^0(y_j) + \sum{}' f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m) + \prod_{j=1}^m f_j^\infty(y_j),$$

where each term of $\sum'$ contains at least one $\alpha_j\neq 0$ and one $\alpha_j\neq\infty$. Write then

$$\begin{aligned}
T^1_{\vec{b}}(\vec{f})(z) \;=\;& (b_1(z)-b^1_B)T(\vec{f})(z) - T((b_1(\cdot)-b^1_B)f_1^0,\cdots,f_m^0)(z)\\
&-\sum{}' T((b_1(\cdot)-b^1_B)f_1^{\alpha_1},\cdots,f_m^{\alpha_m})(z) - T((b_1(\cdot)-b^1_B)f_1^\infty,\cdots,f_m^\infty)(z).
\end{aligned}$$

Take a $c_1=\sum' T((b_1(\cdot)-b^1_B)f_1^{\alpha_1},\cdots,f_m^{\alpha_m})(x_0) + T((b_1(\cdot)-b^1_B)f_1^\infty,\cdots,f_m^\infty)(x_0)$, then

$$\begin{aligned}
&\bigg(\frac{1}{|B|}\int_B \Big||T^1_{\vec{b}}(\vec{f})(z)|^\delta - |c_1|^\delta\Big| dz\bigg)^{1/\delta}\\
&\leq C\bigg(\frac{1}{|B|}\int_B |(b_1(z)-b^1_B)T(\vec{f})(z)|^\delta dz\bigg)^{1/\delta}\\
&\quad + C\bigg(\frac{1}{|B|}\int_B |T((b_1(\cdot)-b^1_B)f_1^0,\cdots,f_m^0)(z)|^\delta dz\bigg)^{1/\delta}
\end{aligned}$$

$$+C\Big(\frac{1}{|B|}\int_B\Big|{\sum}' T((b_1(\cdot)-b_B^1)f_1^{\alpha_1},\cdots,f_m^{\alpha_m})(z)$$
$$-{\sum}' T((b_1(\cdot)-b_B^1)f_1^{\alpha_1},\cdots,f_m^{\alpha_m})(x_0)\Big|^\delta dz\Big)^{1/\delta}$$
$$+C\Big(\frac{1}{|B|}\int_B|T((b_1(\cdot)-b_B^1)f_1^{\infty},\cdots,f_m^{\infty})(z)$$
$$-T((b_1(\cdot)-b_B^1)f_1^{\infty},\cdots,f_m^{\infty})(x_0)|^\delta dz\Big)^{1/\delta}$$
$$:=\sum_{j=1}^4 L_j.$$

Because $0<\delta<1/m$ and $\delta<t<\infty$, there's an u satisfying $1<u<\min\{\frac{t}{\delta},\frac{1}{1-\delta}\}$. Then $\delta u<t$ and $\delta u'>1$. By Hölder's inequality, we can get that

$$\begin{aligned} L_1 &\le C\Big(\frac{1}{B}\int_B|b_1(z)-b_B^1|^{\delta u'}dz\Big)^{\frac{1}{\delta u'}}\Big(\frac{1}{B}\int_B|T(\vec{f})(z)|^{\delta u}dz\Big)^{\frac{1}{\delta u}}\\ &\le C\|b_1\|_{BMO}\Big(\frac{1}{B}\int_B|T(\vec{f})(z)|^{t}dz\Big)^{\frac{1}{t}}\\ &\le C\|b_1\|_{BMO}M_t(T(\vec{f}))(x). \end{aligned}$$

Let $v=s/s_0$, because $s_0<s<\infty$, we can get that $1<v<\infty$. Notice that $0<\delta<r<\infty$, here r is given in Definition 1.2. By Lemma 3.1, Hölder's inequality and Lemma 3.5, we have

$$\begin{aligned} L_2 &\le C|B|^{-1/\delta}\|T((b_1(\cdot)-b_B^1)f_1^0,\cdots,f_m^0)\|_{L^\delta(B)}\\ &\le C|B|^{-1/r}\|T((b_1(\cdot)-b_B^1)f_1^0,\cdots,f_m^0)\|_{L^{r,\infty}(B)}\\ &\le C\Big(\frac{1}{|2B|}\int_{2B}|b_1(y_1)-b_B^1|^{r_1v'}dy_1\Big)^{\frac{1}{r_1v'}}\Big(\frac{1}{|2B|}\int_{2B}|f_1(y_1)|^{r_1v}dy_1\Big)^{\frac{1}{r_1v}}\\ &\quad\times\Big(\frac{1}{|2B|}\int_{2B}|f_2(y_2)|^{r_2}dy_2\Big)^{\frac{1}{r_2}}\times\cdots\times\Big(\frac{1}{|2B|}\int_{2B}|f_m(y_m)|^{r_m}dy_m\Big)^{\frac{1}{r_m}}\\ &\le C\|b_1\|_{BMO}\mathcal{M}_s(\vec{f})(x). \end{aligned}$$

We consider when $\alpha_{j_1}=\cdots=\alpha_{j_l}=\infty$ for some $\{j_1,\cdots,j_l\}\subset\{1,\cdots,m\}$ and $1<l<m$. Without loss of generality, we just think about the case $\alpha_1=\infty$, because other case is similar. For $z\in B$ and $y_{j_1}\in(2B)^c$, $\cdots$, $y_{j_l}\in(2B)^c$, there are $|z-x_0|^\alpha\le\frac{1}{2}|y_{j_\eta}-x_0|$, $\eta=1,\cdots,l$. By Hölder's inequality, the condition (1.1) and

Lemma 3.5, we have

$$\begin{aligned}
&\Big(\frac{1}{|B|}\int_B |T((b_1(\cdot)-b_B^1)f_1^{\alpha_1},\cdots,f_m^{\alpha_m})(z)-T((b_1(\cdot)-b_B^1)f_1^{\alpha_1},\cdots,f_m^{\alpha_m})(x_0)|^\delta dz\Big)^{\frac{1}{\delta}}\\
\le\ & C\frac{1}{|B|}\int_B\int_{(\mathbb{R}^n)^m}\frac{|z-x_0|^\varepsilon}{(|y_1-x_0|+\cdots+|y_m-x_0|)^{mn+\varepsilon/\alpha}}|b_1(y_1)-b_B^1|\prod_{j=1}^m|f_j^{\alpha_j}(y_j)|d\vec{y}dz\\
\le\ & Cr_B^\varepsilon\prod_{j\in\{1,\cdots,m\}\backslash\{j_1,\cdots,j_l\}}\int_{2B}|f_j(y_j)|dy_j\\
&\times\sum_{k=1}^\infty\int_{(2^{k+1}B)^l\backslash(2^kB)^l}\frac{|b_1(y_1)-b_B^1|\prod_{j\in\{j_1,\cdots,j_l\}}|f_j(y_j)|dy_j}{(|x_0-y_{j_1}|+\cdots+|x_0-y_{j_l}|)^{mn+\varepsilon/\alpha}}\\
\le\ & Cr_B^\varepsilon\prod_{j\in\{1,\cdots,m\}\backslash\{j_1,\cdots,j_l\}}\int_{2B}|f_j(y_j)|dy_j\sum_{k=1}^\infty(2^kr_B)^{-(mn+\frac{\varepsilon}{\alpha})}\\
&\times\int_{(2^{k+1}B)^l}|b_1(y_1)-b_B^1|\prod_{j\in\{j_1,\cdots,j_l\}}|f_j(y_j)|dy_j\\
\le\ & C\|b_1\|_{BMO}r_B^{\varepsilon-\frac{\varepsilon}{\alpha}}\sum_{k=1}^\infty k2^{\frac{-k\varepsilon}{\alpha}}\prod_{j=1}^m\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_j(y_j)|^sdy_j\Big)^{\frac{1}{s}}\\
\le\ & C\|b_1\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}$$

From what has been discussed above, we can get that

$$L_3\le C\|b_1\|_{BMO}\mathcal{M}_s(\vec{f})(x).$$

For $z\in B$ and $y_1,\cdots,y_m\in(2B)^c$, there are $|z-x_0|^\alpha\le\frac{1}{2}|y_j-x_0|$, $j=1,\cdots,m$. By Hölder's inequality, the condition (1.1) and Lemma 3.5, we have

$$\begin{aligned}
L_4\ \le\ & C\frac{1}{|B|}\int_B|T((b_1(\cdot)-b_B^1)f_1^\infty,\cdots,f_m^\infty)(z)-T((b_1(\cdot)-b_B^1)f_1^\infty,\cdots,f_m^\infty)(x_0)|dz\\
\le\ & C\frac{1}{|B|}\int_B\int_{(\mathbb{R}^n)^m\backslash(2B)^m}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+\cdots+|x_0-y_m|)^{mn+\varepsilon/\alpha}}|b_1(y_1)-b_B^1|\\
&\times\prod_{j=1}^m|f_j(y_j)|d\vec{y}dz\\
\le\ & Cr_B^\varepsilon\sum_{k=1}^\infty(2^kr_B)^{-(mn+\frac{\varepsilon}{\alpha})}\int_{(2^{k+1}B)^m}|b_1(y_1)-b_B^1|\prod_{j=1}^m|f_j(y_j)|d\vec{y}\\
\le\ & C\|b_1\|_{BMO}r_B^{\varepsilon-\frac{\varepsilon}{\alpha}}\sum_{k=1}^\infty k2^{\frac{-k\varepsilon}{\alpha}}\prod_{j=1}^m\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_j(y_j)|^sdy_j\Big)^{\frac{1}{s}}\\
\le\ & C\|b_1\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}$$

Case 2: $0 < r_B < 1$.

Since $0 < l/q < \alpha$, there exists a θ such that $l/q < \theta < \alpha$. Denote by $\tilde{B} = B(x_0, r_B^\theta)$. Write

$$f_j = f_j\chi_{2\tilde{B}} + f_j\chi_{(2\tilde{B})^c} := \tilde{f_j^0} + \tilde{f_j^\infty}, \quad j = 1, \cdots, m.$$

Then

$$\prod_{j=1}^m f_j(y_j) = \prod_{j=1}^m \tilde{f_j^0}(y_j) + \sum{}' f_1^{\tilde{\alpha}_1}(y_1)\cdots f_m^{\tilde{\alpha}_m}(y_m) + \prod_{j=1}^m \tilde{f_j^\infty}(y_j).$$

Write then

$$\begin{aligned}T_{\vec{b}}^1(\vec{f})(z) &= (b_1(z) - b_B^1)T(\vec{f})(z) - T((b_1(\cdot) - b_B^1)\tilde{f_1^0}, \cdots, \tilde{f_m^0})(z)\\ &\quad -\sum{}' T((b_1(\cdot) - b_B^1)f_1^{\tilde{\alpha}_1}, \cdots, f_m^{\tilde{\alpha}_m})(z) - T((b_1(\cdot) - b_B^1)\tilde{f_1^\infty}, \cdots, \tilde{f_m^\infty})(z).\end{aligned}$$

Take a $\tilde{c}_1 = \sum{}' T\Big((b_1(\cdot) - b_B^1)f_1^{\tilde{\alpha}_1}, \cdots, f_m^{\tilde{\alpha}_m}\Big)(x_0) + T\Big((b_1(\cdot) - b_B^1)\tilde{f_1^\infty}, \cdots, \tilde{f_m^\infty}\Big)(x_0)$, then

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B \Big||T_{\vec{b}}^1(\vec{f})(z)|^\delta - |\tilde{c}_1|^\delta\Big| dz\right)^{1/\delta}\\ &\le C\left(\frac{1}{|B|}\int_B |(b_1(z) - b_B^1)T(\vec{f})(z)|^\delta dz\right)^{1/\delta}\\ &\quad + C\left(\frac{1}{|B|}\int_B |T((b_1(\cdot) - b_B^1)\tilde{f_1^0}, \cdots, \tilde{f_m^0})(z)|^\delta dz\right)^{1/\delta}\\ &\quad + C\left(\frac{1}{|B|}\int_B \Big|\sum{}' T((b_1(\cdot) - b_B^1)f_1^{\tilde{\alpha}_1}, \cdots, f_m^{\tilde{\alpha}_m})(z)\right.\\ &\qquad\qquad \left. -\sum{}' T((b_1(\cdot) - b_B^1)f_1^{\tilde{\alpha}_1}, \cdots, f_m^{\tilde{\alpha}_m})(x_0)\Big|^\delta dz\right)^{1/\delta}\\ &\quad + C\left(\frac{1}{|B|}\int_B |T((b_1(\cdot) - b_B^1)\tilde{f_1^\infty}, \cdots, \tilde{f_m^\infty})(z)\right.\\ &\qquad\qquad \left. -T((b_1(\cdot) - b_B^1)\tilde{f_1^\infty}, \cdots, \tilde{f_m^\infty})(x_0)|^\delta dz\right)^{1/\delta}\\ &:= \sum_{j=1}^4 \tilde{L}_j.\end{aligned}$$

For the same estimate of L_1, we have

$$\tilde{L_1} \le C\|b_1\|_{BMO} M_t(T(\vec{f}))(x).$$

Suppose $v = s/s_0$ and $\varepsilon_1 = n(\theta/l - 1/q)$, then $v > 1$ and $\varepsilon_1 > 0$. Notice that $0 < \delta < q < \infty$, by Lemma 3.1, Hölder's inequality, Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned}
\tilde{L_2} &\le C|B|^{-1/\delta}\|T((b_1(\cdot)-b_B^1)\tilde{f_1^0},\cdots,\tilde{f_m^0})\|_{L^\delta(B)}\\
&\le C|B|^{-1/q}\|T((b_1(\cdot)-b_B^1)\tilde{f_1^0},\cdots,\tilde{f_m^0})\|_{L^{q,\infty}(B)}\\
&\le C|B|^{-1/q}|\tilde{B}|^{1/l}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|b_1(y_1)-b_B^1|^{l_1v'}dy_1\Big)^{\frac{1}{l_1v'}}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|f_1(y_1)|^{l_1v}dy_1\Big)^{\frac{1}{l_1v}}\\
&\quad\times\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|f_2(y_2)|^{l_2}dy_2\Big)^{\frac{1}{l_2}}\cdots\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|f_m(y_m)|^{l_m}dy_m\Big)^{\frac{1}{l_m}}\\
&\le Cr_B^{n(\frac{\theta}{l}-\frac{1}{q})}\|b_1\|_{BMO}(1+(1-\theta)\ln\frac{1}{r_B})\prod_{j=1}^m\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|f_j(y_j)|^s dy_j\Big)^{\frac{1}{s}}\\
&\le Cr_B^{n(\frac{\theta}{l}-\frac{1}{q})-\varepsilon_1}\|b_1\|_{BMO}\prod_{j=1}^m\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|f_j(y_j)|^s dy_j\Big)^{\frac{1}{s}}\\
&\le C\|b_1\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}$$

We consider when $\alpha_{j_1} = \cdots = \alpha_{j_l} = \infty$ for some $\{j_1,\cdots,j_l\} \subset \{1,\cdots,m\}$ and $1 < l < m$. Let's just think about the case $\alpha_1 = \infty$, because other case is similar. For $z \in B$ and $y_{j_1} \in (2\tilde{B})^c, \cdots, y_{j_l} \in (2\tilde{B})^c$, there are $|z-x_0|^\alpha \le \frac{1}{2}|y_{j_\eta}-x_0|$, $\eta = 1,\cdots,l$. Suppose $\varepsilon_2 = \varepsilon(\alpha-\theta)/\alpha$, then $\varepsilon_2 > 0$. By Hölder's inequality, the condition (1.1), Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned}
&\Big(\frac{1}{|B|}\int_B|T((b_1(\cdot)-b_B^1)f_1^{\tilde{\alpha}_1},\cdots,f_m^{\tilde{\alpha}_m})(z)-T((b_1(\cdot)-b_B^1)f_1^{\tilde{\alpha}_1},\cdots,f_m^{\tilde{\alpha}_m})(x_0)|^\delta dz\Big)^{\frac{1}{\delta}}\\
\le\ & C\frac{1}{|B|}\int_B\int_{(\mathbb{R}^n)^m}\frac{|z-x_0|^\varepsilon}{(|y_1-x_0|+\cdots+|y_m-x_0|)^{mn+\varepsilon/\alpha}}|b_1(y_1)-b_B^1|\prod_{j=1}^m|f_j^{\tilde{\alpha}_j}(y_j)|d\vec{y}dz\\
\le\ & Cr_B^\varepsilon\prod_{j\in\{1,\cdots,m\}\setminus\{j_1,\cdots,j_l\}}\int_{2\tilde{B}}|f_j(y_j)|dy_j\\
&\times\sum_{k=1}^\infty\int_{(2^{k+1}\tilde{B})^l\setminus(2^k\tilde{B})^l}\frac{|b_1(y_1)-b_B^1|\prod_{j\in\{j_1,\cdots,j_l\}}|f_j(y_j)|dy_j}{(|x_0-y_{j_1}|+\cdots+|x_0-y_{j_l}|)^{mn+\varepsilon/\alpha}}\\
\le\ & Cr_B^\varepsilon\sum_{k=1}^\infty(2^kr_B^\theta)^{-(mn+\frac{\varepsilon}{\alpha})}\int_{(2^{k+1}\tilde{B})^m}|b_1(y_1)-b_B^1|\prod_{j=1}^m|f_j(y_j)|dy_j\\
\le\ & Cr_B^{\varepsilon-\frac{\varepsilon\theta}{\alpha}}\sum_{k=1}^\infty2^{\frac{-k\varepsilon}{\alpha}}\|b_1\|_{BMO}\Big(1+\Big|\ln\frac{2^{k+1}r_B^\theta}{r_B}\Big|\Big)\prod_{j=1}^m\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_j(y_j)|^s dy_j\Big)^{\frac{1}{s}}
\end{aligned}$$

$$\begin{aligned}
&\leq C\|b_1\|_{BMO} r_B^{\varepsilon-\frac{\varepsilon\theta}{\alpha}-\varepsilon_2}\sum_{k=1}^{\infty} k2^{\frac{-k\varepsilon}{\alpha}}\mathcal{M}_s(\vec{f})(x)\\
&\leq C\|b_1\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}$$

From what has been discussed above, we can get that

$$\tilde{L_3} \leq C\|b_1\|_{BMO}\mathcal{M}_s(\vec{f})(x).$$

For $z \in B$ and $y_1, \cdots, y_m \in (2\tilde{B})^c$, there are $|z-x_0|^\alpha \leq \frac{1}{2}|y_j - x_0|$, $j = 1, \cdots, m$. By Hölder's inequality, the condition (1.1), Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned}
\tilde{L_4} &\leq C\frac{1}{|B|}\int_B \sum_{k=1}^{\infty}\int_{(2^{k+1}\tilde{B})^m\setminus(2^k\tilde{B})^m}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+\cdots+|x_0-y_m|)^{mn+\varepsilon/\alpha}}|b_1(y_1)-b_B^1|\\
&\quad\times\prod_{j=1}^m |f_j(y_j)|d\vec{y}dz\\
&\leq Cr_B^\varepsilon\sum_{k=1}^{\infty}(2^k r_B^\theta)^{-\frac{\varepsilon}{\alpha}}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|b_1(y_1)-b_B^1|^{s'}dy_1\Big)^{\frac{1}{s'}}\\
&\quad\times\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_1(y_1)|^s dy_1\Big)^{\frac{1}{s}}\prod_{j=2}^m\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_j(y_j)|dy_j\Big)\\
&\leq Cr_B^{\varepsilon-\frac{\varepsilon\theta}{\alpha}}\sum_{k=1}^{\infty}2^{\frac{-k\varepsilon}{\alpha}}\|b_1\|_{BMO}\Big(1+\Big|\ln\frac{2^{k+1}r_B^\theta}{r_B}\Big|\Big)\prod_{j=1}^m\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_j(y_j)|^s dy_j\Big)^{\frac{1}{s}}\\
&\leq C\|b_1\|_{BMO} r_B^{\varepsilon-\frac{\varepsilon\theta}{\alpha}-\varepsilon_2}\sum_{k=1}^{\infty} k2^{\frac{-k\varepsilon}{\alpha}}\mathcal{M}_s(\vec{f})(x)\\
&\leq C\|b_1\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}$$

$T_{\vec{b}}^j(\vec{f})(z)(j = 2\cdots m)$ can be dealt with by using the same method. Finally, combining the above situation, we have

$$\begin{aligned}
M_\delta^\sharp(T_{\vec{b}}(\vec{f}))(x) &\sim \sup_{B\ni x}\inf_{a\in\mathbb{C}}\Big(\frac{1}{|B|}\int_B\big||T_{\vec{b}}(\vec{f})|^\delta - a\big|dz\Big)^{1/\delta}\\
&\leq C\|\vec{b}\|_{BMO^m}\Big(M_t(T(\vec{f}))(x)+\mathcal{M}_s(\vec{f})(x)\Big),
\end{aligned}$$

which completes the proof of the theorem. □

Proof of Theorem 2.4.

Denote $\vec{q} = \frac{\vec{p}}{s_0} = (\frac{p_1}{s_0}, \cdots, \frac{p_m}{s_0})$. Since $\omega \in A_{\vec{q}}$, by Lemma 3.3, each $\psi_j = \omega_j^{-\frac{1}{q_j-1}}$ belong to A_∞, where $q_j = \frac{p_j}{s_0}$, $j = 1, \cdots, m$. There are constants c_j, $t_j > 1$, depending on the A_∞ constand of ψ_j, such that by inverse Hölder's inequality, we have

$$\Big(\frac{1}{|B|}\int_B \omega_j^{-\frac{t_j}{q_j-1}}\Big)^{\frac{1}{t_j}} \le \frac{c_j}{|B|}\int_B \omega_j^{-\frac{1}{q_j-1}}$$

for any ball B. Pick a $d_j > 1$ that makes

$$\frac{t_j}{q_j-1} = \frac{1}{\frac{q_j}{d_j}-1}.$$

We can get that $q_j > d_j > 1$, $j = 1, \cdots, m$.

Let $d = \min\{d_1, \cdots, d_m\}$ and $c = \max\{c_1, \cdots, c_m\}$. We have for $q = \frac{p}{s_0}$,

$$\begin{aligned}
&\Big(\frac{1}{|B|}\int_B v_{\vec{\omega}}\Big)^{1/\frac{q}{d}} \prod_{j=1}^m \Big(\frac{1}{|B|}\int_B \omega_j^{-\frac{1}{\frac{q_j}{d}-1}}\Big)^{1-1/(\frac{q_j}{d})}\\
&= \Big(\frac{1}{|B|}\int_B v_{\vec{\omega}}\Big)^{\frac{d}{q}} \prod_{j=1}^m \Big(\frac{1}{|B|}\int_B \omega_j^{-\frac{1}{\frac{q_j}{d}-1}}\Big)^{(\frac{q_j}{d}-1)\frac{d}{q_j}}\\
&\le \Big(\frac{1}{|B|}\int_B v_{\vec{\omega}}\Big)^{\frac{d}{q}} \prod_{j=1}^m \Big(\frac{1}{|B|}\int_B \omega_j^{-\frac{1}{\frac{q_j}{d_j}-1}}\Big)^{(\frac{q_j}{d_j}-1)\frac{d}{q_j}}\\
&= \Big(\frac{1}{|B|}\int_B v_{\vec{\omega}}\Big)^{\frac{d}{q}} \prod_{j=1}^m \Big(\frac{1}{|B|}\int_B \omega_j^{-\frac{t_j}{q_j-1}}\Big)^{(\frac{q_j-1}{t_j})\frac{d}{q_j}}\\
&\le C^{dm}\Big(\frac{1}{|B|}\int_B v_{\vec{\omega}}\Big)^{\frac{d}{q}} \prod_{j=1}^m \Big(\frac{1}{|B|}\int_B \omega_j^{-\frac{1}{q_j-1}}\Big)^{(q_j-1)\frac{d}{q_j}}\\
&\le C^{dm}[\omega]^d_{A_{\vec{q}}}.
\end{aligned}$$

So $\vec{\omega} \in A_{\frac{\vec{q}}{d}}$ is true. Let $s = s_0 d$, then it follows from $d > 1$ that $s > s_0$ and $s < p_j$, $j = 1, \cdots, m$. We have

$$\vec{\omega} \in A_{\frac{\vec{q}}{d}} = A_{\frac{\vec{p}}{s}}.$$

Take δ and t such that $0 < \delta < t < 1/m$, then by Lemma 3.2, Theorem 2.3, Theorem 2.1, Hölder's inequality and Lemma 3.4, we have

$$\begin{aligned}
&\|T_{\vec{b}}(\vec{f})\|_{L^p(v_{\vec{\omega}})} \le \|M_\delta(T_{\vec{b}}(\vec{f}))\|_{L^p(v_{\vec{\omega}})} \le C\|M^\sharp_\delta(T_{\vec{b}}(\vec{f}))\|_{L^p(v_{\vec{\omega}})}\\
&\le C\|\vec{b}\|_{BMO^m}\Big(\|M_t(T(\vec{f}))\|_{L^p(v_{\vec{\omega}})} + \|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{\omega}})}\Big)\\
&\le C\|\vec{b}\|_{BMO^m}\Big(\|\mathcal{M}_{s_0}(\vec{f})\|_{L^p(v_{\vec{\omega}})} + \|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{\omega}})}\Big)\\
&\le C\|\vec{b}\|_{BMO^m}\|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{\omega}})} \le C\|\vec{b}\|_{BMO^m}\Big\|\mathcal{M}(\vec{f}^s)\Big\|^{\frac{1}{s}}_{L^{\frac{p}{s}}(v_{\vec{\omega}})}
\end{aligned}$$

$$\leq C\|\vec{b}\|_{BMO^m}\prod_{j=1}^{m}\||f_j|^s\|^{\frac{1}{s}}_{L^{p_j/s}(\omega_j)} = C\|\vec{b}\|_{BMO^m}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\omega_j)},$$

which completes the proof of the theorem. □

Proof of Theorem 2.5.

To simplify the process, we will only consider the case $m = 2$.

Let $b_1, b_2 \in BMO$, f_1, f_2 be bounded measurable functions with compact support. For any ball $B = B(x_0, r_B)$ containing x with $r_B > 0$, we have

$$\begin{aligned} T_{\Pi\vec{b}}(\vec{f})(x) &= (b_1(x) - b_B^1)(b_2(x) - b_B^2)T(f_1, f_2)(x) \\ &\quad -(b_1(x) - b_B^1)T(f_1, (b_2(\cdot) - b_B^2)f_2)(x) \\ &\quad -(b_2(x) - b_B^2)T((b_1(\cdot) - b_B^1)f_1, f_2)(x) \\ &\quad +T((b_1(\cdot) - b_B^1)f_1, (b_2(\cdot) - b_B^2)f_2)(x) \\ &= -(b_1(x) - b_B^1)(b_2(x) - b_B^2)T(f_1, f_2)(x) \\ &\quad +(b_1(x) - b_B^1)T^2_{b_2(\cdot)-b_B^2}(f_1, f_2)(x) \\ &\quad +(b_2(x) - b_B^2)T^1_{b_1(\cdot)-b_B^1}(f_1, f_2)(x) \\ &\quad +T((b_1(\cdot) - b_B^1)f_1, (b_2(\cdot) - b_B^2)f_2)(x). \end{aligned}$$

We're going to consider two cases, respectively.

Case 1: $r_B \geq 1$. Write

$$f_1 = f_1\chi_{2B} + f_1\chi_{(2B)^c} := f_1^0 + f_1^\infty, \quad f_2 = f_2\chi_{2B} + f_2\chi_{(2B)^c} := f_2^0 + f_2^\infty.$$

Take a

$$\begin{aligned} c_2 &= T((b_1(\cdot) - b_B^1)f_1^0, (b_2(\cdot) - b_B^2)f_2^\infty)(x_0) + T((b_1(\cdot) - b_B^1)f_1^\infty, (b_2(\cdot) - b_B^2)f_2^0)(x_0) \\ &\quad +T((b_1(\cdot) - b_B^1)f_1^\infty, (b_2(\cdot) - b_B^2)f_2^\infty)(x_0) := c_2^1 + c_2^2 + c_2^3. \end{aligned}$$

Then

$$\begin{aligned} &\left(\frac{1}{|B|}\int_B \left||T_{\Pi\vec{b}}(\vec{f})(z)|^\delta - |c_2|^\delta\right|dz\right)^{1/\delta} \\ &\leq C\left(\frac{1}{|B|}\int_B |(b_1(z) - b_B^1)(b_2(z) - b_B^2)T(f_1, f_2)(z)|^\delta dz\right)^{1/\delta} \end{aligned}$$

$$
\begin{aligned}
&+C\Big(\frac{1}{|B|}\int_B |(b_1(z)-b_B^1)T^2_{b_2(\cdot)-b_B^2}(f_1,f_2)(z)|^\delta dz\Big)^{1/\delta}\\
&+C\Big(\frac{1}{|B|}\int_B |(b_2(z)-b_B^2)T^1_{b_1(\cdot)-b_B^1}(f_1,f_2)(z)|^\delta dz\Big)^{1/\delta}\\
&+C\Big(\frac{1}{|B|}\int_B |T((b_1(\cdot)-b_B^1)f_1^0,(b_2(\cdot)-b_B^2)f_2^0)(z)|^\delta dz\Big)^{1/\delta}\\
&+C\Big(\frac{1}{|B|}\int_B |T((b_1(\cdot)-b_B^1)f_1^0,(b_2(\cdot)-b_B^2)f_2^\infty)(z)-c_2^1|^\delta dz\Big)^{1/\delta}\\
&+C\Big(\frac{1}{|B|}\int_B |T((b_1(\cdot)-b_B^1)f_1^\infty,(b_2(\cdot)-b_B^2)f_2^0)(z)-c_2^2|^\delta dz\Big)^{1/\delta}\\
&+C\Big(\frac{1}{|B|}\int_B |T((b_1(\cdot)-b_B^1)f_1^\infty,(b_2(\cdot)-b_B^2)f_2^\infty)(z)-c_2^3|^\delta dz\Big)^{1/\delta}\\
:=\;& J_1+J_2+J_3+J_{41}+J_{42}+J_{43}+J_{44}.
\end{aligned}
$$

Since $0<\delta<\frac{1}{2}$ and $0<\delta<\varepsilon<\infty$, there exists a v such that $1<v<\min\{\frac{\varepsilon}{\delta},\frac{1}{1-\delta}\}$. We have $v\delta<\varepsilon$ and $v'\delta>1$. Take $q_1,q_2\in(1,\infty)$ such that $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{v'}$, then $\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{v}=1$, $q_1\delta>1$ and $q_2\delta>1$. By Hölder's inequality, we can get that

$$
\begin{aligned}
J_1 &\le C\Big(\frac{1}{|B|}\int_B |b_1(z)-b_B^1|^{\delta q_1}dz\Big)^{\frac{1}{\delta q_1}}\Big(\frac{1}{|B|}\int_B |b_2(z)-b_B^2|^{\delta q_2}dz\Big)^{\frac{1}{\delta q_2}}\\
&\quad\times\Big(\frac{1}{|B|}\int_B |T(f_1,f_2)(z)|^{\delta v}dz\Big)^{\frac{1}{\delta v}}\\
&\le C\|b_1\|_{BMO}\|b_2\|_{BMO}M_\varepsilon(T(f_1,f_2))(x).
\end{aligned}
$$

By Hölder's inequality, we have

$$
\begin{aligned}
J_2 &\le C\Big(\frac{1}{|B|}\int_B |b_1(z)-b_B^1|^{\delta v'}dz\Big)^{\frac{1}{\delta v'}}\Big(\frac{1}{|B|}\int_B |T^2_{b_2(\cdot)-b_B^2}(f_1,f_2)(z)|^{\delta v}dz\Big)^{\frac{1}{\delta v}}\\
&\le C\|b_1\|_{BMO}M_{\delta v}(T^2_{b_2(\cdot)-b_B^2}(f_1,f_2))(x)\\
&= C\|b_1\|_{BMO}M_\varepsilon(T^2_{b_2}(f_1,f_2))(x).
\end{aligned}
$$

Similarly, we can get that

$$
J_3 \le C\|b_2\|_{BMO}M_\varepsilon(T^1_{b_1}(f_1,f_2))(x).
$$

Suppose $t=s/s_0$. Since $s_0<s<\infty$, we have $1<t<\infty$, $r_1t\le s$ and $r_2t\le s$. Notice that $0<\delta<r<\infty$, by Lemma 3.1, Hölder's inequality and Lemma 3.5, we can get that

$$\begin{aligned}
J_{41} &\leq C|B|^{-\frac{1}{r}}\|T((b_1(\cdot)-b_B^1)f_1^0,(b_2(\cdot)-b_B^2)f_2^0)\|_{L^{r,\infty}(B)}\\
&\leq C\Big(\frac{1}{|2B|}\int_{2B}|b_1(y_1)-b_B^1|^{r_1}|f_1(y_1)|^{r_1}dy_1\Big)^{\frac{1}{r_1}}\\
&\quad\times\Big(\frac{1}{|2B|}\int_{2B}|b_2(y_2)-b_B^2|^{r_2}|f_2(y_2)|^{r_2}dy_2\Big)^{\frac{1}{r_2}}\\
&\leq C\Big(\frac{1}{|2B|}\int_{2B}|b_1(y_1)-b_B^1|^{r_1t'}dy_1\Big)^{\frac{1}{r_1t'}}\Big(\frac{1}{|2B|}\int_{2B}|f_1(y_1)|^{r_1t}dy_1\Big)^{\frac{1}{r_1t}}\\
&\quad\times\Big(\frac{1}{|2B|}\int_{2B}|b_2(y_2)-b_B^2|^{r_2t'}dy_2\Big)^{\frac{1}{r_2t'}}\Big(\frac{1}{|2B|}\int_{2B}|f_2(y_2)|^{r_2t}dy_2\Big)^{\frac{1}{r_2t}}\\
&\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}$$

For $z\in B$ and $y_2\in(2B)^c$, there is $|z-x_0|^\alpha\leq\frac{1}{2}|y_2-x_0|$. By Hölder's inequality, the condition (1.1) and Lemma 3.5, we have

$$\begin{aligned}
J_{42} &\leq C\Big(\frac{1}{|B|}\int_B|T((b_1(\cdot)-b_B^1)f_1^0,(b_2(\cdot)-b_B^2)f_2^\infty)(z)\\
&\quad-T((b_1(\cdot)-b_B^1)f_1^0,(b_2(\cdot)-b_B^2)f_2^\infty)(x_0)|dz\\
&\leq C\frac{1}{|B|}\int_B\int_{2B}\int_{(2B)^c}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+|x_0-y_2|)^{2n+\frac{\varepsilon}{\alpha}}}|b_1(y_1)-b_B^1||b_2(y_2)-b_B^2|\\
&\quad\times|f_1(y_1)||f_2(y_2)|dy_2dy_1dz\\
&\leq Cr_B^\varepsilon\Big(\int_{2B}|b_1(y_1)-b_B^1||f_1(y_1)|dy_1\Big)\\
&\quad\times\Big(\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}\frac{1}{|x_0-y_2|^{2n+\frac{\varepsilon}{\alpha}}}|b_2(y_2)-b_B^2||f_2(y_2)|dy_2\Big)\\
&\leq Cr_B^\varepsilon\sum_{k=1}^\infty(2^kr_B)^{-(2n+\frac{\varepsilon}{\alpha})}\Big(\int_{2^{k+1}B}|b_1(y_1)-b_B^1||f_1(y_1)|dy_1\Big)\\
&\quad\times\Big(\int_{2^{k+1}B}|b_2(y_2)-b_B^2||f_2(y_2)|dy_2\Big)\\
&\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}r_B^{\varepsilon-\frac{\varepsilon}{\alpha}}\sum_{k=1}^\infty k^2 2^{-\frac{k\varepsilon}{\alpha}}\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_1(y_1)|^sdy_1\Big)^{\frac{1}{s}}\\
&\quad\times\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_2(y_2)|^sdy_2\Big)^{\frac{1}{s}}\\
&\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}$$

Similarly, we can get that

$$J_{43} \leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).$$

For $z \in B$, $y_1 \in (2B)^c$ and $y_2 \in (2B)^c$, there are $|z-x_0|^\alpha \leq \frac{1}{2}|y_1-x_0|$ and $|z-x_0|^\alpha \leq \frac{1}{2}|y_2-x_0|$. By Hölder's inequality, the condition (1.1) and Lemma 3.5, we have

$$\begin{aligned}
J_{44} &\leq C\frac{1}{|B|}\int_B\int_{(2B)^c}\int_{(2B)^c}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+|x_0-y_2|)^{2n+\frac{\varepsilon}{\alpha}}}|b_1(y_1)-b_B^1||b_2(y_2)-b_B^2| \\
&\quad\times|f_1(y_1)||f_2(y_2)|dy_2dy_1dz \\
&\leq Cr_B^\varepsilon\sum_{k=1}^\infty\int_{(2^{k+1}B)^2\setminus(2^kB)^2}\frac{1}{(|x_0-y_1|+|x_0-y_2|)^{2n+\frac{\varepsilon}{\alpha}}}|b_1(y_1)-b_B^1||b_2(y_2)-b_B^2| \\
&\quad\times|f_1(y_1)||f_2(y_2)|dy_2dy_1 \\
&\leq Cr_B^\varepsilon\sum_{k=1}^\infty(2^kr_B)^{-\frac{\varepsilon}{\alpha}}\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b_1(y_1)-b_B^1|^{s'}dy_1\Big)^{\frac{1}{s'}} \\
&\quad\times\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_1(y_1)|^sdy_1\Big)^{\frac{1}{s}}\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b_2(y_2)-b_B^2|^{s'}dy_1\Big)^{\frac{1}{s'}} \\
&\quad\times\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_2(y_2)|^sdy_2\Big)^{\frac{1}{s}} \\
&\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}r_B^{\varepsilon-\frac{\varepsilon}{\alpha}}\sum_{k=1}^\infty k^22^{-\frac{k\varepsilon}{\alpha}}\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_1(y_1)|^sdy_1\Big)^{\frac{1}{s}} \\
&\quad\times\Big(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_2(y_2)|^sdy_2\Big)^{\frac{1}{s}} \\
&\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}$$

Case 2: $0 < r_B < 1$.

Since $0 < l/q < \alpha$, there exists a θ such that $l/q < \theta < \alpha$. Denote by $\tilde{B} = B(x_0, r_B^\theta)$. Write

$$f_1 = f_1\chi_{2\tilde{B}} + f_1\chi_{(2\tilde{B})^c} := \tilde{f_1^0} + \tilde{f_1^\infty}, \quad f_2 = f_2\chi_{2\tilde{B}} + f_2\chi_{(2\tilde{B})^c} := \tilde{f_2^0} + \tilde{f_2^\infty}.$$

Take a

$$\begin{aligned}
\tilde{c_2} &= T((b_1(\cdot)-b_B^1)\tilde{f_1^0}, (b_2(\cdot)-b_B^2)\tilde{f_2^\infty})(x_0) + T((b_1(\cdot)-b_B^1)\tilde{f_1^\infty}, (b_2(\cdot)-b_B^2)\tilde{f_2^0})(x_0) \\
&\quad+T((b_1(\cdot)-b_B^1)\tilde{f_1^\infty}, (b_2(\cdot)-b_B^2)\tilde{f_2^\infty})(x_0) := \tilde{c_2^1}+\tilde{c_2^2}+\tilde{c_2^3}.
\end{aligned}$$

Then

$$
\begin{aligned}
&\Big(\frac{1}{|B|}\int_B \big||T_{\Pi\vec{b}}(\vec{f})(z)|^\delta - |\tilde{c_2}|^\delta\big| dz\Big)^{1/\delta}\\
&\le C\Big(\frac{1}{|B|}\int_B |(b_1(z)-b_B^1)(b_2(z)-b_B^2)T(f_1,f_2)(z)|^\delta dz\Big)^{1/\delta}\\
&\quad +C\Big(\frac{1}{|B|}\int_B |(b_1(z)-b_B^1)T^2_{b_2(\cdot)-b_B^2}(f_1,f_2)(z)|^\delta dz\Big)^{1/\delta}\\
&\quad +C\Big(\frac{1}{|B|}\int_B |(b_2(z)-b_B^2)T^1_{b_1(\cdot)-b_B^1}(f_1,f_2)(z)|^\delta dz\Big)^{1/\delta}\\
&\quad +C\Big(\frac{1}{|B|}\int_B |T((b_1(\cdot)-b_B^1)\tilde{f_1^0},(b_2(\cdot)-b_B^2)\tilde{f_2^0})(z)|^\delta dz\Big)^{1/\delta}\\
&\quad +C\Big(\frac{1}{|B|}\int_B |T((b_1(\cdot)-b_B^1)\tilde{f_1^0},(b_2(\cdot)-b_B^2)\tilde{f_2^\infty})(z)-\tilde{c_2^1}|^\delta dz\Big)^{1/\delta}\\
&\quad +C\Big(\frac{1}{|B|}\int_B |T((b_1(\cdot)-b_B^1)\tilde{f_1^\infty},(b_2(\cdot)-b_B^2)\tilde{f_2^0})(z)-\tilde{c_2^2}|^\delta dz\Big)^{1/\delta}\\
&\quad +C\Big(\frac{1}{|B|}\int_B |T((b_1(\cdot)-b_B^1)\tilde{f_1^\infty},(b_2(\cdot)-b_B^2)\tilde{f_2^\infty})(z)-\tilde{c_2^3}|^\delta dz\Big)^{1/\delta}\\
&:= \tilde{J_1}+\tilde{J_2}+\tilde{J_3}+\tilde{J_{41}}+\tilde{J_{42}}+\tilde{J_{43}}+\tilde{J_{44}}.
\end{aligned}
$$

Similarly to J_1, J_2, J_3, we can get that

$$\tilde{J_1} \le C\|b_1\|_{BMO}\|b_2\|_{BMO}M_\varepsilon(T(f_1,f_2))(x),$$

$$\tilde{J_2} \le C\|b_1\|_{BMO}M_\varepsilon(T^2_{b_2}(f_1,f_2))(x),$$

$$\tilde{J_3} \le C\|b_2\|_{BMO}M_\varepsilon(T^1_{b_1}(f_1,f_2))(x).$$

Since $s_0 < s < \infty$, let's suppose that $t = s/s_0$, then we have $1 < t < \infty$. Denote $\varepsilon_1 = \frac{n}{2}(\frac{\theta}{l} - \frac{1}{q}) > 0$. Noticing that $0 < \delta < q < \infty$, by Hölder's inequality, Lemma 3.5 and Lemma 3.6, we have

$$
\begin{aligned}
\tilde{J_{41}} &\le C|B|^{-\frac{1}{q}}\|T((b_1(\cdot)-b_B^1)\tilde{f_1^0},(b_2(\cdot)-b_B^2)\tilde{f_2^0})\|_{L^{q,\infty}(B)}\\
&\le C|B|^{-\frac{1}{q}}|\tilde{B}|^{\frac{1}{l}}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|b_1(y_1)-b_B^1|^{l_1}|f_1(y_1)|^{l_1}dy_1\Big)^{\frac{1}{l_1}}\\
&\quad\times\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|b_2(y_2)-b_B^2|^{l_2}|f_2(y_2)|^{l_2}dy_1\Big)^{\frac{1}{l_2}}\\
&\le Cr_B^{n(\frac{\theta}{l}-\frac{1}{q})}\Big(1+\Big|\ln\frac{2r_B^\theta}{r_B}\Big|\Big)\|b_1\|_{BMO}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|f_1(y_1)|^s dy_1\Big)^{\frac{1}{s}}
\end{aligned}
$$

$$\times\Big(1+\Big|\ln\frac{2r_B^\theta}{r_B}\Big|\Big)\|b_2\|_{BMO}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|f_2(y_2)|^s dy_1\Big)^{\frac{1}{s}}$$
$$\leq\ Cr_B^{n(\frac{\theta}{l}-\frac{1}{q})-2\varepsilon_1}\|b_1\|_{BMO}\|b_2\|_{BMO}\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|f_1(y_1)|^s dy_1\Big)^{\frac{1}{s}}$$
$$\times\Big(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}|f_2(y_2)|^s dy_1\Big)^{\frac{1}{s}}$$
$$\leq\ C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).$$

For $z\in B$ and $y_2\in(2\tilde{B})^c$, there is $|z-x_0|^\alpha\leq\frac{1}{2}|y_2-x_0|$. Let $\varepsilon_2=\frac{1}{2}\Big(\varepsilon-\frac{\theta\varepsilon}{\alpha}\Big)>0$. By Hölder's inequality, the condition (1.1), Lemma 3.5 and Lemma 3.6, we have

$$\tilde{J_{42}}\ \leq\ C\frac{1}{|B|}\int_B\int_{2\tilde{B}}\int_{(2\tilde{B})^c}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+|x_0-y_2|)^{2n+\frac{\varepsilon}{\alpha}}}|b_1(y_1)-b_B^1||b_2(y_2)-b_B^2|$$
$$\times|f_1(y_1)||f_2(y_2)|dy_2dy_1dz$$
$$\leq\ Cr_B^\varepsilon\Big(\int_{2\tilde{B}}|b_1(y_1)-b_B^1||f_1(y_1)|dy_1\Big)$$
$$\times\Big(\sum_{k=1}^\infty\int_{2^{k+1}\tilde{B}\backslash 2^k\tilde{B}}\frac{1}{|x_0-y_2|^{2n+\frac{\varepsilon}{\alpha}}}|b_2(y_2)-b_B^2||f_2(y_2)|dy_2\Big)$$
$$\leq\ Cr_B^\varepsilon\sum_{k=1}^\infty(2^kr_B^\theta)^{-\frac{\varepsilon}{\alpha}}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|b_1(y_1)-b_B^1|^{s'}dy_1\Big)^{\frac{1}{s'}}$$
$$\times\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_1(y_1)|^s dy_1\Big)^{\frac{1}{s}}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|b_2(y_2)-b_B^2|^{s'}dy_1\Big)^{\frac{1}{s'}}$$
$$\times\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_2(y_2)|^s dy_2\Big)^{\frac{1}{s}}$$
$$\leq\ Cr_B^{\varepsilon-\frac{\varepsilon\theta}{\alpha}}\sum_{k=1}^\infty 2^{-\frac{k\varepsilon}{\alpha}}\Big(1+\Big|\ln\frac{2^{k+1}r_B^\theta}{r_B}\Big|\Big)\|b_1\|_{BMO}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_1(y_1)|^s dy_1\Big)^{\frac{1}{s}}$$
$$\times\Big(1+\Big|\ln\frac{2^{k+1}r_B^\theta}{r_B}\Big|\Big)\|b_2\|_{BMO}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_2(y_2)|^s dy_2\Big)^{\frac{1}{s}}$$
$$\leq\ C\|b_1\|_{BMO}\|b_2\|_{BMO}r_B^{\varepsilon-\frac{\varepsilon\theta}{\alpha}-2\varepsilon_2}\sum_{k=1}^\infty k^2 2^{-\frac{k\varepsilon}{\alpha}}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_1(y_1)|^s dy_1\Big)^{\frac{1}{s}}$$
$$\times\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_2(y_2)|^s dy_2\Big)^{\frac{1}{s}}$$
$$\leq\ C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).$$

Similarly, we can get that

$$\tilde{J_{43}}\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).$$

For $z \in B$, $y_1 \in (2\tilde{B})^c$ and $y_2 \in (2\tilde{B})^c$, there are $|z-x_0|^\alpha \leq \frac{1}{2}|y_1-x_0|$ and $|z-x_0|^\alpha \leq \frac{1}{2}|y_2-x_0|$. By Hölder's inequality, the condition (1.1), Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned}
\tilde{J_{44}} \leq\ & C\frac{1}{|B|}\int_B\int_{(2\tilde{B})^c}\int_{(2\tilde{B})^c}\frac{|z-x_0|^\varepsilon}{(|x_0-y_1|+|x_0-y_2|)^{2n+\frac{\varepsilon}{\alpha}}}|b_1(y_1)-b_B^1||b_2(y_2)-b_B^2| \\
& \times|f_1(y_1)||f_2(y_2)|dy_2dy_1dz \\
\leq\ & Cr_B^\varepsilon\sum_{k=1}^\infty\int_{(2^{k+1}\tilde{B})^2\backslash(2^k\tilde{B})^2}\frac{1}{(|x_0-y_1|+|x_0-y_2|)^{2n+\frac{\varepsilon}{\alpha}}}|b_1(y_1)-b_B^1||b_2(y_2)-b_B^2| \\
& \times|f_1(y_1)||f_2(y_2)|dy_2dy_1 \\
\leq\ & Cr_B^\varepsilon\sum_{k=1}^\infty(2^kr_B^\theta)^{-\frac{\varepsilon}{\alpha}}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|b_1(y_1)-b_B^1|^{s'}dy_1\Big)^{\frac{1}{s'}} \\
& \times\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_1(y_1)|^sdy_1\Big)^{\frac{1}{s}}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|b_2(y_2)-b_B^2|^{s'}dy_1\Big)^{\frac{1}{s'}} \\
& \times\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_2(y_2)|^sdy_2\Big)^{\frac{1}{s}} \\
\leq\ & C\|b_1\|_{BMO}\|b_2\|_{BMO}r_B^{\varepsilon-\frac{\varepsilon\theta}{\alpha}-2\varepsilon_2}\sum_{k=1}^\infty k^22^{-\frac{k\varepsilon}{\alpha}}\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_1(y_1)|^sdy_1\Big)^{\frac{1}{s}} \\
& \times\Big(\frac{1}{|2^{k+1}\tilde{B}|}\int_{2^{k+1}\tilde{B}}|f_2(y_2)|^sdy_2\Big)^{\frac{1}{s}} \\
\leq\ & C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}$$

Combining these two cases, we have

$$\begin{aligned}
& M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f})(x)) = M^\sharp(|T_{\Pi\vec{b}}(\vec{f})|^\delta)^{\frac{1}{\delta}}(x) \sim \sup_{B\ni x}\inf_{a\in\mathbb{C}}\Big(\frac{1}{|B|}\int_B\big||T_{\Pi\vec{b}}(\vec{f})|^\delta-a\big|dz\Big)^{1/\delta} \\
& \leq C\|b_1\|_{BMO}\|b_2\|_{BMO}(\mathcal{M}_s(\vec{f})(x)+M_\varepsilon(T(f_1,f_2))(x)) \\
& +C(\|b_1\|_{BMO}M_\varepsilon(T_{b_2}^2(f_1,f_2))(x)+\|b_2\|_{BMO}M_\varepsilon(T_{b_1}^1(f_1,f_2))(x)),
\end{aligned}$$

which completes the proof of the theorem. □

Proof of Theorem 2.6.

Since $\vec{\omega} \in A_{\frac{\vec{P}}{s_0}}$, as in the proof of Theorem 2.4, we can find $d > 1$ and $s = s_0d$, such that $\vec{\omega} \in A_{\frac{\vec{P}}{s}}$ and $s < p_j$, for $j = 1, \cdots, m$.

Take $\delta, \varepsilon_1, \cdots, \varepsilon_m$ such that $0 < \delta < \varepsilon_1 < \cdots \varepsilon_m < 1/m$. By Theorem 2.5, we have

$$\|M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))\|_{L^p(v_{\vec{\omega}})} \le C \prod_{j=1}^m \|b_j\|_{BMO}\Big(\|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{\omega}})} + \|M_{\varepsilon_1}(T(\vec{f}))\|_{L^p(v_{\vec{\omega}})}\Big)$$

$$+C\sum_{j=1}^{m-1}\sum_{\psi\in C_j^m}\prod_{i=1}^{j}\|b_{\psi(i)}\|_{BMO}\|M_{\varepsilon_1}^\sharp(T_{\Pi\vec{b}_{\psi'}}(\vec{f}))\|_{L^p(v_{\vec{\omega}})}.$$

In order to reduce the dimension of the BMO function in the commutator, Theorem 2.5 is applied again to $\|M_{\varepsilon_1}^\sharp(T_{\Pi\vec{b}_{\psi'}}(\vec{f}))\|_{L^p(v_{\vec{\omega}})}$. According to Definition 1.8, $\psi = \{\psi(1), \cdots, \psi(j)\}$, $\psi' = \{\psi(j+1), \cdots, \psi(m)\}$ and $A_h = \{\psi_1 : \psi_1 \subset \psi'\}$, where ψ_1 is any finite subset of ψ' composed of different elements. Denote $\psi_1' = \psi' - \psi_1$. By Theorem 2.5, we have

$$\begin{aligned}\|M_{\varepsilon_1}^\sharp(T_{\Pi\vec{b}_{\psi'}}(\vec{f}))\|_{L^p(v_{\vec{\omega}})} \;\le\; & C\prod_{k=j+1}^{m}\|b_{\psi(k)}\|_{BMO}\Big(\|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{\omega}})} + \|M_{\varepsilon_2}(T(\vec{f}))\|_{L^p(v_{\vec{\omega}})}\Big)\\ & +C\sum_{h=1}^{m-j-1}\sum_{\psi_1\in A_h}\prod_{i=1}^{h}\|b_{\psi_1(i)}\|_{BMO}\|M_{\varepsilon_2}(T_{\Pi\vec{b}_{\psi_1'}}(\vec{f}))\|_{L^p(v_{\vec{\omega}})}.\end{aligned}$$

Repeat the above process and apply Theorem 2.3, we have

$$\begin{aligned}&\|M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))\|_{L^p(v_{\vec{\omega}})}\\ &\le C\prod_{j=1}^m\|b_j\|_{BMO}\Big(C_{m+1}(m,n)\|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{\omega}})} + C_1(m,n)\|M_{\varepsilon_1}(T(\vec{f}))\|_{L^p(v_{\vec{\omega}})}\\ &+C_2(m,n)\|M_{\varepsilon_2}(T(\vec{f}))\|_{L^p(v_{\vec{\omega}})} + \cdots + C_m(m,n)\|M_{\varepsilon_m}(T(\vec{f}))\|_{L^p(v_{\vec{\omega}})}\Big),\end{aligned}$$

where $C_1(m,n), C_2(m,n), \cdots, C_m(m,n), C_{m+1}(m,n)$ are all finite real numbers associated with m and n. By Lemma 3.2, Theorem 2.1 and Lemma 3.4, we have

$$\begin{aligned}&\|T_{\Pi\vec{b}}(\vec{f})\|_{L^p(v_{\vec{\omega}})} \le \|M_\delta(T_{\Pi\vec{b}}(\vec{f}))\|_{L^p(v_{\vec{\omega}})} \le \|M_\delta^\sharp(T_{\Pi\vec{b}}(\vec{f}))\|_{L^p(v_{\vec{\omega}})}\\ &\le C\prod_{j=1}^m\|b_j\|_{BMO}\Big(C_{m+1}(m,n)\|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{\omega}})} + C_1(m,n)\|M_{\varepsilon_1}(T(\vec{f}))\|_{L^p(v_{\vec{\omega}})}\\ &\quad +C_2(m,n)\|M_{\varepsilon_2}(T(\vec{f}))\|_{L^p(v_{\vec{\omega}})} + \cdots + C_m(m,n)\|M_{\varepsilon_m}(T(\vec{f}))\|_{L^p(v_{\vec{\omega}})}\Big)\\ &\le C\prod_{j=1}^m\|b_j\|_{BMO}\Big(C_{m+1}(m,n)\|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{\omega}})} + C_{m+2}(m,n)\|\mathcal{M}_{s_0}(\vec{f})\|_{L^p(v_{\vec{\omega}})}\Big)\\ &\le C\prod_{j=1}^m\|b_j\|_{BMO}\|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{\omega}})} \le C\prod_{j=1}^m\|b_j\|_{BMO}\Big\|\mathcal{M}(\vec{f}^s)\Big\|_{L^{\frac{p}{s}}(v_{\vec{\omega}})}^{\frac{1}{s}}\end{aligned}$$

$$\leq C \prod_{j=1}^{m} \|b_j\|_{BMO} \prod_{j=1}^{m} \||f_j|^s\|^{\frac{1}{s}}_{L^{\frac{p_j}{s}}(\omega_j)}$$
$$= C \prod_{j=1}^{m} \|b_j\|_{BMO} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\omega_j)},$$

which completes the proof of the theorem. □

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