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# SKEW POLYNOMIAL RING OF THE RING OF MORITA CONTEXT

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Abstract. Ghahramani proved that the skew polynomial ring of the formal triangular matrix ring is isomorphic to a formal triangular matrix ring. We aim to generalize this work to the skew polynomial ring of a ring of Morita context. Let $M$ be a ring of Morita context and $M[z;\theta,d]$ be a skew polynomial ring over $M$. By studying a particular ring homomorphism $\theta$ and a skew derivation $d$ on $M$, one can show that $M[z;\theta,d]$ is isomorphic to a ring of Morita context that is constructed by skew polynomial modules and skew polynomial rings. In this article, we use the definition of skew polynomial module that was introduced in the work of Ghahramani.

## 1. Introduction

Throughout this paper all rings are associative with unity, all modules are unital, and every ring homomorphism preserves the unity element.

Skew polynomial rings were introduced by Oystein Ore in the 1930s. Before the formal definition by Ore, these rings were studied because of their relation to differential equations and operator theory. The structure of these rings has also been studied by P.M. Cohn, T.Y. Lam, A. Leroy, N. Jacobson, A. Ozturk, and many others [5]. Recently, skew polynomial rings have been used to construct codes [2].

Let $R$ be a ring and $\alpha$ be a ring endomorphism of $R$. The *$\alpha$-derivation* of $R$ is an additive map $\delta : R \to R$ such that $\delta(r_1 r_2) = \alpha(r_1)\delta(r_2) + \delta(r_1)r_2$ for every $r_1, r_2 \in R$.

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Recall that $R[x;\alpha,\delta]$ is *the skew polynomial ring over* $R$ or *Ore extension of* $R$, whose elements are polynomials over $R$ with the usual addition and multiplication is subjected to the relation $xr = \alpha(r)x + \delta(r)$ ([3],[4]). It was proved by Ghahramani that the skew polynomial ring of formal triangular matrix ring $T[z;\theta,d]$ is isomorphic to a particular formal triangular matrix ring [3].

Recall that a *Morita context* is a set $M = (R, V, W, S)$ and two mappings $(\ ,\ )$, $[\ ,\ ]$, with $R, S$ are rings, $V$ is an $(R, S)$-bimodule, and $W$ is an $(S, R)$-bimodule. The mapping $(\ ,\ ) : V \otimes_S W \to R$ is an $R - R$ bilinear map and $[\ ,\ ] : W \otimes_R V \to S$ is an $S - S$ bilinear map. These mappings satisfy the associative condition, that is:

$$1_V \otimes [\ ,\ ] = (\ ,\ ) \otimes 1_V : V \otimes_S W \otimes_R V \to V$$

$$[\ ,\ ] \otimes 1_W = 1_W \otimes (\ ,\ ) : W \otimes_R V \otimes_S W \to W$$

([1],[6]). Hereafter, the mappings $(v, w)$ and $[w, v]$ will be written only as $vw$ and $wv$, respectively, for every $w \in W$ and $v \in V$. If $M$ is a Morita context then

$$\mathcal{M} = \left\{ \begin{pmatrix} r & v \\ w & s \end{pmatrix} : r \in R, v \in V, w \in W, s \in S \right\},$$

with usual matrix operations, is a ring. The ring $\mathcal{M}$ will be called *the ring of Morita context*. Note that the formal triangular matrix ring is a special case of the ring of Morita context.

We will generalize the result in [3] for rings of Morita context. We will prove that for a particular ring endomorphism $\theta$, the skew polynomial ring $\mathcal{M}[z;\theta,d]$ is isomorphic to a ring of Morita context

$$(R[x;\alpha,\delta_R], V[y;\gamma,\tau_V], W[x;\eta,\tau_W], S[y;\beta,\delta_S])$$

such that both $R[x;\alpha,\delta_R]$ and $S[y;\beta,\delta_S]$ are skew polynomial rings and also both the $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$ bimodule $V[y;\gamma,\tau_V]$ and the $(S[y;\beta,\delta_S], R[x;\alpha,\delta_R])$ bimodule $W[x;\eta,\tau_W]$ are skew polynomial modules.

The organization of this article is as follows. In section 2, we explain some terminology. In section 3, we introduce a particular class of homomorphisms and skew derivations of rings of Morita context. We also introduce the Morita context

$$(R[x;\alpha,\delta_R], V[y;\gamma,\tau_V], W[x;\eta,\tau_W], S[y;\beta,\delta_S])$$

. In section 4, we study the structure of $\mathcal{M}[z;\theta,d]$.

## 2. Terminology

Here are some notions introduced in [3]:

(1) Let $R, S, A, B$ be rings, $V$ be an $(R,S)$-bimodule, $X$ be an $(A,B)$-bimodule, and $\alpha : R \to A, \beta : S \to B$ be ring homomorphisms. The additive mapping $\gamma : V \to X$ is called a *bimodule homomorphism relative to* $(\alpha,\beta)$ if

$$\gamma(rv) = \alpha(r)\gamma(v) \quad \text{and} \quad \gamma(vs) = \gamma(v)\beta(s)$$

for each $r \in R$, $s \in S$ and $v \in V$.

(2) Let $V$ be an $(R,S)$-bimodule, $\gamma : V \to V$ be a bimodule homomorphism relative to $(\alpha,\beta)$, $\delta_R : R \to R$ be an $\alpha$-derivation and $\delta_S : S \to S$ be a $\beta$-derivation. The additive mapping $\tau : V \to V$ is called a *skew generalized derivation relative to* $(\delta_R, \delta_S)$ if

$$\tau(rv) = \alpha(r)\tau(v) + \delta_R(r)v \quad \text{and} \quad \tau(vs) = \gamma(v)\delta_S(s) + \tau(v)s$$

for all $r \in R$, $s \in S$ and $v \in V$.

(3) Let $V$ be an $(R,S)$-bimodule, the mappings $\alpha : R \to R$, $\beta : S \to S$ be ring endomorphisms, and $\gamma : V \to V$ be a bimodule homomorphism relative to $(\alpha,\beta)$. Suppose that $\delta_R : R \to R$ is an $\alpha$-derivation, $\delta_S : S \to S$ is a $\beta$-derivation, and $\tau : V \to V$ is a skew generalized derivation relative to $(\delta_R, \delta_S)$. We call an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule $V'$ *a skew polynomial module* $V[y;\gamma,\tau]$ if
   (a) $V'$ contains $V$ as an $(R,S)$-subbimodule
   (b) for each $v \in V$, we have $xv = \gamma(v)y + \tau(v)$
   (c) each element $p \in V[y;\gamma,\tau]$ is uniquely written as $p = v_0 + v_1 y + \cdots + v_k y^k$ with $v_i \in V$ and $y^i \in S[y,\beta,\delta_s]$ for all $i \in \mathbb{N}$, $1 \leq i \leq k$

**Remark 1.** *The conditions in skew generalized derivation relative to $(\delta_R, \delta_S)$ might look strange because they are not symmetric as defined by Ghahramani. But, Ghahramani needs the conditions to define the skew derivative [3].*

The following terminology will be used in this article. Let $\mathcal{M}$ be the ring of Morita context $(R, V, W, S)$. The notations $E_{11}$, $E_{22}$, $rE_{11}$, $vE_{12}$, $wE_{21}$, $sE_{22}$ denote the elements $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}$ of $\mathcal{M}$, respectively. The notation $(m)_{ij}$ denotes the $(i, j)$-th entry of $m \in \mathcal{M}$. The notation $(r, v, w, s)$ denotes the elements $\begin{pmatrix} r & v \\ w & s \end{pmatrix}$ of $\mathcal{M}$.

## 3. The Morita Context $(R[x; \alpha, \delta_R], V[y; \gamma, \tau_V], W[x; \eta, \tau_W], S[y; \beta, \delta_S])$

Before we discuss the Morita Context $(R[x; \alpha, \delta_R], V[y; \gamma, \tau_V], W[x; \eta, \tau_W], S[y; \beta, \delta_S])$, we need to discuss the homomorphism and the skew derivation of the Morita ring of Morita context. By using a similar technique as the one developed in [3] we have the following two propositions:

**Proposition 3.1.** *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be rings of Morita context $(R, V, W, S)$ and $(A, X, Y, B)$ respectively and $\theta : \mathcal{M}_1 \to \mathcal{M}_2$ be a mapping. The following statements are equivalent:*

i. *$\theta$ is a ring homomorphism and satisfies $\theta(E_{11}) = E_{11}$ and $\theta(E_{22}) = E_{22}$*

ii. *There are mappings $\alpha : R \to A$, $\beta : S \to B$, $\gamma : V \to X$ and $\eta : W \to Y$ that satisfy:*
   - *$\alpha, \beta$ are ring homomorphisms*
   - *$\gamma, \eta$ are bimodule homomorphisms relative to $(\alpha, \beta)$ and $(\beta, \alpha)$, respectively,*
   - *$\alpha, \beta$ satisfy $\alpha(vw) = \gamma(v)\eta(w)$, $\beta(wv) = \eta(w)\gamma(v)$ for all $v \in V, w \in W$*

   *such that*

$$\theta(r, v, w, s) = (\alpha(r), \gamma(v), \eta(w), \beta(s)).$$

   *for all $(r, v, w, s) \in \mathcal{M}$.*

*Proof.* If (ii) holds, we can directly show that $\theta$ is a ring homomorphism. Since $\alpha$ and $\beta$ are ring homomorphism then $\alpha(1) = 1$ and $\beta(1) = 1$. Since $\gamma$ and $\eta$ are additive then $\gamma(0) = 0$ and $\eta(0) = 0$. Therefore, $\theta(E_{11}) = E_{11}$ and $\theta(E_{22}) = E_{22}$.

If $(i)$ holds. From $rE_{11} = rE_{11}E_{11} = E_{11}rE_{11}$ we have a mapping $\alpha : R \to A$ such that $\theta(rE_{11}) = \alpha(r)E_{11}$ for all $r \in R$. From $(r_1 + r_2)E_{11} = r_1E_{11} + r_2E_{11}$ and $(r_1r_2)E_{11} = r_1E_{11}r_2E_{11}$ we get that $\alpha$ is a ring homomorphism. Similarly, we have a ring homomorphism $\beta : S \to Y$ such that $\theta(sE_{22}) = \beta(s)E_{22}$ for all $s \in S$. From $vE_{12} = vE_{12}E_{22} = E_{11}vE_{12}$, $(v_1 + v_2)E_{12} = v_1E_{12} + v_2E_{12}$, $(rv)E_{12} = rE_{11}vE_{12}$, and $(vs)E_{12} = vE_{12}sE_{22}$, we get a mapping $\gamma : V \to X$ that is a bimodule homomorphism relative to $(\alpha, \beta)$ and satisfying $\theta(vE_{12}) = \gamma(v)E_{12}$ for all $v \in V$. Similarly, we get a mapping $\eta : W \to Y$ that is a bimodule homomorphism relative to $(\beta, \alpha)$ and satisfying $\theta(wE_{21}) = \eta(w)E_{21}$ for all $w \in W$. From $(vw)E_{11} = vE_{12}wE_{21}$ and $(wv)E_{22} = wE_{21}vE_{12}$ we have that $\alpha$ and $\beta$ satisfying the relation $\alpha(vw) = \gamma(v)\eta(w)$ and $\beta(wv) = \eta(w)\gamma(v)$ for all $v \in V, w \in W$. Lastly, since every $(r, v, w, s) \in \mathcal{M}$ can be written in the form $rE_{11} + vE_{12} + wE_{21} + sE_{22}$ then we have $\theta(r, v, w, s) = \theta(rE_{11}) + \theta(vE_{12}) + \theta(wE_{21}) + \theta(sE_{22}) = (\alpha(r), \gamma(v), \eta(w), \beta(s))$. □

**Proposition 3.2.** *Let $\mathcal{M}$ be a ring of Morita context $(R, V, W, S)$ and $\theta : \mathcal{M} \to \mathcal{M}$ be a ring endomorphism satisfying (ii) in Proposition 3.1. The function $d : \mathcal{M} \to \mathcal{M}$ is a $\theta$-derivation if and only if there are $v_0 \in V$, $w_0 \in W$, functions $\delta_R : R \to R$, $\delta_S : S \to S$, $\tau_V : V \to V$, $\tau_W : W \to W$ that satisfy:*

i. *$\delta_R$ and $\delta_S$ are the $\alpha$-derivation and the $\beta$-derivation, respectively, for mappings $\alpha$ and $\beta$ as in the previous proposition*

ii. *$\tau_V$ and $\tau_W$ are skew generalized derivations relative to $(\delta_R, \delta_S)$ and $(\delta_S, \delta_R)$, respectively*

iii. *$\delta_R$ and $\delta_S$ satisfy*
   - *$\delta_R(vw) = \gamma(v)\tau_W(w) + \tau_V(v)w$ and*
   - *$\delta_S(wv) = \eta(w)\tau_V(v) + \tau_W(w)v$*

   *for all $v \in V$, $w \in W$, and mappings $\gamma$ and $\eta$ as in the previous proposition.*

iv. $d(E_{11}) = \begin{pmatrix} 0 & v_0 \\ w_0 & 0 \end{pmatrix}$

*such that*

$$d\begin{pmatrix} r & v \\ w & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau_V(v) \\ \tau_W(w) & \delta_S(s) \end{pmatrix} + \theta\begin{pmatrix} r & v \\ w & s \end{pmatrix}\begin{pmatrix} 0 & v_0 \\ -w_0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & v_0 \\ -w_0 & 0 \end{pmatrix}\begin{pmatrix} r & v \\ w & s \end{pmatrix}$$

*for all $(r, v, w, s) \in \mathcal{M}$.*

*Proof.* Let $d$ be a $\theta$-derivation with $\theta(E_{11}) = E_{11}$ and $\theta(E_{22}) = E_{22}$. Since $I = E_{11} + E_{22}$ then $\theta(I) = I$. Note that $d(I) = d(II) = \theta(I)d(I) + d(I)I = d(I) + d(I)$. Therefore $d(I) = 0$. Since $d$ is additive then $d(E_{11}) + d(E_{22}) = 0$. Similar to the proof of Proposition 3.1, from $E_{11} = E_{11}E_{11}$ we get

$$d(E_{11}) = \begin{pmatrix} 0 & v_0 \\ w_0 & 0 \end{pmatrix} \quad \text{for some} \quad v_0 \in V \text{ and } w_0 \in W.$$

From $rE_{11} = rE_{11}E_{11} = E11rE_{11}$, $(r_1 + r_2)E_{11} = r_1E_{11} + r_2E_{22}$, and $(r_1r_2)E_{11} = r_1E_{11}r_2E_{22}$, we get a function $\delta_R : R \to R$ defined by $\delta_R(r) = (d(rE_{11}))_{11}$ for all $r \in R$ and the function $\delta_R$ is an $\alpha$-derivation. In particular,

$$d\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \alpha(r)v_0 \\ w_0 r & 0 \end{pmatrix}$$

for all $r \in R$. Similarly, we get a function $\delta_S : S \to S$ defined by $\delta_R(s) = (d(sE_{22}))_{22}$ for all $s \in S$ and the function $\delta_S$ is a $\beta$-derivation. In particular

$$d\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & -v_0 s \\ -\beta(s)w_0 & \delta_S(s) \end{pmatrix}$$

for all $s \in S$. From $vE_{12} = vE_{12}E_{22} = E_{11}vE_{12}$, we get a function $\tau_V : V \to V$ defined by $\tau_V(v) = (d(vE_{12}))_{12}$ for all $v \in V$. From $wE_{21} = wE_{21}E_{11} = E_{22}wE_{21}$, we get a function $\tau_W \to W$ defined by $\tau_W(w) = (d(wE_{21}))_{21}$ for all $w \in W$. In particular,

$$d\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\gamma(v)w_0 & \tau_V(v) \\ 0 & w_0 v \end{pmatrix} \text{ dan } d\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} = \begin{pmatrix} -v_0 w & 0 \\ \tau_W(w) & \eta(w)v_0 \end{pmatrix}$$

for all $v \in V$ and $w \in W$. Since $d$ is additive, we can prove both $\tau_V$ and $\tau_W$ are additive. From $(sw)E_{21} = sE_{22}wE_{21}$, $(wr)E_{21} = wE_{21}rE_{11}$, $(rv)E_{12} = rE_{11}vE_{21}$, $(vs)E_{12} = vE_{12}sE_{22}$, $(vw)E_{11} = vE_{12}wE_{21}$, and $(wv)W_{22} = wE_{21}vE_{12}$ we get that $\tau_V$ and $\tau_W$ are skew generalized derivations relative to $(\delta_R, \delta_S)$ and $(\delta_S, \delta_R)$, respectively. Moreover, $\delta_R$ and $\delta_S$ satisfying $\delta_R(vw) = \gamma(v)\tau_W(w) + \tau_V(v)w$ and $\delta_S(wv) = \eta(w)\tau_V(v) + \tau_W(w)v$ for all $v \in V$, $w \in W$. Lastly, for all $(r, v, w, s) \in \mathcal{M}$, we have

$$\begin{aligned} d\begin{pmatrix} r & v \\ w & s \end{pmatrix} &= d\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} \\ &= \begin{pmatrix} \delta_R(r) - \gamma(v)w_0 - v_0 w & \alpha(r)v_0 - v_0 s + \tau_V(v) \\ w_o r - \beta(s)w_0 + \tau_W(w) & \delta_S(s) + w_0 v + \eta(w)v_0 \end{pmatrix} \\ &= \begin{pmatrix} \delta_R(r) & \tau_V(v) \\ \tau_W(w) & \delta_S(s) \end{pmatrix} + \theta\begin{pmatrix} r & v \\ w & s \end{pmatrix}\begin{pmatrix} 0 & v_0 \\ -w_0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & v_0 \\ -w_0 & 0 \end{pmatrix}\begin{pmatrix} r & v \\ w & s \end{pmatrix} \end{aligned}$$

The proof for the converse can be checked through direct calculations. □

Note that the homomorphism described in Proposition 3.1 is a ring isomorphism if and only if $\alpha, \beta$ are ring isomorphisms and $\gamma, \eta$ are bijective mappings satisfying condition (ii) of the proposition. Now, we are ready to discuss the Morita context $(R[x;\alpha,\delta_R], V[y;\gamma,\tau_V], W[x;\eta,\tau_W], S[y;\beta,\delta_S])$.

Let $M = (R, V, W, S)$ be a Morita context, the $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule $V[y;\gamma,\tau_V]$ and the $(S[y;\beta,\delta_S], R[x;\alpha,\delta_R])$-bimodule $W[x;\eta,\tau_W]$ be skew polynomial modules such that the mappings $\alpha, \beta, \gamma, \eta$ satisfy (ii) in Proposition 3.1 and the mappings $\delta_R, \delta_S, \tau_V, \tau_W$ satisfy (iii) in Proposition 3.2.

Since $M = (R, V, W, S)$ is a Morita context, we can define a natural operation from $V[y;\gamma,\tau_V] \otimes_{S[y;\beta,\delta_S]} W[x;\eta,\tau_W]$ to $R[x;\alpha,\delta_R]$ by the definition of skew polynomial modules $V[y;\gamma,\tau_V]$ and $W[x;\eta,\tau_W]$. For example, we can define

$$(vy)(wx) = v(\eta(w)x + \tau_W(w))x = v\eta(w)x^2 + v\tau_W(w)x$$

In the same way, we define the operation from $W[x;\eta,\tau_W] \otimes_{R[x;\alpha,\delta_R]} V[y;\gamma,\tau_V]$ to $S[y;\beta,\delta_S]$. Now, we only need to check that both operations are sufficient to define the Morita Context $(R[x;\alpha,\delta_R], V[y;\gamma,\tau_V], W[x;\eta,\tau_W], S[y;\beta,\delta_S])$. Note that:

(1) For all $v, v' \in V$, $w \in W$ and nonnegative integers $j$ and $k$, we have $vwx^j \in R[x;\alpha,\delta_R]$ and $(vwx^j)v'y^k = vwP(y)y^k = v(wP(y)y^k) = v(wx^j v' y^k)$ with $P(y) = x^j v'$ is a polynomial in $y$, coefficients in $V$, and the second equality follows from the property of the Morita context $M$.

(2) For all $v, v' \in V$, $w \in W$ and nonnegative integers $j$ and $k$, we have

$$\begin{aligned}(vywx^j)v'y^k &= (v\eta(w)x^{j+1} + v\tau_W(w)x^j)v'y^k \\ &= v(\eta(w)x + \tau_W(w))x^j v'y^k \\ &= v(\eta(w)x + \tau_W(w)) \sum v_m y^{m+k} \\ &= v \sum (\eta(w)x + \tau_W(w)) v_m y^{m+k}\end{aligned}$$

with $x^j v' y^k = \sum v_m y^{m+k} \in V[y; \gamma, \tau_V]$. Observe that

$$\begin{aligned}(\eta(w)x + \tau_W(w))v_m y^{m+k} &= (\eta(w)\gamma(v_m)y + \eta(w)\tau_V(v_m) + \tau_W(w)v_m)y^{m+k} \\ &= (\beta(wv_m)y + \delta_S(wv_m))\, y^{m+k} \\ &= ywv_m y^{m+k}.\end{aligned}$$

Then

$$(vywx^j)v'y^k = v \sum ywv_m y^{m+k} = (vy)\left(w \sum v_m y^{m+k}\right) = (vy)\left(wx^j v' y^k\right)$$

(3) If $(vy^t wx^j)v'y^k = vy^t(wx^j v'y^k)$ for all $v, v' \in V$, $w \in W$, nonnegative integers $j$,$k$ and nonnegative integer $t < n$, then we can prove $(vy^n wx^j)v'y^k = vy^n(wx^j v'y^k)$ in the same way.

Therefore, by induction, $(vy^i wx^j)v'y^k = vy^i(wx^j v'y^k)$ holds for all $v, v' \in V$, $w \in W$, nonnegative integers $i, j, k$. Thus for all $\sum v_i y^i, \sum v'_k y^k \in V[y; \gamma, \tau_V]$ and $\sum w_j x^j \in W[x; \eta, \tau_W]$, we have the following chain of equalities

$$\begin{aligned}&\left(\sum_i v_i y^i \sum_j w_j x^j\right) \sum_k v'_k y^k \\ &= \left(\sum_i \left(v_i y^i \sum_j w_j x^j\right)\right) \sum_k v'_k y^k = \sum_i \left(\left(v_i y^i \sum_j w_j x^j\right) \sum_k v'_k y^k\right) \\ &= \sum_i \left(\sum_k \left(v_i y^i \sum_j w_j x^j\right) v'_k y^k\right) = \sum_i \left(\sum_k \left(\sum_j v_i y^i w_j x^j\right) v'_k y^k\right) \\ &= \sum_i \left(\sum_k \left(\sum_j (v_i y^i w_j x^j) v'_k y^k\right)\right) = \sum_i \left(\sum_k \left(\sum_j v_i y^i (w_j x^j v'_k y^k)\right)\right)\end{aligned}$$

$$= \sum_i \left( \sum_k \left( v_i y^i \sum_j (w_j x^j v'_k y^k) \right) \right) = \sum_i \left( v_i y^i \sum_k \left( \sum_j (w_j x^j v'_k y^k) \right) \right)$$

$$= \sum_i (v_i y^i) \sum_k \left( \sum_j (w_j x^j v'_k y^k) \right) = \sum_i v_i y^i \left( \sum_j w_j x^j \sum_k v'_k y^k \right).$$

Analogously, we also have

$$\left( \sum_i w_i x^i \sum_j v_j y^j \right) \sum_k w'_k x^k = \sum_i w_i x^i \left( \sum_j v_j y^j \sum_k w'_k x^k \right)$$

for all $\sum w_i x^i, \sum w'_k x^k \in W[x; \eta, \tau_w]$ and $\sum v_j y^j \in V[y; \gamma, \tau_V]$.

In the same way, we can also prove that the operation from $V[y; \gamma, \tau_V] \otimes_{S[y;\beta,\delta_S]} W[x; \eta, \tau_W]$ to $R[x; \alpha, \delta_R]$ is bilinear $R[x; \alpha, \delta_R] - R[x; \alpha, \delta_R]$ and the operation from $W[x; \eta, \tau_W] \otimes_{R[x;\alpha,\delta_R]} V[y; \gamma, \tau_V]$ to $S[y; \beta, \delta_S]$ is bilinear $S[y; \beta, \delta_S] - S[y; \beta, \delta_S]$.

We summarize our observations in the following proposition.

**Proposition 3.3.** *Let $M = (R, V, W, S)$ be a Morita context. Let $(R[x; \alpha, \delta_R], S[y; \beta, \delta_S])$-bimodule $V[y; \gamma, \tau_V]$ and $(S[y; \beta, \delta_S], R[x; \alpha, \delta_R])$-bimodule $W[x; \eta, \tau_W]$ be skew polynomial modules. Let the mappings $\alpha, \beta, \gamma, \eta$ satisfy (ii) in Proposition 3.1 and the mappings $\delta_R, \delta_S, \tau_V, \tau_W$ satisfy (iii) in Proposition 3.2. Then*

$$(R[x; \alpha, \delta_R], V[y; \gamma, \tau_V], W[x; \eta, \tau_W], S[y; \beta, \delta_S])$$

*is a Morita context under the operations in skew polynomial module $V[y; \gamma, \tau_V]$, skew polynomial module $W[x; \eta, \tau_W]$ and Morita context $M$.*

## 4. The Skew Polynomial Ring $\mathcal{M}[z; \theta, d]$

By using the same technique as the one developed in [3], we have the following proposition. The proof of this proposition is imilar to the proof of [Proposition 3.5] in [3].

**Proposition 4.1.** *Let $\mathcal{M}$ be a ring of Morita context $(R, V, W, S)$, $\theta : \mathcal{M} \to \mathcal{M}$ be a ring endomorphism, and $d : \mathcal{M} \to \mathcal{M}$ be a $\theta$-derivation. The following statements are equivalent:*

i. *$\theta$ satisfies the condition (ii) in Proposition 3.1 and $d$ satisfies the properties in Proposition 3.2 with $v_0 = 0$ and $w_0 = 0$.*

ii. *$E_{11} z E_{22} = 0$ and $E_{22} z E_{11} = 0$ in $\mathcal{M}[z; \theta, d]$*

iii. $E_{11}z = zE_{11}$ *in* $\mathcal{M}[z;\theta,d]$

iv. $E_{22}z = zE_{22}$ *in* $\mathcal{M}[z;\theta,d]$

*If any of the above conditions holds, then in* $\mathcal{M}[z;\theta,d]$*, we have*

(1) $(E_{11}z)^n = E_{11}z^n = z^nE_{11}$ *and* $(E_{22}z)^n = E_{22}z^n = z^nE_{22}$ *for every positive integer* $n$*.*

(2) *For every* $p(z) \in \mathcal{M}[z;\theta,d]$ *with* $p(z) = \sum m_iz^i$*, we have*

(a) $E_{11}p(z)E_{11} = \sum(m_i)_{11}E_{11}z^i$

(b) $E_{11}p(z)E_{22} = \sum(m_i)_{12}E_{12}z^i$

(c) $E_{22}p(z)E_{11} = \sum(m_i)_{21}E_{21}z^i$

(d) $E_{22}p(z)E_{22} = \sum(m_i)_{22}E_{22}z^i$

Here is the main theorem.

**Theorem 4.1.** *Let* $\mathcal{M}$ *be a ring of Morita context* $(R,V,W,S)$*,* $\theta : \mathcal{M} \to \mathcal{M}$ *be a ring endomorphism satisfying the condition (ii) in Proposition 3.1 and* $d : \mathcal{M} \to \mathcal{M}$ *be a* $\theta$*-derivation satisfying the properties in Proposition 3.2 with* $v_0 = 0$ *and* $w_0 = 0$*, then the skew polynomial ring* $\mathcal{M}[z;\theta,d]$ *is isomorphic to the ring of Morita context*

$$(R[x;\alpha,\delta_R], V[y;\gamma,\tau_V], W[x;\eta,\tau_W], S[y;\beta,\delta_S])$$

*as in Proposition 3.3.*

*Proof.* Let $\mathcal{N}_1$ denote the ring of Morita context

$$(R[x;\alpha,\delta_R], V[y;\gamma,\tau_V], W[x;\eta,\tau_W], S[y;\beta,\delta_S])$$

and $\mathcal{N}_2$ denote the ring of Morita context

$$(E_{11}\mathcal{M}[z;\theta,d]E_{11}, E_{11}\mathcal{M}[z;\theta,d]E_{22}, E_{22}\mathcal{M}[z;\theta,d]E_{11}, E_{22}\mathcal{M}[z;\theta,d]E_{22})$$

where $E_{11}, E_{12}, E_{21}, E_{22}$ are the elements in the ring of Morita context $\mathcal{M}$. By Pierce decomposition, we have $\mathcal{M}[z;\theta,d] \cong \mathcal{N}_2$. So we only need to define the bijective mapping $\Lambda : \mathcal{N}_1 \to \mathcal{N}_2$ satisfying the condition (ii) in Proposition 3.1.

Define $\phi_1 : R \to E_{11}\mathcal{M}[z;\theta,d]E_{11}$ by $\phi_1(r) = E_{11}(rE_{11})E_{11} = rE_{11}$ for all $r \in R$. We can check directly that $E_{11}zE_{11} \in E_{11}\mathcal{M}[z;\theta,d]E_{11}$ satisfies

$$(E_{11}zE_{11})\phi_1(r) = \phi_1(\alpha(r))z + \phi(\delta_R(r))$$

for all $r \in R$. By [4, p. 37], there is a unique ring homomorphism $\varphi_1 : R[x;\alpha,\delta_R] \to E_{11}\mathcal{M}[z;\theta,d]E_{11}$ such that $\varphi_1|_R = \phi$ and $\varphi_1(x) = E_{11}zE_{11}$. In particular,

$$\varphi_1\left(\sum r_i x^i\right) = \sum \phi_1(r_i)(E_{11}zE_{11})^i = \sum r_i E_{11} z^i = E_{11}\left(\sum r_i E_{11} z^i\right)E_{11}.$$

The last equation tells us that $\varphi_1$ is injective. On the other hand, by Proposition 4.1, we have

$$E_{11}\mathcal{M}[z;\theta,d]E_{11} = \left\{\sum r_i E_{11} z^i \mid r_i \in R\right\}.$$

Thus $\varphi_1$ is onto. We conclude that $\varphi_1$ is a ring isomorphism.

Analogously, if we define $\phi_2 : S \to E_{22}\mathcal{M}[z;\theta,d]E_{22}$ by $\phi_2(s) = E_{22}(sE_{22})E_{22} = sE_{22}$, then we have a unique ring isomorphism $\varphi_2 : S[y;\beta,\delta_S] \to E_{22}\mathcal{M}[z;\theta,d]E_{22}$ such that $\varphi_2|_S = \phi_2$ and $\varphi_2(y) = E_{22}zE_{22}$. In particular,

$$\varphi_2\left(\sum s_i y^i\right) = \sum \phi_2(s_i)(E_{22}zE_{22})^i = \sum s_2 E_{22} z^i = E_{22}\left(\sum s_i E_{22} z^i\right)E_{22}.$$

Note that

$$E_{11}\mathcal{M}[z;\theta,d]E_{22} = \left\{\sum v_i E_{12} z^i \mid v_i \in V\right\}.$$

Define $\varphi_3 : V[y;\gamma,\tau_V] \to E_{11}\mathcal{M}[z;\theta,d]E_{22}$ by $\varphi_3\left(\sum v_i y^i\right) = \sum v_i E_{12} z^i$ for all $\sum v_i y^i \in V[y;\gamma,\tau_V]$. Clearly, $\varphi_3$ is an additive bijective mapping. Let $r \in R \subset R[x;\alpha,\delta_R]$ and $\sum v_i y^i \in V[y;\gamma,\tau_V]$. Note that

$$\begin{aligned}\varphi_3\left(r\sum v_i y^i\right) &= \varphi_3\left(\sum r v_i y^i\right) = \sum r v_i E_{12} z^i = rE_{11}\sum r v_i E_{12} z^i\\ &= \varphi_1(r)\varphi_3\left(\sum v_i y^i\right).\end{aligned}$$

Therefore, for each $r \in R$ and $v(y) \in V[y;\gamma,\tau_V]$, we have

$$\varphi_3(rv(y)) = \varphi_1(r)\varphi_3(v(y)).$$

Let $rx \in R[x;\alpha,\delta_R]$ and $\sum v_i y^i \in V[y;\gamma,\tau_V]$. Note that

$$\begin{aligned}\varphi_3\left(rx\sum v_i y^i\right) &= \varphi_3\left(\sum r x v_i y^i\right)\\ &= \varphi_3\left(\sum r\left(\gamma(v_i)y^{i+1} + \tau_V(v)y^i\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum r\gamma(v_i)E_{12}z^{i+1} + r\tau_V(v)E_{12}z^i \\
&= (rE_{11})\left(\sum(\gamma(v_i)E_{12}z + \tau_V(v)E_{12})z^i\right) \\
&= (rE_{11})\left(\sum zv_iE_{12}z^i\right) \\
&= (rE_{11}z)\left(\sum v_iE_{12}z^i\right) \\
&= \varphi_1(rx)\varphi_3\left(\sum v_iy^i\right).
\end{aligned}
$$

Therefore, for each $rx \in R[x;\alpha,\delta_R]$ and $v(y) \in V[y;\gamma,\tau_V]$ we have

$$\varphi_3(rxv(y)) = \varphi_1(rx)\varphi_3(v(y)).$$

Let $n$ be a positive integer, assuming that for each $rx^t \in R[x;\alpha,\delta_R]$ with $r \in R$, $t < n$ and $v(y) \in V[y;\gamma,\tau_V]$, we have $\varphi_3(rx^tv(y)) = \varphi_1(rx^t)\varphi_3(v(y))$. In the same way, for every $rx^n \in R[x;\alpha,\delta_R]$ and $\sum v_iy^i \in V[y;\gamma,\tau_V]$ we have the following equality

$$\varphi_3\left(rx^n\sum v_iy^i\right) = \varphi_1(rx^n)\varphi_3\left(\sum v_iy^i\right).$$

By induction, we conclude that the equation $\varphi_3(rx^nv(y)) = \varphi_1(rx^n)\varphi_3(v(y))$ holds for every nonnegative integer $n$, $r \in R$ and $v(y) \in V[y;\gamma,\tau_V]$. By using the additivity of $\varphi_3$, we have the equality

$$\varphi_3(r(x)v(y)) = \varphi_1(r(x))\varphi_3(v(y))$$

, which holds for all $r(x) \in R[x;\alpha,\delta_R]$ and $v(y) \in V[y;\gamma,\tau_V]$. By the same argument (using the property of skew polynomial ring $S[y;\beta,\delta_S]$) we can prove that the equality $\varphi_3(v(y)s(y)) = \varphi_3(v(y))\varphi_2(s(y))$ holds for all $v(y) \in V[y;\gamma,\tau_V]$ and $s(y) \in S[y;\beta,\delta_S]$. Thus, $\varphi_3$ is a bijective bimodule homomorphism relative to $(\varphi_1,\varphi_2)$.

Similarly, we can prove that the mapping $\varphi_4 : W[x;\gamma,\tau_W] \to E_{22}\mathcal{M}[z;\theta,d]E_{22}$, which is defined by $\varphi_4\left(\sum w_ix^i\right) = \sum w_iE_{21}z^i$ for all $\sum w_ix^i \in W[x;\gamma,\tau_W]$, is bijective bimodule homomorphism relative to $(\varphi_2,\varphi_1)$.

Similarly, we can also prove that the equations $\varphi_1(v(y)w(x)) = \varphi_3(v(y))\varphi_4(w(x))$ and $\varphi_4(w(y)v(x)) = \varphi_4(w(y))\varphi_3(v(x))$ hold for all $v(y) \in V[y;\gamma,\tau_V]$ and $w(x) \in W[x;\eta,\tau_W]$.

Define $\Lambda : \mathcal{N}_1 \to \mathcal{N}_2$ by

$$\Lambda(r(x),v(y),w(x),s(y)) = (\varphi_1(r(x)),\varphi_3(v(x)),\varphi_4(w(x)),\varphi_2(s(x)))$$

for all $(r(x), v(y), w(x), s(y)) \in \mathcal{N}_1$. By the above observations, we have $\mathcal{N}_1 \cong \mathcal{N}_2$. □

**Example 4.1.** *Let* $\begin{pmatrix} R & V \\ W & S \end{pmatrix}$ *denote the ring of Morita context* $(R, V, W, S)$.

(1) *Let $R$ be a ring. Note that the set $M = (R, R^{1\times 2}, R^{2\times 1}, R^{2\times 2})$ with the usual matrix operation becomes a Morita context. Let $\mathcal{M}$ be a ring of Morita context $M$. Define the mappings $\theta : \mathcal{M} \to \mathcal{M}$ and $d : \mathcal{M} \to \mathcal{M}$ by:*

$$\theta\,(a, b, c, d) = (a, -b, -c, d) \quad \textit{and} \quad d\,(a, b, c, d) = (0, b, c, 0)$$

*for every $(a, b, c, d) \in \mathcal{M}$. The mappings $\theta$ and $d$ are a ring endomorphism and a $\theta$ derivation, respectively. By Theorem 4.1,*

$$\mathcal{M}[z; \theta, d] \cong \begin{pmatrix} R[x] & R^{1\times 2}[y; -id_{R^{1\times 2}}, id_{R^{1\times 2}}] \\ R^{2\times 1}[x; -id_{R^{2\times 1}}, id_{R^{2\times 1}}] & R^{2\times 2}[y] \end{pmatrix}$$

*with $R[x]$ and $R^{2\times 2}[y]$ as the usual polynomial rings.*

(2) *Let $\mathcal{M}$ be a ring of Morita context $(R, V, W, S)$, and $\theta : \mathcal{M} \to \mathcal{M}$ be a ring endomorphism that satisfy (ii) in Proposition 3.1. Choose the zero mapping in $\mathcal{M}$ for the $\theta$-derivation. By Theorem 4.1,*

$$\mathcal{M}[z; \theta] \cong \begin{pmatrix} R[x; \alpha] & V[y; \gamma] \\ W[x; \eta] & S[y; \beta] \end{pmatrix}.$$

*If $\alpha \neq id_R$ and $\beta \neq id_S$ then both $R[x; \alpha]$ and $S[y; \beta]$ are the polynomial ring of endomorphism type.*

(3) *Let $W = \{0\}$ be a $(S, R)$-bimodule. One can define the Morita context $(R, V, W, S)$ by $v0 = 0 \in R$ and $0v = 0 \in S$ for all $v \in V$. Let $\alpha : R \to R$ and $\beta : R \to R$ be ring homomorphisms, $\gamma : V \to V$ be a bimodule homomorphism relative to $(\alpha, \beta)$, $\delta_R : R \to R$ be an $\alpha$-derivation, $\delta_S : S \to S$ be a $\beta$-derivation, and $\tau : V \to V$ be a skew generalized derivation relative to $(\delta_R, \delta_S)$. Note that the identity mapping in $W$ is a bimodule homomorphism relative to $(\beta, \alpha)$ and a skew generalized derivation relative to $(\delta_S, \delta_R)$. By Theorem 4.1*

$$\mathcal{M}[z; \theta, d] \cong \begin{pmatrix} R[x; \alpha, \delta_R] & V[y; \gamma, \tau] \\ 0 & S[y; \beta, \delta_S] \end{pmatrix}.$$

*This result is confirmed by [Proposition 3.6] in [3].*

The following theorem generalizes Theorem 4.1.

**Theorem 4.2.** *Let $\mathcal{M}$ be a ring of Morita context $(R, V, W, S)$, $\theta : \mathcal{M} \to \mathcal{M}$ be a ring endomorphism satisfying condition (ii) of Proposition 3.1 and $d : \mathcal{M} \to \mathcal{M}$ be a $\theta$-derivation, then the skew polynomial ring $\mathcal{M}[z; \theta, d]$ is isomorphic with the Morita context*

$$(R[x; \alpha, \delta_R], V[y; \gamma, \tau_V], W[x; \eta, \tau_W], S[y; \beta, \delta_S])$$

*as in Proposition 3.3 and the mappings $\delta_R, \delta_S, \tau_V, \tau_W$ satisfying conditions (i), (ii), and (iii) in Proposition 3.2.*

*Proof.* Since $d$ is a $\theta$-derivation, $d$ can be written in the form

$$d\begin{pmatrix} r & v \\ w & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau_V(v) \\ \tau_W(w) & \delta_S(s) \end{pmatrix} + I_a \begin{pmatrix} r & v \\ w & s \end{pmatrix}$$

for all $r \in R$, $v \in V$, $w \in W$, $s \in S$ with $a = \begin{pmatrix} 0 & v_0 \\ -w_0 & 0 \end{pmatrix}$ and mappings $\delta_R, \delta_s, \tau_V, \tau_W$ satisfying (i), (ii), and (iii) in Proposition 3.2. Define $D : \mathcal{M} \to \mathcal{M}$ by

$$D\begin{pmatrix} r & v \\ w & s \end{pmatrix} = \begin{pmatrix} \delta_R(r) & \tau_V(v) \\ \tau_W(w) & \delta_S(s) \end{pmatrix}$$

for all $r \in R$, $v \in V$, $w \in W$, $s \in S$. By Proposition 3.2, $D$ is a $\theta$-derivation. By [3, Proposition 3.2]

$$\mathcal{M}[z; \theta, d] \cong \mathcal{M}[z'; \theta, D]$$

by the isomorphism sending $z$ to $z' - a$. On the other hand, by Theorem 4.1 we have

$$M[z'; \theta, D] \cong \begin{pmatrix} R[x; \alpha, \delta_R] & V[y; \gamma, \tau_V] \\ W[x; \eta, \tau_W] & S[y; \beta, \delta_S] \end{pmatrix}$$

with $(R[x; \alpha, \delta_R], V[y; \gamma, \tau_V], W[x; \eta, \tau_W], S[y; \beta, \delta_S])$ as the Morita context in Proposition 3.3. We conclude that

$$M[z; \theta, d] \cong \begin{pmatrix} R[x; \alpha, \delta_R] & V[y; \gamma, \tau_V] \\ W[x; \eta, \tau_W] & S[y; \beta, \delta_S] \end{pmatrix}.$$

□

In Theorem 4.2, there are two ring isomorphisms:

$$\Lambda_1 : \mathcal{M}[z;\theta,d] \to \mathcal{M}[z';\theta,D]$$

and

$$\Lambda_2 : \mathcal{M}[z';\theta,D] \to \begin{pmatrix} R[x;\alpha,\delta_R] & V[y;\gamma,\tau_V] \\ W[x;\eta,\tau_W] & S[y;\beta,\delta_S] \end{pmatrix}.$$

From [3, Proposition 3.2], the mapping $\Lambda_1$ satisfies $\Lambda_1(E_{11}) = E_{11}$, $\Lambda_1(E_{22}) = E_{22}$, and $\Lambda_1(z) = z' - a$. From Theorem 4.1, the mapping $\Lambda_2$ satisfies $\Lambda_2(E_{11}) = E_{11}$, $\Lambda_2(E_{22}) = E_{22}$, and $\Lambda_2(z' - a) = \Lambda_2(z') - \Lambda_2(a) = (x, 0, 0, y) - a$. Moreover, $\Lambda_1$ and $\Lambda_2$ are identity mappings on $\mathcal{M}$. Therefore,

$$(\Lambda_2 \circ \Lambda_1)(z + a) = (\Lambda_2 \circ \Lambda_1)(z) + a = (x, 0, 0, y)$$

is a diagonal matrix.

The following theorem explains that the homomorphism and skew derivation on the ring of Morita context must satisfy the conditions given in Propositions 3.1 and 3.2, respectively, if the skew polynomial ring of Morita context is isomorphic to a ring of Morita context and satisfies some particular conditions.

**Theorem 4.3.** *Let $\mathcal{M}$ be a ring of Morita context $(R, V, W, S)$, $\theta : \mathcal{M} \to \mathcal{M}$ be a ring endomorphism on $\mathcal{M}$, and $d : \mathcal{M} \to \mathcal{M}$ be a $\theta$-derivation. Let $(A, X, Y, B)$ be a Morita context such that there exists an isomorphism*

$$\varphi : \mathcal{M}[z;\theta,d] \to \begin{pmatrix} A & X \\ Y & B \end{pmatrix}$$

*satisfying $\varphi(E_{11}) = E_{11}$ and $\varphi(E_{22}) = E_{22}$. Assume that there are an invertible diagonal matrix $P$ in $\mathcal{M}$ and a matrix $Q$ in $\mathcal{M}$ such that $\varphi(Pz + Q)$ is a diagonal matrix, then:*

(1) *The mapping $\theta$ satisfies condition (ii) in Proposition 3.1.*
(2) *The mapping $d$ satisfies the conditions in Proposition 3.2.*
(3) *We have the folowing isomorphisms: $A \cong R[x;\alpha,\delta_R]$ and $B \cong S[y;\beta,\delta_S]$. By the isomorphism one can consider $X$ as an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule which is isomorphic to skew polynomial modules $V[y;\gamma,\tau_V]$ and $Y$ as an*

*$(S[y;\beta,\delta_S], R[x;\alpha,\delta_R])$-bimodule that is isomorphic to skew polynomial modules $W[y;\eta,\tau_W]$.*

*Proof.* Note that

$$\varphi(E_{11}(Pz+Q)) = \varphi(E_{11})\varphi(Pz+Q) = E_{11}\varphi(Pz+Q) = \varphi(Pz+Q)E_{11}$$
$$= \varphi(Pz+Q)\varphi(E_{11}) = \varphi((Pz+Q)E_{11}).$$

Since $\varphi$ is an isomorphism, $E_{11}Pz + E_{11}Q = PzE_{11} + QE_{11}$. On the other hand, $PzE_{11} = P\theta(E_{11})z + Pd(E_{11})$, thus we have the equalities $P\theta(E_{11}) = E_{11}P$ and $E_{11}Q = QE_{11} + Pd(E_{11})$. Since $P$ is an invertible diagonal matrix, $P\theta(E_{11}) = E_{11}P$ implies $\theta(E_{11}) = E_{11}$ and $E_{11}Q = QE_{11} + Pd(E_{11})$ implies $d(E_{11}) = (0, v_0, -w_0, 0)$ for some $v_0 \in V$ and $w_0 \in W$. Similarly, from $\varphi(E_{22}(Pz+Q))$ we conclude that $\theta(E_{22}) = E_{22}$. Therefore, the first two statements are proven by Propositions 3.1 and 3.2.

By Theorem 4.2, there is an isomorphism

$$\Lambda : \begin{pmatrix} R[x;\alpha,\delta_R] & V[y;\gamma,\tau_V] \\ W[x;\eta,\tau_W] & S[y;\beta,\delta_S] \end{pmatrix} \to \mathcal{M}[z;\theta,d]$$

such that $\Lambda(E_{11}) = E_{11}$ and $\Lambda(E_{22}) = E_{22}$. Thus

$$\varphi\circ\Lambda : \begin{pmatrix} R[x;\alpha,\delta_R] & V[y;\gamma,\tau_V] \\ W[x;\eta,\tau_W] & S[y;\beta,\delta_S] \end{pmatrix} \to \begin{pmatrix} A & X \\ Y & B \end{pmatrix}$$

is the isomorphism such that $(\varphi\circ\Lambda)(E_{11}) = E_{11}$ and $(\varphi\circ\Lambda)(E_{22}) = E_{22}$. By Proposition 3.1, we have the following ring isomorphisms

$$\varphi_1 : R[x;\alpha,\delta_R] \to A \quad \text{and} \quad \varphi_2 : S[y;\beta,\delta_S] \to B,$$

and the additive bijective mappings

$$\varphi_3 : V[y;\gamma,\tau_V] \to X \quad \text{and} \quad \varphi_2 : W[x;\eta,\tau_W] \to Y,$$

such that $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ satisfy (ii) in Proposition 3.1. By the isomorphism, $X$ can be considered as a $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule by

$$r(x)ts(y) = \varphi_1(r(x))t\varphi_2(s(y))$$

for every $r(x) \in R[x;\alpha,\delta_R]$, $t \in X$, $s(y) \in S[y;\beta,\delta_S]$. Moreover, $\varphi_3$ is a bimodule isomorphism. Therefore, $X$ is a $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule isomorphic to $V[y;\gamma,\tau_V]$. In the same way, $Y$ can be considered a $(S[y;\beta,\delta_S], R[x;\alpha,\delta_R])$-bimodule and isomorphic to $W[x;\eta,\tau_W]$. □

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