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TOPOLOGICAL INVARIANTS OF GENERALIZED SPLITTING GRAPHS AND k-SHADOW GRAPHS

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ABSTRACT. A topological index or an invariant can be defined as a function from a set of graphs to the real line. Topological indices are invariant under graph isomorphism. This paper deals with the general expressions for various topological indices of two derived graphs called generalized splitting graphs and k-shadow graphs. In particular this manuscript discuss about the first Zagreb index, second Zagreb index, F-index, hyper-Zagreb index, symmetric division degree index, first and the second multiplicative Zagreb indices and a lower bound for the irregularity index of generalized splitting graphs and k-shadow graphs.

1. Introduction

In this article we use the notation $\Gamma(V,E)$ for a simple, undirected, connected and finite graph, where V is the vertex set and E is the edge set with $|V| = n$ and $|E| = m$. The study of topological invariants are closely related with molecular graphs and these invariants provides an idea about the topology of the molecular graph. Thus it helps the chemists to explain the physical properties of chemical compounds.

The first and the second Zagreb indices were introduced by Gutman and Trinajestic in [5]. These Zagreb indices are defined as follows

$$M_1(\Gamma) = \sum_{x\in V(\Gamma)} (d_\Gamma(x))^2$$

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and

$$M_2(\Gamma) = \sum_{xy \in E(\Gamma)} d_\Gamma(x) d_\Gamma(y)$$

where $d_\Gamma(x)$ denotes the degree of the vertex x in Γ. The *forgotten topological index* or *F-index* was introduced by Furtula and Gutman in [4] and is defined as

$$F(\Gamma) = \sum_{x \in V(\Gamma)} d_\Gamma(x)^3 = \sum_{xy \in E(\Gamma)} [d_\Gamma(x)^2 + d_\Gamma(y)^2]$$

Shridel et al. in [14] introduced the hyper-Zagreb index as

$$HM(\Gamma) = \sum_{xy \in E(\Gamma)} (d_\Gamma(x) + d_\Gamma(y))^2$$

The *symmetric division degree index* was introduced in [17] as

$$SDD(\Gamma) = \sum_{xy \in E(\Gamma)} \frac{d_\Gamma(x)^2 + d_\Gamma(y)^2}{d_\Gamma(x) d_\Gamma(y)}$$

The first and the second multiplicative Zagreb indices are defined as follows

$$\prod_1(\Gamma) = \prod_{x \in V(\Gamma)} (d_\Gamma(x))^2$$

and

$$\prod_2(\Gamma) = \prod_{xy \in E(\Gamma)} d_\Gamma(x) d_\Gamma(y)$$

They were introduced by Todeschini et al. in [15] and [16].

The *irregularity index* of graph Γ in [2] is defined as

$$irr(\Gamma) = \sum_{xy \in E(\Gamma)} |d_\Gamma(x) - d_\Gamma(y)|$$

where $|d_\Gamma(x) - d_\Gamma(y)|$ is called the imbalance of the edge xy.

The idea of *splitting graph* was proposed by E. Sampathkumar et al. in [10]. The *splitting graph* $S'(\Gamma)$ of a graph Γ is obtained by putting a new vertex v' corresponding to each vertex $v \in V(\Gamma)$ such that vertex v' is adjacent to all the vertices in $N(v)$, the open neighbourhood of v. Thus we have the following observations:

(i) $|V(S'(\Gamma))| = 2n$

(ii) $|E(S'(\Gamma))| = 3m$

(iii) $d_{S'(\Gamma)}(v) = 2d_\Gamma(v)$ and $d_{S'(\Gamma)}(v') = d_\Gamma(v)$

M.E. Addel-Aal in [1] introduced the concept of *k-splitting graph* and *k-shadow graph*. The *generalized splitting graph* or the *k-splitting graph* $S^k(\Gamma)$ is defined as the graph obtained by adding k new vertices corresponding to each vertex $v \in V(\Gamma)$ such that all the new vertices corresponding to a vertex $v \in V(\Gamma)$ are adjacent to all the vertices in $N(v)$, the open neighbourhood of v.
The Figure 1 depicts the graph $S^2(P_3)$.

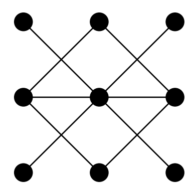


FIGURE 1. The 2-splitting graph of the path P_3

Observation 1.1. *The following observations are immediate consequences of the above definition:*

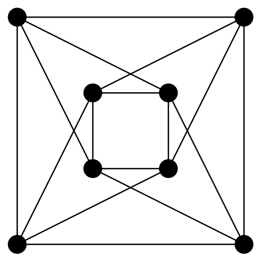
(i) $|V(S^k(\Gamma))| = (k+1)|V(\Gamma)|$
(ii) $|E(S^k(\Gamma))| = (2k+1)|E(\Gamma)|$
(iii) $d_{S^k(\Gamma)}(v) = (k+1)d_\Gamma(v) \quad \forall v \in V(\Gamma)$
(iv) *If* $v' \in V(S^k(\Gamma)) \setminus V(\Gamma)$ *is one of the new vertices added corresponding to the vertex* $v \in V(\Gamma)$*, then* $d_{S^k(\Gamma)}(v') = d_\Gamma(v)$.

The *k-shadow graph* $D_k(\Gamma)$ of a connected graph Γ is obtained by taking k copies of Γ, say $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ and joining each vertex $u \in V(\Gamma_i)$ to the neighbourhood of the corresponding vertex $v \in V(\Gamma_j)$ for $1 \le i, j \le k$ and $i \neq j$.

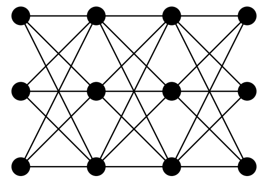
Observation 1.2. *We have the following obvious results:*

(i) $|V(D_k(\Gamma))| = kn$
(ii) $|E(D_k(\Gamma))| = k^2m$
(iii) *If* $v' \in V(\Gamma_i)$ *for some* $i \in \{1, \ldots, k\}$*, then* $d_{D_k(\Gamma)}(v') = kd_\Gamma(v)$*, where* v *is the corresponding vertex of* Γ.

Figure 2 shows the 2-shadow graph $D_2(C_4)$ and the 3-shadow graph $D_3(P_4)$.



(A) The 2-shadow graph of C_4



(B) The 3-shadow graph of P_4

FIGURE 2. Examples for the 2-shadow graph and the 3-shadow graph

Now we find the general expressions for the degree based topological invariants such as the first Zagreb index, second Zagreb index, F-index, symmetric division degree index, the first and the second multiplicative Zagreb indices, irregularity index of *generalised splitting graphs* and *k-shadow graph*. Recently in [3], [6], [9], [7],[8], [11], [12] and [13] computed general expressions for various topological invariants of different graph operations.

2. Topological Invariants for Generalized splitting Graphs

In this section we deal with generalized splitting graph of a graph Γ. We begin this section with the first Zagreb index $M_1(\Gamma)$.

Theorem 2.1. *For any simple connected graph* Γ*,*

$$M_1(S^k(\Gamma)) = (k^2 + 3k + 1)M_1(\Gamma).$$

Proof. The first Zagreb index for a graph Γ is given by

$$M_1(\Gamma) = \sum_{x \in V(\Gamma)} (d_\Gamma(x))^2.$$

By using Observation 1.1 we evaluate $M_1(S^k(\Gamma))$.

$$\begin{aligned}
M_1(S^k(\Gamma)) &= \sum_{x\in V(S^k(\Gamma))} (d_{S^k(\Gamma)}(x))^2 \\
&= \sum_{v\in V(\Gamma)} ((k+1)d_\Gamma(v))^2 + \sum_{v'\in V(S^k(\Gamma))/V(\Gamma)} (d_\Gamma(v))^2 \\
&= (k+1)^2 \sum_{v\in V(\Gamma)} (d_\Gamma(v))^2 + k \sum_{v\in V(\Gamma)} (d_\Gamma(v))^2 \\
&= (k^2+3k+1) \sum_{v\in V(\Gamma)} (d_\Gamma(v))^2 \\
&= (k^2+3k+1)M_1(\Gamma).
\end{aligned}$$

□

Theorem 2.2. *If $S^k(\Gamma)$ is a k-Splitting graph, then*

$$M_2(S^k(\Gamma)) = (3k^2+4k+1)M_2(\Gamma).$$

Proof. The second Zagreb index for a graph Γ is given by

$$M_2(\Gamma) = \sum_{xy\in E(\Gamma)} (d_\Gamma(x)d_\Gamma(y)).$$

Now we find $M_2(S^k(\Gamma))$ by using Observation 1.1.

$$\begin{aligned}
M_2(S^k(\Gamma)) &= \sum_{xy\in E(S^k(\Gamma))} (d_{S^k(\Gamma)}(x) \times d_{S^k(\Gamma)}(y)) \\
&= \sum_{xy\in E(\Gamma)} ((k+1)d_\Gamma(x) \times (k+1)d_\Gamma(y)) + 2k \left(\sum_{xy\in E(\Gamma)} ((k+1)d_\Gamma(x) \times d_\Gamma(y)) \right)
\end{aligned}$$

$$\begin{aligned}
M_2(S^k(\Gamma)) &= (k+1)^2 \left(\sum_{xy\in E(\Gamma)} (d_\Gamma(x) \times d_\Gamma(y)) \right) + 2k(k+1) \left(\sum_{xy\in E(\Gamma)} (d_\Gamma(x) \times d_\Gamma(y)) \right) \\
&= ((k+1)^2 + 2k(k+1)) \left(\sum_{xy\in E(\Gamma)} (d_\Gamma(x) \times d_\Gamma(y)) \right) \\
&= ((k+1)^2 + 2k(k+1))M_2(\Gamma) \\
&= (3k^2+4k+1)M_2(\Gamma).
\end{aligned}$$

□

Theorem 2.3. *For a k-Splitting graph $S^k(\Gamma)$,*

$$F(S^k(\Gamma)) = (k^3 + 3k^2 + 4k + 1)F(\Gamma).$$

Proof. The F-index for a graph Γ is given by

$$F(\Gamma) = \sum_{x\in(\Gamma)} (d_\Gamma(x))^3 = \sum_{xy\in E(\Gamma)} ((d_\Gamma(x))^2 + (d_\Gamma(y))^2).$$

By using the observations on *k-splitting graph* we evaluate $F(S^k(\Gamma))$.

$$\begin{aligned}
F(S^k(\Gamma)) &= \sum_{xy\in E(S^k(\Gamma))} ((d_{S^k(\Gamma)}(x))^2 + (d_{S^k(\Gamma)}(y))^2) \\
&= \sum_{xy\in E(\Gamma)} (((k+1)d_\Gamma(x))^2 + ((k+1)d_\Gamma(y))^2) \\
&+ k\left(\sum_{xy\in E(\Gamma)} [(((k+1)d_\Gamma(x))^2 + (d_\Gamma(y))^2) + ((d_\Gamma(x))^2 + ((k+1)d_\Gamma(y))^2)]\right) \\
&= (k+1)^2\left(\sum_{xy\in E(\Gamma)} ((d_\Gamma(x))^2 + (d_\Gamma(y))^2)\right) \\
&+ k((k+1)^2+1)\left(\sum_{xy\in E(\Gamma)} ((d_\Gamma(x))^2 + (d_\Gamma(y))^2)\right) \\
&= ((k+1)^2 + k((k+1)^2+1))\left(\sum_{xy\in E(\Gamma)} ((d_\Gamma(x))^2 + (d_\Gamma(y))^2)\right) \\
&= (k^3 + 3k^2 + 4k + 1)F(\Gamma).
\end{aligned}$$

□

Theorem 2.4. *If Γ is a simple connected graph, then*

$$HM(S^k(\Gamma)) = (k+1)^2 HM(\Gamma) + k((k+1)^2+1)F(\Gamma) + 4k(k+1)M_2(\Gamma).$$

Proof. The hyper Zagreb index for a graph Γ is given by

$$HM(\Gamma) = \sum_{xy\in E(\Gamma)} (d_\Gamma(x) + d_\Gamma(y))^2.$$

By using the results in Observation 1.1 we compute $HM(S^k(\Gamma))$.

$$\begin{aligned}
HM(S^k(\Gamma)) &= \sum_{xy\in E(S^k(\Gamma))} (d_{S^k(\Gamma)}(x) + d_{S^k(\Gamma)}(y))^2 \\
&= \sum_{xy\in E(\Gamma)} ((k+1)d_\Gamma(x) + (k+1)d_\Gamma(y))^2 \\
&+ k\sum_{xy\in E(\Gamma)} \left[((k+1)d_\Gamma(x) + d_\Gamma(y))^2 + (d_\Gamma(x) + (k+1)d_\Gamma(y))^2\right] \\
&= (k+1)^2 \sum_{xy\in E(\Gamma)} (d_\Gamma(x) + d_\Gamma(y))^2 \\
&+ k\sum_{xy\in E(\Gamma)} \left[((k+1)^2+1)((d_\Gamma(x))^2 + (d_\Gamma(y))^2) + 4(k+1)d_\Gamma(x)d_\Gamma(y)\right] \\
&= (k+1)^2 \sum_{xy\in E(\Gamma)} (d_\Gamma(x) + d_\Gamma(y))^2 \\
&+ k((k+1)^2+1) \sum_{xy\in E(\Gamma)} ((d_\Gamma(x))^2 + (d_\Gamma(y))^2) + 4k(k+1)\sum_{xy\in E(\Gamma)} d_\Gamma(x)d_\Gamma(y) \\
&= (k+1)^2 HM(\Gamma) + k((k+1)^2+1)F(\Gamma) + 4k(k+1)M_2(\Gamma).
\end{aligned}$$

□

Theorem 2.5. *If Γ is any simple connected graph, then*

$$SDD(S^k(\Gamma)) = \frac{(k^3+2k^2+3k+1)}{k+1} SDD(\Gamma).$$

Proof. By the definition of symmetric division degree index of a graph and using the Observation 1.1 we obtain the following expression.

$$\begin{aligned}
SDD(S^k(\Gamma)) &= \sum_{xy\in E(S^k(\Gamma))} \frac{((d_{S^k(\Gamma)}(x))^2 + (d_{S^k(\Gamma)}(y))^2)}{d_{S^k(\Gamma)}(x)d_{S^k(\Gamma)}(y)} \\
&= \sum_{xy\in E(\Gamma)} \frac{(((k+1)d_\Gamma(x))^2 + ((k+1)d_\Gamma(y))^2)}{(k+1)d_\Gamma(x)(k+1)d_\Gamma(y)} \\
&+ k\sum_{xy\in E(\Gamma)} \left[\frac{(((k+1)d_\Gamma(x))^2 + (d_\Gamma(y))^2)}{(k+1)d_\Gamma(x)d_\Gamma(y)} + \frac{((d_\Gamma(x))^2 + ((k+1)d_\Gamma(y))^2)}{d_\Gamma(x)(k+1)d_\Gamma(y)}\right] \\
&= \sum_{xy\in E(\Gamma)} \frac{(d_\Gamma(x))^2 + (d_\Gamma(y))^2}{d_\Gamma(x)d_\Gamma(y)} \\
&+ k\sum_{xy\in E(\Gamma)} \frac{((k+1)^2+1)}{k+1}\frac{((d_\Gamma(x))^2 + (d_\Gamma(y))^2)}{d_\Gamma(x)d_\Gamma(y)}
\end{aligned}$$

$$= SDD(\Gamma) + k\frac{((k+1)^2+1)}{k+1}SDD(\Gamma)$$
$$= \frac{(k^3+2k^2+3k+1)}{k+1}SDD(\Gamma).$$

□

Theorem 2.6. *If $S^k(\Gamma)$ is a k-splitting graph, then*

$$irr(S^k(\Gamma)) \geq (k^2+3k+1)irr(\Gamma).$$

Proof. The irregularity index of a graph Γ is given by

$$irr(\Gamma) = \sum_{xy\in E(\Gamma)} |d_\Gamma(x) - d_\Gamma(y)|.$$

We use Observation 1.1 to obtain the inequality.

$$\begin{aligned}
irr(S^k(\Gamma)) &= \sum_{xy\in E(S^k(\Gamma))} \left|d_{S^k(\Gamma)}(x) - d_{S^k(\Gamma)}(y)\right| \\
&= \sum_{xy\in E(\Gamma)} |(k+1)d_\Gamma(x) - (k+1)d_\Gamma(y)| \\
&+ k\sum_{xy\in E(\Gamma)} [|(k+1)d_\Gamma(x) - d_\Gamma(y)| + |d_\Gamma(x) + (k+1)d_\Gamma(y)|] \\
&\geq (k+1)\sum_{xy\in E(\Gamma)} |d_\Gamma(x) - d_\Gamma(y)| + k\sum_{xy\in E(\Gamma)} |(k+2)d_\Gamma(x) - (k+2)d_\Gamma(y)| \\
&= (k+1)irr(\Gamma) + k(k+2)\sum_{xy\in E(\Gamma)} |d_\Gamma(x) - d_\Gamma(y)| \\
&= (k+1)irr(\Gamma) + k(k+2)irr(\Gamma) = (k^2+3k+1)irr(\Gamma).
\end{aligned}$$

□

Theorem 2.7. *If $S^k(\Gamma)$ is a k-splitting graph, then*

$$\prod_1(S^k(\Gamma)) = (k+1)^{2n}\left[\prod_1(\Gamma)\right]^{k+1}$$

where n is the order of Γ.

Proof. The first multiplicative Zagreb index for a graph Γ is given by

$$\prod_1(\Gamma) = \prod_{x\in V(\Gamma)} (d_\Gamma(x))^2$$

By using Observation 1.1 we obtain the following expression for $\prod_1(S^k(\Gamma))$.

$$\begin{aligned}
\prod_1(S^k(\Gamma)) &= \prod_{x\in V(S^k(\Gamma))} (d_{S^k(\Gamma)}(x))^2 \\
&= \prod_{v\in V(\Gamma)} ((k+1)d_\Gamma(v))^2 \prod_{v'\in V(S^k(\Gamma))} (d_\Gamma(v))^2 \\
&= (k+1)^{2n} \prod_{v\in V(\Gamma)} (d_\Gamma(v))^2 \left[\prod_{v\in V(\Gamma)} (d_\Gamma(v))^2\right]^k \\
&= (k+1)^{2n} \left[\prod_{v\in V(\Gamma)} (d_\Gamma(v))^2\right]^{k+1} \\
&= (k+1)^{2n} \left[\prod_1(\Gamma)\right]^{k+1}.
\end{aligned}$$

□

Theorem 2.8. *For a k-splitting graph $S^k(\Gamma)$,*

$$\prod_2(S^k(\Gamma)) = (k+1)^{2m(k+1)} \left[\prod_2(\Gamma)\right]^{2k+1}$$

, where m is the size of the graph Γ.

Proof. The second Multiplicative Zagreb index for a graph Γ is given by

$$\prod_2(\Gamma) = \prod_{xy\in E(\Gamma)} (d_\Gamma(x)d_\Gamma(y)).$$

Similarly, as in the above theorem, by using Observation 1.1 we get

$$\prod_2(S^k(\Gamma)) = \prod_{xy\in E(S^k(\Gamma))} (d_{S^k(\Gamma)}(x) \times d_{S^k(\Gamma)}(y))$$

$$= \prod_{xy\in E(\Gamma)} ((k+1)d_\Gamma(x) \times (k+1)d_\Gamma(y))$$

$$\times \left[\prod_{xy\in E(\Gamma)} ((k+1)d_\Gamma(x) \times d_\Gamma(y))\,(d_\Gamma(x) \times (k+1)d_\Gamma(y)) \right]^k$$

$$= (k+1)^{2m} \left[\prod_{xy\in E(\Gamma)} (d_\Gamma(x) \times d_\Gamma(y)) \right] \times \left[\prod_{xy\in E(\Gamma)} ((k+1)^2(d_\Gamma(x))^2 \times (d_\Gamma(y))^2) \right]^k$$

$$= (k+1)^{2m} \left[\prod_{xy\in E(\Gamma)} (d_\Gamma(x) \times d_\Gamma(y)) \right] \times (k+1)^{2mk} \left[\prod_{xy\in E(\Gamma)} d_\Gamma(x) \times d_\Gamma(y) \right]^{2k}$$

$$= (k+1)^{2m(k+1)} \left[\prod_2 (\Gamma) \right]^{2k+1}.$$

□

3. Topological Invariants for k-shadow graph

In this section, we compute the topological invariants of the k-shadow graph.

Theorem 3.1. *For a simple connected graph Γ,*

$$M_1(D_k(\Gamma)) = k^3 M_1(\Gamma).$$

Proof. To prove the result we use the Observation 1.2.

$$\begin{aligned} M_1(D_k(\Gamma)) &= \sum_{x\in V(D_k(\Gamma))} (d_{D_k(\Gamma)}(x))^2 \\ &= k \sum_{v\in V(\Gamma)} ((k)d_\Gamma(v))^2 \\ &= k^3 \sum_{v\in V(\Gamma)} (d_\Gamma(v))^2 \\ &= k^3 M_1(\Gamma). \end{aligned}$$

□

Theorem 3.2. *If $D_k(\Gamma)$ is a k-shadow graph, then*

$$M_2(D_k(\Gamma)) = k^4 M_2(\Gamma).$$

Proof. By using the Observation 1.2 we get

$$\begin{aligned} M_2(D_k(\Gamma)) &= \sum_{xy\in E(D_k(\Gamma))} (d_{D_k(\Gamma)}(x) \times d_{D_k(\Gamma)}(y)) \\ &= k^2 \sum_{xy\in E(\Gamma)} ((k)d_\Gamma(x) \times (k)d_\Gamma(y)) \\ &= k^2 \sum_{xy\in E(\Gamma)} ((k^2)d_\Gamma(x) \times d_\Gamma(y)) \\ &= k^4 \sum_{xy\in E(\Gamma)} (d_\Gamma(x) \times d_\Gamma(y)) \\ &= k^4 M_2(\Gamma). \end{aligned}$$

□

Now we calculate the forgotten topological index of $D_k(\Gamma)$.

Theorem 3.3. *If $D_k(\Gamma)$ is a k-shadow graph, then*

$$F(D_k(\Gamma)) = k^4 F(\Gamma).$$

Proof. Using the Observation 1.2 we find that

$$\begin{aligned} F(D_k(\Gamma)) &= \sum_{xy\in E(D_k(\Gamma))} ((d_{D_k(\Gamma)}(x))^2 + (d_{D_k(\Gamma)}(y))^2) \\ &= k^2 \sum_{xy\in E(\Gamma)} ((k \times d_\Gamma(x))^2 + (k \times d_\Gamma(y))^2) \\ &= k^2 \sum_{xy\in E(\Gamma)} k^2 \times ((d_\Gamma(x))^2 + (d_\Gamma(y))^2) \\ &= k^4 \sum_{xy\in E(\Gamma)} ((d_\Gamma(x))^2 + (d_\Gamma(y))^2) \\ &= k^4 F(\Gamma). \end{aligned}$$

□

Now we compute the hyper Zagrab index of the k-shadow graph.

Theorem 3.4. *If $D_k(\Gamma)$ is a k-shadow graph, then*

$$HM(D_k(\Gamma)) = k^4 HM(\Gamma).$$

Proof. By using Observation 1.2 we obtain the following expression.

$$
\begin{aligned}
HM(D_k(\Gamma)) &= \sum_{xy\in E(D_k(\Gamma))} (d_{D_k(\Gamma)}(x) + d_{D_k(\Gamma)}(y))^2 \\
&= k^2 \sum_{xy\in E(\Gamma)} (k \times d_\Gamma(x) + k \times d_\Gamma(y))^2 \\
&= k^4 \sum_{xy\in E(\Gamma)} (d_\Gamma(x) + d_\Gamma(y))^2 \\
&= k^4 HM(\Gamma).
\end{aligned}
$$

□

The next theorem computes symmetric division degree index of $D_k(\Gamma)$.

Theorem 3.5. *For any k-shadow graph $D_k(\Gamma)$,*

$$SDD(D_k(\Gamma)) = k^2 SDD(\Gamma).$$

Proof. To obtain the result we use the Observation 1.2.

$$
\begin{aligned}
F(D_k(\Gamma)) &= \sum_{xy\in E(D_k(\Gamma))} \frac{((d_{D_k(\Gamma)}(x))^2 + (d_{D_k(\Gamma)}(y))^2)}{(d_{D_k(\Gamma)}(x) \times d_{D_k(\Gamma)}(y))} \\
&= k^2 \sum_{xy\in E(\Gamma)} \frac{((k \times d_\Gamma(x))^2 + (k \times d_\Gamma(y))^2)}{(k \times d_\Gamma(x) \times k \times d_\Gamma(y))} \\
&= k^2 \sum_{xy\in E(\Gamma)} \frac{(k)^2((d_\Gamma(x))^2 + (d_\Gamma(y))^2)}{(k)^2(d_\Gamma(x) \times d_\Gamma(y))} \\
&= k^2 \sum_{xy\in E(\Gamma)} \frac{((d_\Gamma(x))^2 + (d_\Gamma(y))^2)}{(d_\Gamma(x) \times d_\Gamma(y))} \\
&= k^2 SDD(\Gamma).
\end{aligned}
$$

□

Theorem 3.6. *If $D_k(\Gamma)$ is a k-shadow graph, then*

$$irr(D_k(\Gamma)) = k^3 irr(\Gamma).$$

Proof. By using Observation 1.2 we get

$$\begin{aligned} irr(D_k(\Gamma)) &= \sum_{xy\in E(D_k(\Gamma))} \left|d_{D_k(\Gamma)}(x) - d_{D_k(\Gamma)}(y)\right| \\ &= k^2 \sum_{xy\in E(\Gamma)} |k \times d_\Gamma(x) - k \times d_\Gamma(y)| \\ &= k^3 \sum_{xy\in E(\Gamma)} |d_\Gamma(x) - d_\Gamma(y)| \\ &= k^3 irr(\Gamma). \end{aligned}$$

□

Theorem 3.7. *For a simple connected graph* Γ*,*

$$\prod_1(D_k(\Gamma)) = k^{2kn}\left[\prod_1(\Gamma)\right]^k$$

where n *is the order of the graph* Γ*.*

Proof. To obtain this expression we use the Observation 1.2.

$$\begin{aligned} \prod_1(D_k(\Gamma)) &= \prod_{x\in V(D_k(\Gamma))} (d_{D_k(\Gamma)}(x))^2 \\ &= \left[\prod_{x\in V(\Gamma)} ((k)d_\Gamma(x))^2\right]^k \\ &= \left[\prod_{x\in V(\Gamma)} (k)^2(d_\Gamma(x))^2\right]^k \\ &= \left[k^{2n}\prod_{x\in V(\Gamma)} (d_\Gamma(x))^2\right]^k \\ &= k^{2kn}\left[\prod_{x\in V(\Gamma)} (d_\Gamma(x))^2\right]^k \\ &= k^{2kn}\left[\prod_1(\Gamma)\right]^k. \end{aligned}$$

□

Theorem 3.8. *If $D_k(\Gamma)$ is a k-shadow graph, then*

$$\prod_2 (D_k(\Gamma)) = k^{2mk^2}\left[\prod_2(\Gamma)\right]^{k^2}$$

,where m is the size of the graph Γ.

Proof. We use Observation 1.2 to obtain the following expression

$$\begin{aligned}
\prod_2 (D_k(\Gamma)) &= \prod_{xy\in E(D_k(\Gamma))} (d_{D_k(\Gamma)}(x)\times d_{D_k(\Gamma)}(y)) \\
&= \left[\prod_{xy\in E(\Gamma)} ((k)d_\Gamma(x)\times (k)d_\Gamma(y))\right]^{k^2} \\
&= \left[\prod_{xy\in E(\Gamma)} (k)^2(d_\Gamma(x)\times d_\Gamma(y))\right]^{k^2} \\
&= \left[k^{2m}\prod_{xy\in E(\Gamma)} (d_\Gamma(x)\times d_\Gamma(y))\right]^{k^2} \\
&= k^{2mk^2}\left[\prod_{xy\in E(\Gamma)} (d_\Gamma(x)\times d_\Gamma(y))\right]^{k^2} \\
&= k^{2mk^2}\left[\prod_2(\Gamma)\right]^{k^2}.
\end{aligned}$$

□

4. Conclusion

This paper discusses about the general formulas for certain topological invariants such as the first Zagreb index, second Zagreb index, F-index, symmetric division degree index, hyper Zagreb idex, first and second multiplicative Zagreb indices and irregularity idex of generalized splitting graphs and k-shadow graphs and a lower bound for the irregularity of the generalized splitting graph of a graph. It is an open problem to characterize the graphs Γ for which

$$irr(S^k(\Gamma)) = (k^2+3k+1)irr(\Gamma).$$

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