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TOTALLY AND C-TOTALLY REAL SUBMANIFOLDS OF SASAKIAN MANIFOLDS AND SASAKIAN SPACE FORMS

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Abstract. In the present paper, we study totally real submanifolds of Sasakian manifolds. Also, we study totally and C-totally real submanifolds of Sasakian space forms with respect to Levi-Civita connection as well as quarter symmetric metric connection. We have obtained some results in this regard. Among these results, we have made an important deduction that the scalar curvatures of a C-totally real submanifold of a Sasakian space form with respect to both the aforesaid connections are the same.

1. Introduction

Let $(\tilde{M}, g)$ be an n-dimensional Riemannian manifold endowed with an endomorphism ϕ of its tangent bundle $T\tilde{M}$, a vector field ξ and a 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1, \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),\ \eta(X) = g(X,\xi),\ g(X,\phi Y) = -g(\phi X, Y) \tag{1.2}$$

for all vector fields $X, Y \in \Gamma(T\tilde{M})$.

The n-dimensional Riemannian manifold $\tilde{M}$ is said to have a contact Riemannian structure (ϕ, ξ, η, g) if in addition

$$d\eta(X,Y) = g(\phi X, Y). \tag{1.3}$$

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Moreover, if the structure is normal, i.e., if

$$[\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y] = -2d\eta(X, Y)\xi, \tag{1.4}$$

then the contact Riemannian structure is called a Sasakian structure and $\tilde{M}$ is called a *Sasakian manifold*. Thus we have for all vector fields X, Y on $\tilde{M}$,

$$\tilde{\nabla}_X \xi = -\phi X, \ (\tilde{\nabla}_X \phi)Y = -\eta(Y)X + g(X, Y)\xi, \tag{1.5}$$

where $\tilde{\nabla}$ denotes the Riemannian connection on g.

A plane section σ in $T_p\tilde{M}$ of a Sasakian manifold $\tilde{M}$ is called a ϕ-section if it is planned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\tilde{K}(\sigma)$ with respect to a ϕ-section σ is called a ϕ-sectional curvature c, then the Sasakian manifold $\tilde{M}$ is called a *Sasakian space form* and is denoted by $\tilde{M}(c)$.

Let M be a submanifold of dimension m of an n-dimensional Riemannian manifold $\tilde{M}$ $(m < n)$ with induced metric g. Also let ∇ and $\nabla^\perp$ be the induced connection on the tangent bundle TM and the normal bundle $T^\perp M$ of M respectively, then the Gauss and Weingerten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.6}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{1.7}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where h is the second fundamental form and A_V is the shape operator (corresponding to the normal vector field V) for the immersion of M into $\tilde{M}$ and they are related by

$$g(h(X, Y), V) = g(A_V X, Y). \tag{1.8}$$

The equation of Gauss is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)) \tag{1.9}$$

for any vector fields X, Y, Z, W of M.

The normal component of $\tilde{R}(X,Y)Z$ is given by

$$(\tilde{R}(X,Y)Z)^N = (\tilde{\nabla}_X h)(Y,Z) - (\tilde{\nabla}_Y h)(X,Z), \tag{1.10}$$

also for any vector fields X, Y, Z of M we have,

$$(\tilde{\nabla}_X h)(y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{1.11}$$

For any vector fields X, Y and normal vector fields U, V of M we have,

$$\tilde{R}(X,Y,U,V) = g(R^{\perp}(X,Y)U,V) - g([A_U, A_V]X, Y), \tag{1.12}$$

where $R^{\perp}$ is the curvature tensor of the normal induced connection $\nabla^{\perp}$.

Now for a Sasakian space form $\tilde{M}(c)$ the curvature tensor $\tilde{R}$ is given by [1]:

$$\tilde{R}(X,Y)Z = \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] + \frac{c-1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \tag{1.13}$$
$$+g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z]$$

for any vector fields $X, Y, Z \in \tilde{M}(c)$.

A submanifold M of a contact metric manifold $\tilde{M}$ is called *totally real* if each tangent space of M is mapped into the normal space by the contact metric structure ϕ on M. A submanifold M of a contact metric manifold $\tilde{M}$ is called a *C-totally* real submanifold if every tangent vector of M belongs to the contact distribution [13]. Thus a submanifold M in a contact metric manifold is C-totally real if ξ is a normal vector field on M. A direct consequence of this definition is that $\phi(TM) \subset T^{\perp}M$, i.e. a C-totally real submanifold is totally real.

B. Y. Chen and K. Ogiue studied some fundamental properties of totally real manifolds in 1974 [4]. In 1976, K. Yano [15] and in 1977, B. Y. Chen et al. [3] studied properties of totally real submanifolds of a Kaehler manifold. Noriaki Sato discussed properties of totally real submanifolds of a complex space form with nonzero parallel mean curvature vector in 1995 [12]. Zhong Hua Hou studied properties of totally real submanifolds of a nearly Kahler manifold in 2001 [8]. In 2003, Ion Mihai [11] discussed ideal C-totally real submanifolds in Sasakian space forms. Totally real and C-totally real submanifolds of an $(LCS)_n$-manifold were discussed by S. K. Hui and T. Pal in 2017 [9]. Motivated from their work in this paper, we have established some new results on totally real manifolds of a Sasakian manifold and totally and C-totally real submanifolds of a Sasakian space form.

The notion of semi-symmetric linear connection on a smooth manifold was introduced by A. Friedman and J. Schouten [5]. Then H. A. Hayden introduced the idea of metric connection with torsion on a Riemannian manifold [7]. Thereafter K. Yano studied semi-symmetric metric connection on a Riemannian manifold systematically [14]. As a generalisation of semi-symmetric connection, S. Golab [6] introduced the idea of quarter symmetric linear connection on smooth manifolds.

Now from (1.9) and (1.13) we get $\forall X, Y, Z, W \in \Gamma(TM)$,
for a totally real submanifold M of a Sasakian space form $\tilde{M}(c)$,

$$R(X,Y,Z,W) = g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)) \tag{1.14}$$

$$+\tfrac{c+3}{4}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + \tfrac{c-1}{4}[\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W)$$

$$+\eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z)]$$

since $g(\phi X, Y) = 0 \ \forall X, Y \in \Gamma(TM)$,

and for a C-totally real submanifold M of a Sasakian space form $\tilde{M}(c)$,

$$R(X,Y,Z,W) = g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)) \tag{1.15}$$

$$+\tfrac{c+3}{4}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$

since $\eta(X) = 0,\ g(\phi X, Y) = 0 \ \forall X, Y \in \Gamma(TM)$.

We have used the above two equations in the third section of this paper to obtain some results related to totally and C-totally real submanifolds of a Sasakian space form.

2. Totally Real Submanifolds of Sasakian Manifolds

This section deals first with the general study of totally real submanifolds of a Sasakian manifold and next with the study of totally real submanifolds of a Sasakian manifold with parallel mean curvature vector.

Let M be an n-dimensional totally real submanifold of a $2m$-dimensional Kaehler manifold $\tilde{M}$. If there exists a $2r$-dimensional parallel holomorphic subbundle Q of the normal bundle $T^{\perp}M$ (then $\phi Q \subset Q$), then for any section V in Q and vector fields X, Y in M we have,

$$g(h(X,Y),V) = g(A_V X, Y) = g(-\tilde{\nabla}_X V + \nabla^{\perp}_X V, Y) = -g(\tilde{\nabla}_X V, Y) = g(V, \tilde{\nabla}_X Y)$$
$$= g(\phi V, \tilde{\nabla}_X \phi Y) = g(\phi V, \nabla^{\perp}_X \phi Y) = -g(\nabla^{\perp}_X \phi V, \phi Y) = -g(\tilde{\nabla}_X \phi V, \phi Y) - g(A_{\phi V} X, \phi Y)$$
$$= -g(\nabla_X \phi V, \phi Y) - g(h(X, \phi V), \phi Y) - g(h(X, \phi Y), \phi V)$$
$$= 0.$$

Thus we have:

Theorem 2.1. *Let M be an n-dimensional totally real submanifold of a $2m$-dimensional Sasakian manifold $\tilde{M}$. If Q is a $2r$-dimensional parallel holomorphic subbundle of $T^{\perp}M$, then $h|_Q \equiv 0$.*

Let Q be a holomorphic subbundle of $T^{\perp}M$, then the coholomorphic subbundle Q^c (i.e., the complimentary subbundle of Q in $T^{\perp}M$ where $Q \oplus Q^c = T^{\perp}M$) contains $\phi(TM)$ as its subbundle and Q is parallel if and only if Q^c is parallel (i.e., Q^c is invariant under the parallel translation $\nabla^{\perp}$). Hence from Theorem 2.1 we obtain:

Theorem 2.2. *Let M be an n-dimensional totally real submanifold of a $2m$-dimensional Sasakian manifold $\tilde{M}$. If Q be a parallel coholomorphic subbundle of $T^{\perp}M$, then $Im(h) \subset Q$, where $Im(h) = \{h(X,Y) : X, Y \in TM\}$.*

Theorem 2.3. *Let M be an n-dimensional totally real submanifold of a $2m$-dimensional Sasakian manifold $\tilde{M}$, then $Im(h) \subset \phi(TM) \Rightarrow (\phi(TM))^c$ is parallel.*

Proof. Let V be any section of the holomorphic subbundle $(\phi(TM))^c$ and $Im(h) \subset \phi(TM)$, then we have $h|_{(\phi(TM))^c} = 0$ and hence for $X, Y \in TM$,
$0 = g(h(X,Y), \phi V) = g(\tilde{\nabla}_X Y, \phi V) = -g(\phi\tilde{\nabla}_X Y, V) = -g(\tilde{\nabla}_X \phi Y, V) = -g(\nabla^{\perp}_X \phi Y, V)$
$= g(\nabla^{\perp}_X Y, \phi V)$. Therefore, $(\phi(TM))^c$ is parallel. □

Next let us assume that M is an n-dimensional totally real submanifold of a $2n$-dimensional Kaehler manifold $\tilde{M}$, then from (1.6) and (1.7) we obtain,

$$\tilde{\nabla}_X \phi Y = -A_{\phi Y} X + \nabla^{\perp}_X \phi Y$$
$$\Rightarrow (\tilde{\nabla}_X \phi)Y + \phi(\nabla_X Y + h(X,Y)) = -A_{\phi Y} X + \nabla^{\perp}_X \phi Y$$
$$\Rightarrow [(\tilde{\nabla}_X \phi)Y + \phi h(X,Y)] + \phi(\nabla_X Y) = -A_{\phi Y} X + \nabla^{\perp}_X \phi Y$$

$$\Rightarrow \nabla^{\perp}_X \phi Y = \phi \nabla_X Y, \tag{2.1}$$

$$\text{and } -A_{\phi Y} X = (\tilde{\nabla}_X \phi)Y + \phi h(X,Y)$$
$$\Rightarrow -\phi A_{\phi Y} X = \phi(\tilde{\nabla}_X \phi)Y + \phi^2 h(X,Y)$$
$$\Rightarrow -\phi A_{\phi Y} X = \phi(\tilde{\nabla}_X \phi)Y - h(X,Y) + \eta(h(X,Y)\xi)$$

$$\Rightarrow \phi A_{\phi Y} X = h(X,Y). \tag{2.2}$$

$$\nabla_X \phi V = \tilde{\nabla}_X \phi V - h(X, \phi V)$$
$$= (\tilde{\nabla}_X \phi)V + \phi(\tilde{\nabla}_X V) - h(X, \phi V)$$
$$= (\tilde{\nabla}_X \phi)V + \phi(-A_V X + \nabla^{\perp}_X V) - h(X, \phi V)$$
$$= (\tilde{\nabla}_X \phi)V - \phi(A_V X) + \phi(\nabla^{\perp}_X V) - h(X, \phi V)$$
$$= [(\tilde{\nabla}_X \phi)V - \phi(A_V X) - h(X, \phi V)] + \phi(\nabla^{\perp}_X V)$$

$$\Rightarrow \nabla_X \phi V = \phi(\nabla^{\perp}_X V). \tag{2.3}$$

From (2.1) we can conclude:

Theorem 2.4. *Let M be an n-dimensional totally real submanifold of a $2n$-dimensional Sasakian manifold $\tilde{M}$, then the normal bundle $T^{\perp}M$ admits a parallel non-trivial (local) section if and only if the tangent bundle admits a parallel non-trivial (local) section.*

3. Totally and C-totally Real Submanifolds of Sasakian Space Forms

In this section we study totally and C-totally real submanifolds of Sasakian space forms with respect to Levi-Civita connection as well as quarter symmetric metric connection.

A linear connection $\tilde{\nabla}$ in an n-dimensional smooth manifold $\tilde{M}$ is said to be a quarter symmetric connection [6] if its torsion tensor T is of the form:

$$\begin{aligned} T(X,Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y, \end{aligned}$$

where η is a 1-form and ϕ is a tensor of type $(1,1)$. In particular, if $\phi X = X$ then the quarter symmetric connection reduces to the semi-symmetric connection. Further if the quarter symmetric connection $\tilde{\nabla}$ satisfies the condition $(\tilde{\nabla}_X g)(Y,Z) = 0$ for all smooth vector fields X, Y, Z on $\tilde{M}$, then $\tilde{\nabla}$ is said to be a quarter symmetric metric connection.

Let M be an m-dimensional submanifold of an n-dimensional Sasakian space form $\tilde{M}$ and $\{e_i\}_{i=1}^n$ be an orthonormal basis of the tangent space $\tilde{M}$ such that, refracting to M, $\{e_i\}_{i=1}^m$ is the orthonormal basis to the tangent space $T_x M$ with respect to the induced connection.

We write $h_{ij}^r = g(h(e_i, e_j), e_r)$.

Then the length of h, i.e., $\|h\|$ satisfies–

$$\|h\|^2 = \frac{1}{m}\sum_{i,j=1}^m g(h(e_i,e_j), h(e_i,e_j)).$$

The quarter symmetric metric connection $\tilde{\tilde{\nabla}}$ and Riemannian connection $\tilde{\nabla}$ on a Sasakian space form $\tilde{M}$ are related by the equation (3.6) given in [10] as

$$\tilde{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{3.1}$$

Let L be a k-plane section of $T_x M$ and X be a unit vector in L. We choose an orthonormal basis $\{e_i\}_{i=1}^k$ of L such that $e_1 = X$, then the Ricci curvature Ric_L of L

at X is defined by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k},$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. Such a curvature is called a k-Ricci curvature.

The scalar curvature τ of the k-plane section L is given by

$$\tau(L) = \sum_{1\leq i<j\leq k} K_{ij}.$$

For each integer $k,\ 2 \leq k \leq n$, the Riemannian invariant Θ_k on an n-dimensional Riemannian manifold M is defined by

$$\Theta_k(x) = \frac{1}{k-1} inf_{L,X} Ric_L(X),\ x \in M,$$

where L runs over all k-plane sections in T_xM and X runs over all unit vectors in L.

The relative null space for a submanifold M of a Riemannian manifold at a point $x \in M$ is defined by $N_x = \{X \in T_xM : h(X,Y) = 0, Y \in T_xM\}$.

Now we prove the following:

Theorem 3.1. *Let M be a totally real m-dimensional submanifold of an n-dimensional Sasakian space form $(m < n)$, then*

$$2\tau = m^2\|H\|^2 - \|h\|^2 + \frac{(c+3)(m-1)m}{8} - \frac{(c-1)(m-1)}{2}, \tag{3.2}$$

where τ is the scalar curvature of M.

Proof. From (1.14) we have $\forall X, Y, Z, W \in \Gamma(TM)$,

$R(X,Y,Z,W) = g(h(Y,Z), h(X,W)) - g(h(X,Z), h(Y,W))$
$+\frac{c+3}{4}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + \frac{c-1}{4}[\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W)$
$+\eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z)]$.

Applying the above equation for $X = W = e_i,\ Y = Z = e_j$ and taking summation over $1 \leq i < j \leq m$, we get,

$2\tau = m^2\|H\|^2 - \|h\|^2 + \frac{c+3}{4}[(m-1) + (m-2) + ... + 3 + 2 + 1] + \frac{c-1}{4}[-(m-1) - (m-1)]$
$= m^2\|H\|^2 - \|h\|^2 + \frac{(c+3)(m-1)m}{8} - \frac{(c-1)(m-1)}{2}$. ∎

Corollary 3.1. *Let M be a C-totally real m-dimensional submanifold of an n-dimensional Sasakian space form $(m < n)$, then*

$$2\tau = m^2\|H\|^2 - \|h\|^2 + \frac{(c+3)(m-1)m}{8}, \tag{3.3}$$

where τ is the scalar curvature of M.

Proof. Applying (1.15) for $X = W = e_i,\ Y = Z = e_j$ and taking summation over $1 \leq i < j \leq m$, we get,

$$2\tau = m^2\|H\|^2 - \|h\|^2 + \tfrac{(c+3)(m-1)m}{8}.$$

□

Now let M be a submanifold of dimension m $(m < n)$ of an n-dimensional Sasakian space form $\tilde{M}$ with respect to the quarter symmetric metric connection $\bar{\bar{\nabla}}$ and $\bar{\nabla}$ be the induced connection of M associated to the quarter symmetric metric connection. Also let $\bar{h}$ be the second fundamental form of M with respect to $\bar{\nabla}$, then the Gauss formula can be written as

$$\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{h}(X, Y) \tag{3.4}$$

and hence by virtue of (1.6) and (3.1) we get,

$$\bar{\nabla}_X Y + \bar{h}(X, Y) = \nabla_X Y + h(X, Y) + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{3.5}$$

If M is a totally real submanifold of $\tilde{M}$, then for any $X \in TM$, $\phi X \in T^{\perp}M$ and hence $g(\phi X, Y) = 0$ for $X, Y \in TM$. So, equating the normal part from (3.5) we get,

$$\bar{h}(X, Y) = h(X, Y) + \eta(Y)\phi X. \tag{3.6}$$

Further, if M is C-totally real submanifold of $\tilde{M}$, then $\xi \in T^{\perp}M$ and hence $\eta(Y) = 0$ for all $Y \in TM$. So, from (3.6) we get,

$$\bar{h}(X, Y) = h(X, Y). \tag{3.7}$$

Let U be a unit tangent vector at $x \in \tilde{M}$ and $\{e_i\}_{i=1}^{n}$ be an orthonormal basis of the tangent space of $\tilde{M}$ such that $e_1 = U$ refracting to M, $\{e_i\}_{i=1}^{m}$ is the orthonormal basis to the tangent space T_xM with respect to the induced symmetric metric connection, then we have the following:

Theorem 3.2. *Let M be a totally real submanifold of a Sasakian space form $\tilde{M}$ with respect to the quarter symmetric metric connection, then*

$$2\bar{\tau} = m^2\|H\|^2 - \|h\|^2 + \frac{(c+3)m(m-1)}{8} + \frac{(c+3)}{4}[1-\eta^2(U)m], \tag{3.8}$$

where $\bar{\tau}$ is the scalar curvature of M with respect to the induced connection associated to the quarter symmetric metric connection.

Proof. If $\tilde{\bar{R}}$ and $\tilde{R}$ are the curvature tensors of $\tilde{M}$ with respect to the quarter symmetric metric connection $\tilde{\bar{\nabla}}$ and Riemannian connection $\tilde{\nabla}$ respectively, then the equation (1.9) becomes

$$\tilde{\bar{R}}(X,Y,Z,W) = \bar{R}(X,Y,Z,W) + g(\bar{h}(X,Z),\bar{h}(Y,W)) - g(\bar{h}(X,W),\bar{h}(Y,Z)). \tag{3.9}$$

Similarly the equation $\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z$ becomes

$$\tilde{\bar{R}}(X,Y)Z = \tilde{\bar{\nabla}}_X\tilde{\bar{\nabla}}_Y Z - \tilde{\bar{\nabla}}_Y\tilde{\bar{\nabla}}_X Z - \tilde{\bar{\nabla}}_{[X,Y]}Z.$$

Applying (3.1) and then (1.5) on the above equation we obtain,

$$\tilde{\bar{R}}(X,Y)Z = \tilde{R}(X,Y)Z - g(Z,\phi Y)\phi X + g(Z,\phi X)\phi Y + [-g(Y,Z)\eta(X) + g(X,Z)\eta(Y)]\xi + \eta(Z)[-\eta(Y)X + \eta(X)Y].$$

Hence we have,

$$\tilde{\bar{R}}(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) + [-g(\phi Y,Z)g(\phi X,W) + g(\phi X,Z)g(\phi Y,W)] + \eta(W)[-g(Y,Z)\eta(X) + g(X,Z)\eta(Y)] + \eta(Z)[-\eta(Y)g(X,W) + \eta(X)g(Y,W)]. \tag{3.10}$$

Now from (3.9) and (3.10) we get,

$$\bar{R}(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) + [-g(\phi Y,Z)g(\phi X,W) + g(\phi X,Z)g(\phi Y,W)] + \eta(W)[-g(Y,Z)\eta(X) + g(X,Z)\eta(Y)] + \eta(Z)[-\eta(Y)g(X,W) + \eta(X)g(Y,W)] - g(\bar{h}(X,Z),\bar{h}(Y,W)) + g(\bar{h}(X,W),\bar{h}(Y,Z)). \tag{3.11}$$

Using (1.13) and $g(\phi X,Y) = 0$ (since M is totally real) on (3.11) we obtain,

$\bar{R}(X,Y,Z,W) = g(\bar{h}(X,W),\bar{h}(Y,Z)) - g(\bar{h}(X,Z),\bar{h}(Y,W)) + \frac{c+3}{4}[g(Y,Z)g(X,W)$

$- g(X,Z)g(Y,W)] + \frac{c+3}{4}[\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z)$
$- \eta(X)\eta(W)g(Y,Z)].$

Using (3.6) on the above equation we get,

$$\bar{R}(X,Y,Z,W) = g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \tag{3.12}$$

$+\eta(Z)g(h(X,W),\phi Y) + \eta(W)g(\phi X, h(Y,Z)) - \eta(Z)g(\phi X, h(Y,W))$
$-\eta(W)g(h(X,Z),\phi Y) + \frac{c+3}{4}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + \frac{c+3}{4}[\eta(X)\eta(Z)g(Y,W)$
$-\eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z)].$

Applying the above equation for $X = W = e_i,\ Y = Z = e_j$ and taking summation over $1 \le i < j \le m$ we obtain,
$2\bar{\tau} = m^2\|H\|^2 - \|h\|^2 + \frac{c+3}{4}.\frac{m(m-1)}{2} + \frac{c+3}{4}[-\{-1+\eta^2(U)\}m - (m-1)]$
$\Rightarrow 2\bar{\tau} = m^2\|H\|^2 - \|h\|^2 + \frac{(c+3)m(m-1)}{8} + \frac{(c+3)}{4}[1 - \eta^2(U)m].$ □

Corollary 3.2. *Let M be a C-totally real submanifold of a Sasakian space form $\tilde{M}$ with respect to the quarter symmetric metric connection, then*

$$2\bar{\tau} = m^2\|H\|^2 - \|h\|^2 + \frac{(c+3)m(m-1)}{8}, \tag{3.13}$$

where $\bar{\tau}$ is the scalar curvature of M with respect to the induced connection associated to the quarter symmetric metric connection.

Proof. If M a is C-totally real submanifold of $\tilde{M}$, then $\eta(X) = 0\ \forall X \in TM$ and hence (3.12) implies
$\bar{R}(X,Y,Z,W) = g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) + \frac{c+3}{4}[g(Y,Z)g(X,W)$
$- g(X,Z)g(Y,W)].$

Applying the above equation for $X = W = e_i,\ Y = Z = e_j$ and taking summation over $1 \le i < j \le m$, we obtain the equation (3.13). □

From Corollary 3.1 and Corollary 3.2 we can state:

Theorem 3.3. *Let M be a C-totally real submanifold of a Sasakian space form $\tilde{M}$, then the scalar curvatures of M with respect to the induced Levi-Civita connection* *and the induced quarter symmetric metric connection are same.*

Next we prove the following:

Theorem 3.4. *Let M be an m-dimensional totally real submanifold of an n-dimensional Sasakian space form $\tilde{M}$, $(m < n)$. We have–*
(i) For each unit vector $X \in T_xM$,

$$2Ric(X) \leq \frac{m^2}{2}\|H\|^2 - \frac{c+3}{8}(3m^2-15m+16) - \frac{c-1}{2}[-m+3+2(m-2)\eta^2(X)]; \tag{3.14}$$

(ii) In case of $H(x) = 0$, a unit tangent vector X at x satisfies the equality case of (3.14) *if and only if X lies in the relative null space N_x at x;*
(iii) The equality case of (3.14) *holds identically for all unit tangent vectors at x if and only if either x is a totally geodesic point or $m = 2$ and x is a totally umbilical point.*

Proof. (i) Let $X \in T_xM$ be a unit tangent vector at x. We choose an orthonormal basis $\{e_1, e_2, ..., e_{m-1}, e_m, e_{m+1}, ..., e_n\}$ such that e_i's are tangent to M at x for $i = 1, ..., m$ and $e_1 = X$, then from (3.2) we have,
$m^2\|H\|^2 = 2\tau + \sum_{r=m+1}^{n}\{(h^r_{11})^2 + (h^r_{22} + ... + h^r_{mm})^2\} + 2\sum_{r=m+1}^{n}\sum_{i<j}(h^r_{ij})^2$
$-2\sum_{r=m+1}^{n}\sum_{2\leq i<j\leq m} h^r_{ii}h^r_{jj} - \frac{(c+3)m(m-1)}{8} + \frac{(c-1)(m-1)}{2}$

$$\Rightarrow m^2\|H\|^2 = 2\tau + \frac{1}{2}\sum_{r=m+1}^{n}\{(h^r_{11} + h^r_{22} + ... + h^r_{mm})^2 + (h^r_{11} - h^r_{22} - ... - h^r_{mm})^2\} \tag{3.15}$$

$+2\sum_{r=m+1}^{n}\sum_{i<j}(h^r_{ij})^2 - 2\sum_{r=m+1}^{n}\sum_{2\leq i<j\leq m} h^r_{ii}h^r_{jj} - \frac{(c+3)m(m-1)}{8} + \frac{(c-1)(m-1)}{2}$.

From (1.14) we find,

$$K_{ij} = \sum_{r=m+1}^{n}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2] + \frac{c+3}{4} - \frac{c-1}{4}[\eta^2(e_i) + \eta^2(e_j)],$$

and consequently,

$$\sum_{2\leq i<j\leq m} K_{ij} = \sum_{r=m+1}^{n}\sum_{2\leq i<j\leq m}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2] + \frac{(c+3)}{4}(m-2)^2 \tag{3.16}$$

$-\frac{c-1}{2}(m-2)[1-\eta^2(X)] + \frac{c-1}{4}(m-2)[\eta^2(e_2) + \eta^2(e_m)]$.

Using (3.16) in (3.15) we obtain,

$m^2\|H\|^2 \geq 2\tau + \frac{m^2}{2}\|H\|^2 + 2\sum_{r=m+1}^{n}\sum_{j=2}^{m}(h_{1j}^r)^2 - 2\sum_{2\leq i<j\leq m} K_{ij} + \frac{c+3}{2}(m-2)^2$
$-(c-1)(m-2)[1-\eta^2(X)] - \frac{(c+3)m(m-1)}{8} + \frac{(c-1)(m-1)}{2}$
$\Rightarrow \frac{m^2}{2}\|H\|^2 \geq 2Ric(X) + \frac{c+3}{8}(3m^2-15m+16) + \frac{c-1}{2}[-m+3+2(m-2)\eta^2(X)]$,
from which we get (3.14).

(ii) Let $H(X)=0$, then the equality holds in (3.14) if and only if
$h_{11}^r = h_{12}^r = ... = h_{1m}^r = 0$ and $h_{11}^r = h_{22}^r + ... + h_{mm}^r,\ r \in \{m+1, m+2, ..., n-1, n\}$,
then $h_{ij}^r = 0\ \forall j \in \{1,2,...,m\}, r \in \{m+1,...,n\}$, i.e., $X \in N_x$.

(iii) Equality in (3.14) holds for every tangent vector at x if and only if
$h_{ij}^r = 0,\ i \neq j$ and $h_{11}^r + h_{22}^r + ... + h_{mm}^r - 2h_{ii}^r = 0$.
We distinguish two cases:
a) $m \neq 2 \Rightarrow x$ is a totally geodesic point;
b) $m = 2 \Rightarrow x$ is a totally umbilical point and the converse is trivial. □

Next we obtain:

Theorem 3.5. *Let M be a totally real submanifold of dimension m of an n-dimensional Sasakian space form $\tilde{M}$ $(m<n)$, then we have,*

$$\|H\|^2 \geq \frac{2\tau}{m(m-1)} - \frac{1}{8}(c+3) + \frac{1}{2m}(c-1). \tag{3.17}$$

Proof. We choose an orthonormal basis $\{e_1, ..., e_m, e_{m+1}, ...e_n\}$ at x such that e_{m+1} is parallel to the mean curvature vector $H(x)$ and $e_1, ..., e_m$ diagonalise the shape operator $A_{e_{m+1}}$, then the shape operator takes the form

$$A_{e_{m+1}} = \begin{pmatrix} a_1 & 0 & 0 & ... & 0 \\ 0 & a_2 & 0 & ... & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & ... & a_n \end{pmatrix}, \tag{3.18}$$

where $A_{e_r} = (h^r_{ij}),\ i,j = 1,...,m;\ r = m+2,...,n,\ traceA_{e_r} = \sum_{i=1}^m h^r_{ii} = 0.$

Now from (3.2) we get,

$$(3.19)\quad m^2\|H\|^2 = 2\tau + \sum_{i=1}^m a_i^2 + \sum_{r=m+2}^n \sum_{i,j=1}^m (h^r_{ij})^2 - \frac{(c+3)}{8}m(m-1) + \frac{c-1}{2}(m-1).$$

Also we have,

$$0 \le \sum_{i<j}(a_i - a_j)^2 = (m-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j,$$

from which we get,

$$m^2\|H\|^2 = (\sum_{i=1}^m a_i)^2 = \sum_i a_i^2 + 2\sum_{i<j} a_i a_j \le m\sum_{i=1}^m a_i^2$$

$$(3.20)\qquad \Rightarrow \sum_{i=1}^m a_i^2 \ge m\|H\|^2.$$

Applying (3.20) on (3.19) we obtain,

$$m^2\|H\|^2 \ge 2\tau + m\|H\|^2 - \tfrac{(c+3)}{8}m(m-1) + \tfrac{c-1}{2}(m-1)$$

which implies (3.17). □

Theorem 3.6. *Let M be a totally real submanifold of dimension m of a Sasakian space form $\tilde{M}$ $(m < n)$, then for any integer k, $2 \le k \le m$, and any point $x \in M$, we have,*

$$(3.21)\qquad \|H\|^2(x) \ge \Theta_k(x) - \frac{1}{8}(c+3) + \frac{1}{2m}(c-1).$$

Proof. Let $\{e_i\}_{i=1}^m$ be an orthonormal basis of T_xM. We denote the k-plane section spanned by $\{e_{i_r}\}_{r=1}^k$, by $L_{i_1,...,i_k}$, then from the relation (3.5) given in [2] we have,

$$(3.22)\qquad \tau(x) \ge \frac{m(m-1)}{2}\Theta_k(x).$$

Using (3.22) in (3.17) we get (3.21). □

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