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EXISTENCE RESULTS FOR A CLASS OF FIRST-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENTS AND NONLOCAL INITIAL CONDITIONS

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Abstract. This work is concerned with the construction of solutions for a class of first order fractional differential equations with advanced arguments and with nonlocal initial conditions. We also give some examples to illustrate our results.

1. Introduction

The purpose of this work is to study the existence of solutions for a class of first order fractional differential equations with advanced arguments subject to integral initial conditions. More specifically, we consider the nonlinear initial value problem

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}u(t) = f(t, u\left(t\right), u\left(\theta\left(t\right)\right)),\ t \in J = [0, T], \\ u\left(0\right) = \int_0^T g(s)u(s)ds, \end{cases} \tag{1.1}$$

where ${}^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order α with $0 < \alpha \leq 1$, $T > 0$, $\theta : J \to J$ is continuous with $\theta\left(t\right) \leq t$, for all $t \in J$, $f : J \times \mathbb{R}^2 \to \mathbb{R}$ and $g : J \to \mathbb{R}_+$ are continuous functions.

Fractional differential equations arise in many scientific fields such as viscoelasticity, electrical circuits, electroanalytical chemistry, biology, control theory, electromagnetic theory, biomedical problems and so on (see [32], [27], [25], [20] and the references cited in [24]). Indeed, fractional differential equations are more realistic than ordinary

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differential equations when studying some models of viscoelasticity and plasticity (see [5], [32, Chapter 10 Section 10.2] and [34]). On the other hand differential equations with advanced arguments arises in the problem of analyzing the dynamics of an overhead current collection system for an electric locomotive (see [31]), number theory, electrodynamics, quantum mechanics and engineering applications (see [7], [13], [26], [36] and the references therein).

Differential equations with advanced arguments in the complex plane were first studied by P. Flamant [12]. More precisely in [12] P. Flamant consider the problem

$$\begin{cases} u'(z) = a(z)u(\frac{z}{\sigma}) + b(z), z \in D, \\ u(0) = c_0, \end{cases} \tag{1.2}$$

where $\sigma \in \mathbb{C}$ such that $|\sigma| \geq 1$ and $a(z)$, $b(z)$ are analytic in a closed and simply connected region D and $c_0 \in \mathbb{C}$.

By using the method of successive approximation the author proved that the problem (1.2) admits a unique solution analytic in D. He indicated that the preceding method can be applied to the problem

$$\begin{cases} u'(z) = \sum\limits_{j=0}^{n} a_i(z)u(\frac{z}{\sigma_i}) + b(z), z \in D, \\ u(0) = c_0, \end{cases} \tag{1.3}$$

where $\sigma_i \in \mathbb{C}$ such that $|\sigma_i| \geq 1$ and a_i are analytic in D, for all $i = 1, ..., n$.

On the other hand fractional differential equations with deviating arguments have been studied by several authors using the Banach contraction principle, the upper and lower solutions method, the upper and lower solutions method coupled with monotone iterative technique, fixed point theorems in cones and numerical methods(see [2], [3], [4], [10], [15], [16], [21], [22], [23], [30], [35], [37], [38], [39] and the references therein). Let us recall some of them.

In [4], the authors studied the existence of solutions for the problem

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) = f(t, u\left(t\right), u\left(\lambda t\right)),\ t \in J = [0, T], \\ u\left(0\right) = g(u), \end{cases} \tag{1.4}$$

where $f : [0, T] \times X \times X \to X$, $g : C(J; X) \to X$ are continuous with X a Banach space.

By using the Banach contraction principle, the authors proved the existence and uniqueness of solutions for problem (1.4).

In [10], by using the upper and lower solution method coupled with monotone iterative technique, the authors studied the existence and uniqueness of solutions for the problem

$$\begin{cases} D^q u(t) = f(t, u(t), u(\theta(t))), \ t \in J = [0, T], \\ u(0) = \lambda \int_0^1 u(s)ds + d, \end{cases}$$

where $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\theta : J \to J$ is continuous with $\theta(t) \leq t$, for all $t \in J$ and $D^q u(t)$ is the q-th Riemann-Liouville fractional derivative of u with respect to t, which q is such that $0 < q < 1$, $\lambda \geq 0$ and d is a real umber.

In [16], the author studied the problem

$$\begin{cases} D^q u(t) = f(t, u(t), u(\theta(t))), \ t \in J = [0, T], \\ t^{1-q} u(t)\big|_{t=0} = u_0, \end{cases} \tag{1.5}$$

where $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\theta : J \to J$ is continuous with $\theta(t) \leq t$, for all $t \in J$ and $D^q u(t)$ is the q-th Riemann-Liouville fractional derivative of u with respect to t, which q is such that $0 < q < 1$.

By using the method of upper and lower solutions method coupled with monotone iterative technique, the author obtained the existence of extremal solutions for the problem (1.5) and he gave sufficient conditions under which problem (1.5) has a unique solution.

In [23], the authors studied the problem

$$\begin{cases} D_t^q u(t) = f(t, u(t), u(\beta(t))), \ t \in (0, T], \ T > 0, \\ g(\widetilde{u}(0), \widetilde{u}(T)) = 0, \end{cases} \tag{1.6}$$

where $f \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\beta(t) \in C^{\alpha}([0,T], [0,T])$, $\widetilde{u}(0) = t^{1-q} u(t)\big|_{t=0}$, $\widetilde{u}(T) = t^{1-q} u(t)\big|_{t=T}$ and $D_t^q u(t)$ is the q-th Riemann-Liouville fractional derivative of u with respect to t, which q is such that $0 < q \leq 1$.

By using the method of upper and lower solutions coupled with monotone iterative technique, the authors studied the existence and the multiplicity of the solutions for the problem (1.6).

In [30], the authors studied the problem

$$\left\{\begin{array}{l} D_0^q u(t) + a\left(t\right) f(t, u\left(\theta\left(t\right)\right)),\ 0 < t < 1,\, n-1 < q \leq n \\ u^{(i)}\left(0\right) = 0,\ i = 0, 1, 2, ..., n-2, \\ \left[D_0^\beta u(t)\right]_{t=1} = 0,\ 1 \leq \beta \leq n-2, \end{array}\right. \tag{1.7}$$

where $n > 0$ $(n \in \mathbb{N})$, D_0^q is the standard Riemann-Liouville fractional derivative of order q, $f : [0, \infty) \to [0, \infty)$, $a : [0, 1] \to (0, \infty)$ and $\theta : (0, 1) \to (0, 1]$ are continuous functions.

By using he Banach contraction principle and the Guo-Krasnoselskii fixed point theorem, the authors proved the existence and uniqueness results of positive solutions to problem (1.7).

In [39], by applying fixed point index theory and Leggett-Williams fixed point theorem, the authors established sufficient conditions for the existence of multiple positive solutions for the boundary value problem (1.7).

It is well know that the method of upper and lower solutions coupled with monotone iterative technique has been used to prove existence of solutions for first order differential equations with advanced arguments by various authors (see [3], [10], [15], [16], [22], [23] and [38]). The purpose of this work is to show that it can be applied successfully to problems of type (1.1). We note also that to the best of our knowledge this is the first paper which gives a correct proof to the comparison result (see Lemma 2.4) for first order fractional differential equations with advanced arguments by using the mean value theorem for Caputo's fractional derivative.

The plan of this paper is organized as follows. In Section 2, we give some definitions and preliminary results. The main result is presented and proved in Section 3, followed by some examples in Section 4 illustrating the application of our result and finally in Section 5, we give a conclusion.

2. Preliminary

In this section we give some definitions and preliminary results that will be used in the remainder of this paper.

Definition 2.1. For $0 < q < 1$ and $h \in L^p(J,\mathbb{R})$ with $p \geq 1$. The Riemann-Liouville integral of order q of h is defined by

$$I_{0^+}^q h(t) := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\, ds,$$

where Γ is the Gamma Euler function defined by

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt,$$

where $x \in \mathbb{R}$ with $x > 0$.

Remark 1. *If $q = 0$, we put by definition $I_{0^+}^0 h(t) = h(t)$.*

Definition 2.2. For $0 < q \leq 1$, the Caputo fractional derivative of order q of a function h is defined by

$${}^C D_{0^+}^q h(t) = {}^{RL} D_{0^+}^q (h(t) - h(0)),$$

where ${}^{RL}D_0^q$ is the Riemann-Liouville fractional derivative defined by

$$\begin{aligned} {}^{RL}D_{0^+}^q h(t) &= \frac{d}{dt} I^{1-q} h(t) \\ &= \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} h(s)\, ds. \end{aligned}$$

Notation . *For $h \in C(J,\mathbb{R})$, we note $\|h\|_0$ the usual norm defined by*

$$||h||_0 = \max_{s\in[0,T]} |h(s)|.$$

We have the following result.

Lemma 2.1 (See [18, Lemma 1])**.** *For $0 < q \leq 1$, the Riemann-Liouville fractional integration operator $I_{0^+}^q$ is bounded from $C(J,\mathbb{R})$ to $C(J,\mathbb{R})$ and*

$$\left\| I_{0^+}^q h \right\|_0 \leq \frac{T^q}{\Gamma(q+1)} \|h\|_0.$$

Definition 2.3. The Mittag–Leffler function with two parameters E_{q_1,q_2} is defined by

$$E_{q_1,q_2}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(q_1 n + q_2)},\ q_1 > 0,\ q_2 > 0 \text{ and } x \in \mathbb{R}.$$

Remark 2. *In* [29] *the authors proved that the Mittag–Leffler function with two parameters* $E_{\alpha,\beta}(-x)$ *with* $x \geq 0$ *is completely monotonic, that is*

$$(-1)^n \frac{d^n}{dx^n} E_{q_1,q_2}(-x) \geq 0 \text{ for all } n \in \mathbb{N},$$

if and and only if $0 < q_1 \leq 1$ *and* $q_2 \geq q_1$.

Now, we consider the problem

$$\begin{cases} {}^C D_{0^+}^q u(t) = \widetilde{g}(t, u(t), u(\theta(t))), \ t \in J, \\ u(0) = \widetilde{a}, \end{cases} \tag{2.1}$$

where $0 < q \leq 1$, $\widetilde{g} : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\widetilde{a} \in \mathbb{R}$.
We have the following results.

Lemma 2.2 (See [20, Corollary 3.24])**.** *A function* $u \in C(J, \mathbb{R})$ *is a solution of the Cauchy problem (2.1) if, and only if,* u *satisfies the Volterra integral equation*

$$u(t) = \widetilde{a} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \widetilde{g}(s, u(s), u(\theta(s))) ds, \text{ for all } t \in J,$$

where $0 < q \leq 1$.

Lemma 2.3 (See [11, Corollary 2.4])**.** *For* $0 < q \leq 1$ *and* $a < b$ *and assume* $u \in C([a, b], \mathbb{R})$ *with* ${}^C D_{0^+}^q u \in C([a, b], \mathbb{R})$. *Then there exists some* c *in* (a, b) *such that*

$$\frac{u(b) - u(a)}{(b-a)^q} = \frac{{}^C D_{0^+}^q u(c)}{\Gamma(q+1)}.$$

Notation . *For* $0 < \beta \leq 1$, *we use the notation*

$$C^{\beta,0}(J, \mathbb{R}) = \left\{ u \in C(J, \mathbb{R}) : \ {}^C D_{0^+}^\beta u \in C(J, \mathbb{R}) \right\}.$$

Theorem 2.1. *Assume that the function* f *satisfies the hypothesis*

(H) *There exists a positive constants* L_1 *and* L_2 *such that*

$$|\widetilde{g}(t, u_1, v_1) - \widetilde{g}(t, u_2, v_2)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2|,$$

for all $t \in J$, $u_i \in \mathbb{R}$ *and* $v_i \in \mathbb{R}$ *for* $i = 1, 2$.

Then the problem (2.1) admits a unique solution $u \in C^{q,0}(J, \mathbb{R})$.

Proof. From Lemma 2.2 $u \in C^{q,0}(J,\mathbb{R})$ is a solution of the Cauchy problem (2.1) if, and only if, u satisfies the Volterra integral equation

$$u(t) = \widetilde{a} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \widetilde{g}(s, u(s), u(\theta(s)))ds, \text{ for all } t \in J.$$

We define the operator

$$\begin{aligned} A: \ & C(J,\mathbb{R}) \to C(J,\mathbb{R}) \\ & u \mapsto (Au)(t) = \widetilde{a} + \tfrac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \widetilde{g}(s, u(s), u(\theta(s)))ds. \end{aligned}$$

To prove that the operator A admits a unique fixed point, we use the norm

$$\|x\|_* = \max_{t\in J} e^{-\lambda t} |x(t)|,$$

where $\lambda > 0$ and $x \in C(J,\mathbb{R})$.

Since the norms $\|.\|_*$ and $\|.\|_0$ are equivalent, then $(C(J,\mathbb{R}), \|.\|_*)$ is a Banach space.

Now we are going to prove that A is a contraction on $(C(J,\mathbb{R}), \|.\|_*)$.

Let u_1, $u_2 \in C(J,\mathbb{R})$, then for all $t \in J$, we have

$$\begin{aligned} & e^{-\lambda t} |(Au_1)(t) - (Au_2)(t)| \\ = \ & \frac{e^{-\lambda t}}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} \left(\widetilde{g}(s, u_1(s), u_1(\theta(s))) - \widetilde{g}(s, u_2(s), u_2(\theta(s)))\right) ds \right| \\ \leq \ & \frac{e^{-\lambda t}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left|\widetilde{g}(s, u_1(s), u_1(\theta(s))) - \widetilde{g}(s, u_2(s), u_2(\theta(s)))\right| ds \\ \leq \ & \frac{e^{-\lambda t}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left(L_1 |u_1(s) - u_2(s)| + L_2 |u_1(\theta(s)) - u_2(\theta(s))|\right) ds \\ \leq \ & \frac{e^{-\lambda t}}{\Gamma(q)} (L_1 + L_2) \|u_1 - u_2\|_* \int_0^t e^{\lambda s} (t-s)^{q-1} ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{(L_1+L_2)}{\Gamma(q)} \|u_1-u_2\|_* \int_0^t e^{-\lambda(t-s)} (t-s)^{q-1} \, ds \\
&= \frac{(L_1+L_2)}{\Gamma(q)} \|u_1-u_2\|_* \int_0^t e^{-\lambda\tau} \tau^{q-1} d\tau \\
&= \frac{(L_1+L_2)}{\lambda^q \Gamma(q)} \|u_1-u_2\|_* \int_0^{\lambda t} e^{-\eta} \eta^{q-1} d\eta \\
&\leq \frac{(L_1+L_2)}{\lambda^q \Gamma(q)} \|u_1-u_2\|_* \int_0^{+\infty} e^{-\eta} \eta^{q-1} d\eta \\
&= \frac{(L_1+L_2)}{\lambda^q} \|u_1-u_2\|_* ,
\end{aligned}
$$

which implies that

$$\|Au_1 - Au_2\|_* \leq \frac{(L_1+L_2)}{\lambda^q} \|u_1-u_2\|_* .$$

Then if we choose $\lambda \geq (1+L_1+L_2)^{\frac{1}{q}}$, we obtain

$$\|Au_1 - Au_2\|_* < \|u_1-u_2\|_* ,$$

which means that A is a contraction on $(C(J,\mathbb{R}), \|.\|_*)$.

Therefore by the Banach fixed theorem, the operator A admits a unique fixed point and consequently from Lemma 2.2, it follows that the problem (2.1) admits a unique solution $u \in C^{q,0}(J,\mathbb{R})$. □

Remark 3. *The idea of the proof is similar to that of* [6, Theorem 5.1].

Remark 4. *Theorem 2.1 improve and generalize* [4, Theorem 3.1].

Now, we state and prove the following comparison result.

Lemma 2.4. *Assume that $\theta \in C(J,J)$ with $\theta(t) \leq t$ on J and $u \in C^{q,0}(J,\mathbb{R})$ satisfies the inequalities*

$$
\begin{cases}
{}^{C}D_{0^+}^{\alpha} u(t) \leq -M_1 u(t) - N_1 u(\theta(t)), \ t \in J, \\
u(0) \leq 0,
\end{cases}
$$

where $0 < q \leq 1$ and $M_1 \geq 0$ and $N_1 \geq 0$.

If

$$(M_1 + N_1) \frac{T^\alpha}{\Gamma(1+\alpha)} \leq 1,$$

then $u(t) \leq 0$, *for all* $t \in J$.

Proof. We put by definition

$$u_\varepsilon(t) = u(t) - \varepsilon(1 + t^\alpha),$$

where $\varepsilon > 0$ and $t \in J$.

For all $t \in J$, we have

$$\begin{aligned} {}^C D^\alpha_{0^+} u_\varepsilon(t) &= {}^C D^\alpha_0 u(t) - \varepsilon {}^C D^\alpha_0 (1 + t^\alpha) \\ &= {}^C D^\alpha_0 u(t) - \varepsilon\Gamma(1+\alpha) \\ &\leq -M_1 u(t) - N_1 u(\theta(t)) - \varepsilon\Gamma(1+\alpha) \\ &< -M_1 u_\varepsilon(t) - N_1 u_\varepsilon(\theta(t)). \end{aligned}$$

That is

$$ {}^C D^\alpha_{0^+} u_\varepsilon(t) < -M_1 u_\varepsilon(t) - N_1 u_\varepsilon(\theta(t)), \text{ for all } t \in J. \tag{2.2}$$

On the other hand since $u(0) \leq 0$, we have

$$u_\varepsilon(0) = u(0) - \varepsilon < 0.$$

Now, we are going to prove that

$$u_\varepsilon(t) < 0, \text{ for all } t \in J.$$

Assume that there exists $t_* \in (0, T]$ such that

$$u_\varepsilon(t) < 0, \text{ for all } t \in [0, t_*) \text{ and } u_\varepsilon(t_*) = 0. \tag{2.3}$$

We put by definition

$$u_\varepsilon(\eta) = \min_{t \in [0, t_*]} u_\varepsilon(t) < 0.$$

By Lemma 2.3, there exists $\sigma \in (\eta, t_*)$ such that

$$u_\varepsilon(t_*) - u_\varepsilon(\eta) = {}^C D^\alpha_{0^+} u_\varepsilon(\sigma) \frac{(t_* - \eta)^\alpha}{\Gamma(1+\alpha)}.$$

Then by using (2.2) and (2.3), we obtain

$$-u_{\varepsilon}(\eta) < -\left(M_1 u_{\varepsilon}(\sigma) + N_1 u_{\varepsilon}(\theta(\sigma))\right) \frac{(t_* - \eta)^{\alpha}}{\Gamma(1+\alpha)},$$

which implies that

$$\begin{aligned} -u_{\varepsilon}(\eta) &< -\left(M_1 + N_1\right) u_{\varepsilon}(\eta) \frac{(t_* - \eta)^{\alpha}}{\Gamma(1+\alpha)} \\ &< -\left(M_1 + N_1\right) u_{\varepsilon}(\eta) \frac{T^{\alpha}}{\Gamma(1+\alpha)}. \end{aligned}$$

That is

$$\left(M_1 + N_1\right) \frac{T^{\alpha}}{\Gamma(1+\alpha)} > 1,$$

which is a contradiction with the assumption

$$\left(M_1 + N_1\right) \frac{T^{\alpha}}{\Gamma(1+\alpha)} \leq 1.$$

Then, we have

$$u_{\varepsilon}(t) < 0, \text{ for all } t \in J.$$

That is

$$u_{\varepsilon}(t) < \varepsilon\left(1 + t^{\alpha}\right), \text{ for all } t \in J.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$u(t) \leq 0, \text{ for all } t \in J.$$

□

3. Main result

In this section we give some definitions, we state and prove our result.

Definition 3.1. We say that $\underline{u} \in C^{\alpha,0}(J, \mathbb{R})$ is a lower solution of (1.1) if

$$\begin{cases} {}^{C}D_{0+}^{\alpha} \underline{u}(t) \leq f\left(t, \underline{u}(t), \underline{u}(\theta(t))\right), \ t \in J, \\ \underline{u}(0) \leq \int\limits_0^T g(s)\, \underline{u}(s)\, ds. \end{cases}$$

Definition 3.2. We say that $\overline{u} \in C^{\alpha,0}(J, \mathbb{R})$ is an upper solution of (1.1) if

$$\begin{cases} {}^{C}D_{0+}^{\alpha} \overline{u}(t) \geq f\left(t, \overline{u}(t), \overline{u}(\theta(t))\right), \ t \in J, \\ \overline{u}(0) \geq \int\limits_0^T g(s)\, \overline{u}(s)\, ds. \end{cases}$$

Definition 3.3. We say that u is a solution of (1.1) if $u \in C^{\alpha,0}(J,\mathbb{R})$ and satisfies (1.1).

We have the following result.

Theorem 3.1. *Assume that $\theta \in C(J,J)$ with $\theta(t) \leq t$ on J and $\underline{u}$ and $\overline{u}$ be lower and upper solutions respectively for problem (1.1) such that $\underline{u} \leq \overline{u}$ in J.*
If there exists two constants $M \geq 0$, $N \geq 0$ satisfying

(H1) $f(t,x_1,y_1) - f(t,x_2,y_2) \geq -M(x_1 - x_2) - N(y_1 - y_2)$, *for all* $t \in J$, $\underline{u}(t) \leq x_2 \leq x_1 \leq \overline{u}(t)$ *and* $\underline{u}(\theta(t)) \leq y_2 \leq y_1 \leq \overline{u}(\theta(t))$.

(H2) $(M+N)\dfrac{T^{\alpha}}{\Gamma(1+\alpha)} \leq 1$.

Then the problem (1.1) has a minimal solution u_* and a maximal solution u^* such that for every solution u of (1.1) with $\underline{u} \leq u \leq \overline{u}$ in J, we have

$$\underline{u} \leq u_* \leq u \leq u^* \leq \overline{u} \text{ in } J.$$

Proof. We take $\underline{u}_0 = \underline{u}$, and we define the sequences $(\underline{u}_n)_{n\geq 1}$ by

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}\underline{u}_{n+1}(t) + M\underline{u}_{n+1}(t) + N\underline{u}_{n+1}(\theta(t)) = f_n(t), \; t \in J, \\ \underline{u}_{n+1}(0) = \int_0^T g(s)\underline{u}_n(s)ds, \end{cases} \tag{3.1}$$

where

$$f_n(t) = f(t, \underline{u}_n(t), \underline{u}_n(\theta(t))) + M\underline{u}_n(t) + N\underline{u}_n(\theta(t)).$$

Analogously, we take $\overline{u}_0 = \overline{u}$ and we define the sequences $(\overline{u}_n)_{n\geq 1}$ by

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}\overline{u}_{n+1}(t) + M\overline{u}_{n+1}(t) + N\overline{u}_{n+1}(\theta(t)) = \widetilde{f}_n(t), \; t \in J, \\ \overline{u}_{n+1}(0) = \int_0^T g(s)\overline{u}_n(s)ds, \end{cases} \tag{3.2}$$

where

$$\widetilde{f}_n(t) = f(t, \overline{u}_n(t), \overline{u}_n(\theta(t))) + M\overline{u}_n(t) + N\overline{u}_n(\theta(t)).$$

Step 1: For all $n \in \mathbb{N}$, we have

$$\underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u}_{n+1} \leq \overline{u}_n \text{ in } J.$$

Let

$$w_0(t) := \underline{u}_1(t) - \underline{u}_0(t), \; t \in J.$$

By (3.1) and using the definition of lower solution, we have

$$\begin{cases} {}^{C}D_{0+}^{\alpha} w_0\left(t\right) + M w_0\left(t\right) + N w_0\left(\theta(t)\right) \geq 0,\, t \in J, \\ w_0\left(0\right) \geq 0. \end{cases}$$

Then by Lemma 2.4, we obtain

$$w_0\left(t\right) \geq 0 \text{ for all } t \in J.$$

That is

$$\underline{u}_0 \leq \underline{u}_1 \text{ in } J. \tag{3.3}$$

Similarly, we can prove that

$$\overline{u}_1 \leq \overline{u}_0 \text{ in } J. \tag{3.4}$$

Now, we put by definition

$$p_1\left(t\right) = \underline{u}_1\left(t\right) - \overline{u}_1\left(t\right),\, t \in J.$$

By (3.1) and (3.2), we have

$$\begin{cases} {}^{C}D_{0+}^{\alpha} p_1\left(t\right) + M p_1\left(t\right) + N p_1\left(\theta(t)\right) = f_0\left(t\right) - \widetilde{f}_0\left(t\right),\ t \in J, \\ p_1(0) = \int_0^T g(s)\left(\underline{u}_0(s) - \overline{u}_0\left(s\right)\right) ds, \end{cases}$$

Since $\underline{u}_0 = \underline{u} \leq \overline{u} = \overline{u}_0$ in J and using the hypothesis (H1), we obtain

$$\begin{cases} {}^{C}D_{0+}^{\alpha} p_1\left(t\right) + M p_1\left(t\right) + N p_1\left(\theta(t)\right) \leq 0,\ t \in J, \\ p_1(0) = \int_0^T g(s)\left(\underline{u}_0(s) - \overline{u}_0\left(s\right)\right) ds \leq 0, \end{cases}$$

and then by hypothesis (H2) Lemma 2.4 implies

$$p_1\left(t\right) \leq 0 \text{ for all } t \in J.$$

That is

$$\underline{u}_1 \leq \overline{u}_1 \text{ in } J. \tag{3.5}$$

Then by (3.3), (3.4) and (3.5), we have

$$\underline{u}_0 \leq \underline{u}_1 \leq \overline{u}_1 \leq \overline{u}_0 \text{ in } J.$$

Assume for fixed $n \geq 1$, we have

$$\underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u}_{n+1} \leq \overline{u}_n \text{ in } J,$$

and we show that

$$\underline{u}_{n+1} \leq \underline{u}_{n+2} \leq \overline{u}_{n+2} \leq \overline{u}_{n+1} \text{ in } J.$$

We put by definition

$$w_{n+1}(t) := \underline{u}_{n+2}(t) - \underline{u}_{n+1}(t), \ t \in J.$$

By (3.1)), we have

$$\begin{cases} {}^{C}D^{\alpha}_{0^{+}} w_{n+1}(t) + M w_{n+1}(t) + N w_{n+1}(\theta(t)) = f_{n+1}(t) - f_{n}(t), \ t \in J, \\ w_{n+1}(0) = \int_0^1 g(s) \left(\underline{u}_{n+1}(s) - \underline{u}_{n}(s) \right) ds, \end{cases}$$

Since by the hypothesis of recurrence, we have $\underline{u}_n \leq \underline{u}_{n+1}$ in J and by using the hypothesis (H1), we obtain

$$\begin{cases} {}^{C}D^{\alpha}_{0^{+}} w_{n+1}(t) + M w_{n+1}(t) + N w_{n+1}(\theta(t)) \geq 0, \quad t \in J, \\ w_{n+1}(0) \geq 0, \end{cases}$$

and then by hypothesis (H2) Lemma 2.4 implies

$$w_{n+1}(t) \geq 0 \text{ for all } t \in J.$$

That is

$$\underline{u}_{n+1}(t) \leq \underline{u}_{n+2}(t) \ \text{ for all } t \in J. \tag{3.6}$$

Similarly, we can prove that

$$\overline{u}_{n+2} \leq \overline{u}_{n+1} \text{ in } J, \tag{3.7}$$

and

$$\underline{u}_{n+2} \leq \overline{u}_{n+2} \text{ in } J. \tag{3.8}$$

Then by (3.6), (3.7) and (3.8), we have

$$\underline{u}_{n+1} \leq \underline{u}_{n+2} \leq \overline{u}_{n+2} \leq \overline{u}_{n+1} \text{ in } J.$$

Hence for all $n \in \mathbb{N}$, we have

$$\underline{u}_{n} \leq \underline{u}_{n+1} \leq \overline{u}_{n+1} \leq \overline{u}_{n} \text{ in } J.$$

The proof of **Step 1** is complete.

Step 2: The sequence $(\underline{u}_n)_{n \in \mathbb{N}}$ converges to a minimal solution of (1.1).

By **Step 1** and using Dini's theorem it follows that the sequence of functions $(u_n)_{n\in\mathbb{N}}$ converges uniformly to u_*.

Let $n \in \mathbb{N}^*$ and $t \in J$, then by Lemma 2.2 we have

$$\underline{u}_{n+1}(t) = \underline{u}_{n+1}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (f_n(s) - M\underline{u}_{n+1}(s) - N\underline{u}_{n+1}(\theta(s)))ds.$$

Now, as n tends to $+\infty$, we obtain

$$f_n(s) - M\underline{u}_{n+1}(s) - N\underline{u}_{n+1}(\theta(s)) \to f(s, u_*(s), u_*(\theta(s))).$$

Then by Lemma 2.1, one has

$$u_*(t) - u_*(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u_*(s), u_*(\theta(s)))\, ds, \tag{3.9}$$

and from Lemma 2.2, we deduce

$${}^C D_{0^+}^{\alpha} u_*(t) = f(t, u_*(t), u_*(\theta(t))), \ t \in J.$$

On the other hand it is not difficult to prove that

$$u_*(0) = \int_0^1 g(s) u^*(s) ds,$$

and consequently it follows that u_* is a solution of (1.1).

Now, we prove that if u is another solution of (1.1) such that $\underline{u} \le u \le \overline{u}$, then $u_* \le u$.

Since u is an upper solution of (1.1), then by **Step 1**, we have

$$\forall n \in \mathbb{N}, \ \underline{u}_n \le u.$$

Letting $n \to +\infty$, we obtain

$$u_* = \lim_{n\to+\infty} \underline{u}_n \le u,$$

and consequently it follows that u_* is a minimal solution of problem (1.1).

The proof of **Step 2** is complete.

Similarly, we can prove that the sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ converges to a maximal solution u^* of (1.1).

The proof of Theorem 3.1 is complete. □

4. Applications

In this section we give some examples illustrating the application of our result.

Example 4.1. *We consider the problem*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}} u(t) = \sqrt{t}u(t) - u(t^2) + \cos t, \ t \in \left[0, \frac{1}{2}\right], \\ u(0) = \int\limits_0^{\frac{1}{2}} \sqrt{s}u(s)\, ds, \end{cases} \tag{4.1}$$

We put by definition $\underline{u}(t) = \dfrac{99}{100}\sqrt{t}$ *and* $\overline{u}(t) = \sqrt{t} + 1$, *for all* $t \in \left[0, \frac{1}{2}\right]$.
First $\underline{u}$ *is a lower solution for the problem (4.1) if we have*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}} \underline{u}(t) \leq \sqrt{t}\underline{u}(t) - \underline{u}(t^2) + \cos t, \ t \in \left[0, \frac{1}{2}\right], \\ \underline{u}(0) \leq \int\limits_0^{\frac{1}{2}} \sqrt{s}\underline{u}(s)\, ds. \end{cases}$$

That is

$$\begin{cases} \dfrac{99}{100}\Gamma\left(\frac{3}{2}\right) \leq \cos t, \ t \in \left[0, \frac{1}{2}\right], \\ 0 \leq \dfrac{99}{100} \int\limits_0^{\frac{1}{2}} s ds = \dfrac{99}{800}. \end{cases}$$

Since $\dfrac{99}{100}\Gamma\left(\frac{3}{2}\right) = 0.87736 < 0.87758 = \cos\frac{1}{2} \leq \cos t$ *for all* $t \in \left[0, \frac{1}{2}\right]$, *we obtain* $\underline{u}$ *is an upper solution for the problem (4.1).*
Now $\overline{u}$ *is an upper solution for the problem (4.1) if we have*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}} \overline{u}(t) \geq \sqrt{t}\overline{u}(t) - \overline{u}(t^2) + \cos t, \ t \in \left[0, \frac{1}{2}\right], \\ \overline{u}(0) \geq \int\limits_0^{\frac{1}{2}} \sqrt{s}\overline{u}(s)\, ds. \end{cases}$$

That is

$$\begin{cases} \Gamma\left(\frac{3}{2}\right) \geq \sqrt{t}\left(\sqrt{t}+1\right) - (t+1) + \cos t, \ t \in \left[0, \frac{1}{2}\right], \\ 1 \geq \int\limits_0^{\frac{1}{2}} \sqrt{s}\left(\sqrt{s}+1\right) ds. \end{cases}$$

That is

$$\begin{cases} \Gamma\left(\frac{3}{2}\right) \geq \sqrt{t} - 1 + \cos t, \ t \in \left[0, \frac{1}{2}\right], \\ 1 \geq \frac{1}{8} + \frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} = 0.3607. \end{cases}$$

Since $\sqrt{t} - 1 + \cos t < \frac{1}{\sqrt{2}} = 0.70711 < 0.88623 = \Gamma\left(\frac{3}{2}\right)$, *for all* $t \in \left[0, \frac{1}{2}\right]$, *we obtain* $\overline{u}$ *is an upper solution for the problem (4.1).*

Now if we choose $M = 0$ *and* $N = 1$, *then* $\dfrac{1}{\sqrt{2}\Gamma\left(1+\frac{1}{2}\right)} = 0.79788 \leq 1$ *and the function* $t \mapsto \sqrt{t}u(t) - u(t^2) + \cos t$ *satisfies the assumption of Theorem 3.1 and consequently it follows that the problem (4.1) admits a minimal solution* u_* *and a maximal solution* u^*.

Example 4.2. *We consider the problem*

$$\left\{\begin{array}{l} {}^{C}D_{0^+}^{\frac{1}{3}} u(t) = \dfrac{\sqrt[3]{t}}{3} u(t) - u(t^2) - \dfrac{\sqrt[3]{t}}{3} + 1 + e^{-t},\ t \in \left[0, \frac{1}{3}\right], \\ u(0) = \int\limits_0^{\frac{1}{3}} e^{2s} u(s)\, ds, \end{array}\right. \tag{4.2}$$

We put by definition $\underline{u}(t) = \sqrt[3]{t}$ *and* $\overline{u}(t) = 2\sqrt[3]{t} + 1$, *for all* $t \in \left[0, \frac{1}{3}\right]$.
First $\underline{u}$ *is a lower solution for the problem (4.2) if we have*

$$\left\{\begin{array}{l} {}^{C}D_{0^+}^{\frac{1}{3}} \underline{u}(t) \leq \dfrac{\sqrt[3]{t}}{3} \underline{u}(t) - \underline{u}(t^2) - \dfrac{\sqrt[3]{t}}{3} + 1 + e^{-t},\ t \in \left[0, \frac{1}{3}\right], \\ \underline{u}(0) \leq \int\limits_0^{\frac{1}{3}} e^{2s} \underline{u}(s)\, ds. \end{array}\right.$$

That is

$$\left\{\begin{array}{l} \Gamma\left(\frac{1}{3}+1\right) \leq -\dfrac{2}{3}\sqrt[3]{t^2} - \dfrac{\sqrt[3]{t}}{3} + 1 + e^{-t},\ t \in \left[0, \frac{1}{3}\right], \\ 0 \leq 0.25782. \end{array}\right.$$

Since

$$\Gamma\left(\frac{1}{3}+1\right) = 0.89298,$$

and

$$\min_{t \in \left[0, \frac{1}{3}\right]} \left(-\frac{2}{3}\sqrt[3]{t^2} - \frac{\sqrt[3]{t}}{3} + 1 + e^{-t}\right) = 1.1649,$$

we obtain $\underline{u}$ *is a lower solution for the problem (4.2).*
Now $\overline{u}$ *is an upper solution for the problem (4.2) if we have*

$$\left\{\begin{array}{l} {}^{C}D_{0^+}^{\frac{1}{3}} \overline{u}(t) \geq \dfrac{\sqrt[3]{t}}{3} \overline{u}(t) - \overline{u}(t^2) - \dfrac{\sqrt[3]{t}}{3} + 1 + e^{-t},\ t \in \left[0, \frac{1}{3}\right], \\ \overline{u}(0) \geq \int\limits_0^{\frac{1}{3}} e^{2s} \overline{u}(s)\, ds. \end{array}\right.$$

That is

$$\left\{\begin{array}{l} 2\Gamma\left(\frac{1}{3}+1\right) \geq -\frac{4}{3}\sqrt[3]{t^2} + e^{-t},\ t \in \left[0, \frac{1}{3}\right], \\ 1 \geq \int\limits_0^{\frac{1}{3}} e^{2s}\left(2\sqrt[3]{s}+1\right) ds = 0.9895. \end{array}\right.$$

Since

$$-\frac{4}{3}\sqrt[3]{t^2}+e^{-t}\leq 1, \text{ for all } t\in\left[0,\frac{1}{3}\right],$$

and

$$2\Gamma\left(\frac{1}{3}+1\right)=1.786,$$

we obtain $\overline{u}$ *is an upper solution for the problem (4.2).*

Now if we choose $M=0$ *and* $N=1$*, then* $\dfrac{1}{\sqrt[3]{3}\Gamma\left(1+\frac{1}{3}\right)}=0.77646\leq 1$ *and the function* $t\mapsto\dfrac{\sqrt[3]{t}}{3}u(t)-u(t^2)-\dfrac{\sqrt[3]{t}}{3}+1+e^{-t}$ *satisfies the assumption of Theorem 3.1 and consequently it follows that the problem (4.2) admits a minimal solution* u_* *and a maximal solution* u^*.

Example 4.3. *We consider the problem*

$$(4.3)\qquad\begin{cases} {}^{C}D_{0^+}^{\frac{3}{4}}u(t)=\dfrac{u^2\left(\frac{t}{2}\right)}{2}-u(t)+\dfrac{1}{\Gamma\left(\frac{5}{4}\right)}+\dfrac{2}{3},\ t\in\left[0,\frac{4}{5}\right],\\ u(0)=\int\limits_0^{\frac{4}{5}}s^{\frac{3}{4}}u(s)\,ds,\end{cases}$$

We put by definition $\underline{u}(t)=1-e^{-t}$ *and* $\overline{u}(t)=1+2t^{\frac{3}{4}}$*, for all* $t\in\left[0,\frac{4}{5}\right]$.

First $\underline{u}$ *is a lower solution for the problem (4.3) if we have*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{3}{4}}\underline{u}(t)\leq\dfrac{\underline{u}^2\left(\frac{t}{2}\right)}{2}-\underline{u}(t)+\dfrac{1}{\Gamma\left(\frac{5}{4}\right)}+\dfrac{2}{3},\ t\in\left[0,\frac{4}{5}\right],\\ \underline{u}(0)\leq\int\limits_0^{\frac{4}{5}}s^{\frac{3}{4}}\underline{u}(s)\,ds.\end{cases}$$

That is

$$\begin{cases} t^{\frac{1}{4}}E_{1,\frac{5}{4}}(-t)\leq\frac{3}{2}e^{-t}-e^{-\frac{t}{2}}+\dfrac{1}{\Gamma\left(\frac{5}{4}\right)}+\dfrac{2}{3},\ t\in\left[0,\frac{4}{5}\right],\\ 0\leq\int\limits_0^{\frac{4}{5}}s^{\frac{3}{4}}\left(1-e^{-s}\right)ds=0.14951.\end{cases}$$

From Remark 2, we deduce

$$t^{\frac{1}{4}}E_{1,\frac{5}{4}}(-t)\leq E_{1,\frac{5}{4}}(-t)\leq\frac{1}{\Gamma\left(\frac{5}{4}\right)}=1.1033,\ \text{for all } t\in\left[0,\frac{4}{5}\right],$$

and since

$$\min_{t\in\left[0,\frac{4}{5}\right]}(\frac{3}{2}e^{-t}-e^{-\frac{t}{2}}+\frac{1}{\Gamma\left(\frac{5}{4}\right)}+\frac{2}{3})=1.6033,$$

we obtain $\underline{u}$ is a lower solution for the problem (4.3).

Now $\overline{u}$ is an upper solution for the problem (4.3) if we have

$$\begin{cases} {}^{C}D_{0^+}^{\frac{3}{4}}\overline{u}\left(t\right)\geq\dfrac{\underline{u}^2\left(\frac{t}{2}\right)}{2}-\underline{u}\left(t\right)+\dfrac{1}{\Gamma\left(\frac{5}{4}\right)}+\dfrac{2}{3},\ t\in\left[0,\dfrac{4}{5}\right], \\ \overline{u}(0)\geq\int\limits_0^{\frac{4}{5}}s^{\frac{3}{4}}\overline{u}\left(s\right)ds. \end{cases}$$

That is

$$\begin{cases} 2\Gamma\left(\frac{7}{4}\right)\geq 2\left(\dfrac{t}{2}\right)^{\frac{3}{2}}+2\left(\left(\dfrac{t}{2}\right)^{\frac{3}{4}}-t^{\frac{3}{4}}\right)+\dfrac{1}{\Gamma\left(\frac{5}{4}\right)}+\dfrac{1}{6},\ t\in\left[0,\dfrac{4}{5}\right], \\ 1\geq\int\limits_0^{\frac{4}{5}}s^{\frac{3}{4}}\left(1+2s^{\frac{3}{4}}\right)ds=0.84464. \end{cases}$$

Since

$$2\Gamma\left(\frac{7}{4}\right)=1.8381,$$

and

$$\max_{t\in\left[0,\frac{4}{5}\right]}(2\left(\frac{t}{2}\right)^{\frac{3}{2}}+2\left(\left(\frac{t}{2}\right)^{\frac{3}{4}}-t^{\frac{3}{4}}\right)+\frac{1}{\Gamma\left(\frac{5}{4}\right)}+\frac{1}{6})=\frac{1}{\Gamma\left(\frac{5}{4}\right)}+\frac{1}{6}=1.2699,$$

we obtain $\overline{u}$ is an upper solution for the problem (4.3).

Now if we choose $M=1$ and $N=0$, then $\dfrac{\left(\frac{4}{5}\right)^{\frac{3}{4}}}{\Gamma\left(1+\frac{3}{4}\right)}=0.92039\leq 1$ *and the function* $t\mapsto\dfrac{u^2\left(\frac{t}{2}\right)}{2}-u\left(t\right)+\dfrac{1}{\Gamma\left(\frac{5}{4}\right)}+\dfrac{2}{3}$ *satisfies the assumption of Theorem 3.1 and consequently it follows that the problem (4.3) admits a minimal solution u_* and a maximal solution u^*.*

Example 4.4. *We consider the problem*

$$(4.4)\qquad \begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}}u(t) = \sqrt{t}u(t) - \sqrt{t}u\left(\frac{t}{4}\right) + \frac{\sqrt{\pi}}{2},\ t \in \left[0, \frac{3}{4}\right], \\ u(0) = \int\limits_0^{\frac{3}{4}} \sqrt{s}u(s)\,ds, \end{cases}$$

We put by definition $\underline{u}(t) = \sqrt{t}$ *and* $\overline{u}(t) = 2\sqrt{t} + 1$, *for all* $t \in \left[0, \frac{3}{4}\right]$.

First $\underline{u}$ *is a lower solution for the problem (4.4) if we have*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}}\underline{u}(t) \le \sqrt{t}\underline{u}(t) - \sqrt{t}\underline{u}\left(\frac{t}{4}\right) + \frac{\sqrt{\pi}}{2},\ t \in \left[0, \frac{3}{4}\right], \\ \underline{u}(0) \le \int\limits_0^{\frac{3}{4}} \sqrt{s}\underline{u}(s)\,ds, \end{cases}$$

That is

$$\begin{cases} \frac{\sqrt{\pi}}{2} \le \frac{t}{2} + \frac{\sqrt{\pi}}{2},\ t \in \left[0, \frac{3}{4}\right], \\ 0 \le \int\limits_0^{\frac{3}{4}} s ds = \frac{9}{32}. \end{cases}$$

Since $\frac{t}{2} \ge 0$, *for all* $t \in \left[0, \frac{3}{4}\right]$, *we obtain* $\underline{u}$ *is a lower solution for the problem (4.4).*
Now $\overline{u}$ *is an upper solution for the problem (4.4) if we have*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2}}\overline{u}(t) \ge \sqrt{t}\overline{u}(t) - \sqrt{t}\overline{u}\left(\frac{t}{4}\right) + \frac{\sqrt{\pi}}{2},\ t \in \left[0, \frac{3}{4}\right], \\ \overline{u}(0) \ge \int\limits_0^{\frac{3}{4}} \sqrt{s}\overline{u}(s)\,ds. \end{cases}$$

That is

$$\begin{cases} \sqrt{\pi} \ge t + \frac{\sqrt{\pi}}{2},\ t \in \left[0, \frac{3}{4}\right], \\ \overline{u}(0) = 1 \ge \int\limits_0^{\frac{3}{4}} \sqrt{s}\left(2\sqrt{s} + 1\right)ds = 0.99551. \end{cases}$$

Since

$$\frac{\sqrt{\pi}}{2} = 0.88623,$$

we obtain $\overline{u}$ *is an upper solution for the problem (4.4).*

Now if we choose $M = 1$ *and* $N = 0$, *then* $\dfrac{\sqrt{\frac{3}{4}}}{\Gamma\left(\frac{3}{2}\right)} = 0.97721 \le 1$ *and the function* $t \mapsto \sqrt{t}u(t) - \sqrt{t}u\left(\frac{t}{4}\right) + \frac{\sqrt{\pi}}{2}$ *satisfies the assumption of Theorem 3.1 and consequently it* *follows that the problem (4.4) admits a minimal solution u_* and a maximal solution u^*.*

5. Conclusion

In this paper, based on the upper and lower solutions method and monotone iterative technique, we proved the existence of extremal solutions for a class of fractional differential equations with deviating arguments and integral initial conditions. Several examples are given illustrating the applications of our results. On the other hand, it could be interesting to study the existence of solutions when the function g changes its sign.

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References

[1] C. R. Adams, Linear q-difference equations, *Bull. Amer. Math. Soc.* **37** (1931), 361–400.

[2] J. Alzabut, A. G. M. Selvam, R. A. El-Nabulsi, V. Dhakshinamoorthy and M. E. Samei, Asymptotic stability of nonlinear discrete fractional pantograph equations with non-local initial conditions, *Symmetry*, **13(473)** (2021), 22 pages.

[3] A. Augustynowicz and J. Jankowski, On extremal solutions of differential equations with advanced argument, *Annales Societatis Mathematicae Polonae, Series I: Commentationes Mathematicae* **XLVI(1)** (2006), 17–24.

[4] K. Balachandran, S. Kiruthika and J.J. Trujillo, Existence of solutions of nonlinear fractional pantograph equations, *Acta Mathematica Scientia* **33B** (2013), 712–720.

[5] R. C. L. Bosworth, A definition of plasticity, *Nature* **137** (1946), 447–447.

[6] L. Bourdin, Weighted Hölder continuity of Riemann-Liouville fractional integrals – Application to regularity of solutions to fractional Cauchy problems with Carathéodory dynamics, *Fract. Calc. Appl. Anal.* **22** (2019), 722–749.

[7] G. Derfel, P. J. Grabner and R. F. Tichy, On the asymptotic behaviour of the zeros of the solutions of a functional-differential equation with rescaling, *Operator Theory: Advances and Applications* **263** (2018), 281–295.

[8] M. Derhab, T. Khedim and B. Messirdi, Existence results of first-order differential equations with integral boundary conditions at resonance, *Comm. Appl. Nonlinear Anal.* **24** (2017), 93–106.

[9] M. Derhab and F. Meziane, Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations with integral boundary conditions, *J. Nonlinear Funct. Anal.* **2018** (2018), 1–22.

[10] D. Dhaigude and B. Rizqan, Existence and uniqueness of solutions of fractional differential equations with deviating arguments under integral boundary conditions, *Kyungpook Math. J.* **59** (2019), 191–202.

[11] K. Diethelm, Erratum: The mean value theorems and a Nagumo-type uniqueness theorem for Caputo's fractional calculus, *Fract. Calc. Appl. Anal.* **20** (2017), 1567–1570.

[12] M. P. Flamant, Sur une équation différentielle fonctionnelle linéaire, *Rend. Circ. Matem. Palermo* **48** (1924), 135–208.

[13] L. Fox, D. F. Mayers, J. R. Ockendon, and A. B. Tayler, On a function differential equation, *IMA J. Appl. Math.* **8** (1971), 271–307.

[14] J. K. Hale, Theory of Functional Differential Equations. Springer, New York, 1977.

[15] T. Jankowski, Existence results to delay fractional differential equations with nonlinear boundary conditions, *Appl. Math. Comput.* **219** (2013), 9155–9164.

[16] T. Jankowski, Fractional differential equation with deviating arguments, *Dynam. Systems Appl.* **17** (2008), 677–684.

[17] T. Kato and J. B. McLeod, The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$, *Bull. Amer. Math. Soc.* **77** (1971), 891–937.

[18] A. A. Kilbas and S. A. Marzan, Cauchy problem for differential equation with Caputo derivative, *Fract. Calc. Appl. Anal.* **7** (2004), 297–321.

[19] A. Kilbas and S. A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Differential Equations* **41** (2005), 84–89.

[20] A. A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[21] D. Li and C. Zhang, Long time numerical behaviors of fractional pantograph equations, *Math. Comput. Simul.* **172** (2020), 244-257.

[22] E. Liz and J.J. Nieto, Periodic boundary value problems for a class of functional differential equations, *J. Math. Anal. Appl.* **200** (1996), 680–686.

[23] T. Ma and B. Yan, The multipilicity solutions for nonlinear fractional differential equations of Riemann-Liouville type, *Fract. Calc. Appl. Anal.* **21** (2018), 801–818.

[24] J.A. Tenreiro Machado, Fractional Calculus: Models, Algorithms, Technology. Discontinuity, Nonlinearity, and Complexity **4** (2015), 383–389.

[25] R. L. Magin, Fractional Calculus in Bioengineering. Begell House Publishers, Inc. Connectitut, United States of America, 2006.

[26] K. Mahler, On a special functional equation, *J. London Math. Soc.* **15** (1940), 115–123.

[27] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity, Imperial College Press, London, 2010.

[28] G. R. Morris, A. Feldstein, E. W. Bowen, The Phragmén–Lindelöf principle and a class of functional differential equations, in: L. Weiss (Ed.), Ordinary Differential Equations: 1971 NRL-MRC Conference, Academic Press, New York, 1972, 513–540.

[29] K. S. Miller and S.G. Samko. A note on the complete monotonicity of of the generalized Mittag-Leffler function. *Real Anal. Exchange* **23** (1997), 753–755.

[30] S. Ntouyas, G. Wang and L. Zhang, Positive solutions of arbitrary order nonlinear fractional differential equations with advanced arguments, *Opuscula Math.* **31**, No 3 (2011), 433–442.

[31] J. R. Ockendon and A. B. Tayler, The dynamics of a current collection system for an electric locomotive, *Proc. R. Soc. Lond. A* **322** (1971), 447–468.

[32] I. Podlubny, Fractional Differential Equations. Academic Press, San Diego, 1999.

[33] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach, Yverdon, 1993.

[34] G. W. Scott Blair. Thc role of psychophysics in rheology, *J. of Colloid Sciences* **2** (1947), 21–32.

[35] H. Sebbagh and M. Derhab, The Adomian decomposition method for solving a class of fractional nonhomogeneous multi-pantograph equations with initial conditions, *Comm. Appl. Nonlinear Anal.* **28** (2021), 1–30.

[36] V. Spiridonov, Universal superpositions of coherent states and self-similar potentials, *Phys. Rev. A* **52** (1995), 1909–1935.

[37] D. Vivek, K. Kanagarajan and S. Harikrishnan, Dynamics and stability resuls of fractional pantograph equations with complex order, *J. Appl. Nonlinear Dyn.* **7** (2018), 179–187.

[38] G. Wang, Monotone iterative technique for boundary value problems of a nonlinear fractional differential equation with deviating arguments, *J. Comput. Appl. Math.* **236** (2012), 2425–2430.

[39] G. Wang, L. Zhang and S. K. Ntouyas, Existence of multiple positive solutions of a nonlinear arbitrary order boundary value problem with advanced arguments, *Electron. J. Qual. Theory Differ. Equ.* **15** (2012), 1–13.

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