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APPLICATION OF $\mathcal{T}$-CURVATURE TENSOR IN SPACETIMES

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Abstract. In this paper we show that $\mathcal{T}$-flat spacetime is Einstein with constant curvature and the energy momentum tensor of this spacetime satisfying the Einstein's field equation with the cosmological constant is covariant constant. Then we find the length of the Ricci operator and derive some geometric properties for a $\mathcal{T}$-flat general relativistic viscous fluid spacetime. We also see that for a purely electromagnetic distribution the scalar curvature of a $\mathcal{T}$-flat spacetime satisfying the Einstein's field equation without cosmological constant vanishes. Lastly we study the general relativistic viscous fluid spacetime with the divergence-free $\mathcal{T}$-curvature tensor with respect to some conditions and the possible local cosmological structure is of Petrov type I, D or O.

1. Introduction

This paper is dealt with some investigations in the theory of general relativity with respect to the coordinate vanishing method in differential geometry. In this type of study a spacetime of general relativity is considered like a connected pseudo-Riemannian manifold of dimension four equipped with the Lorentzian metric g having signature (-, +, +, +). The field equation of Einstein [3] follows that the energy momentum tensor is of divergence free. If the energy momentum tensor is covariant constant then this demand is fulfilled. Chaki and Roy [11] had proved that a general relativistic spacetime admitting the covariant constant energy momentum tensor is Ricci symmetric. Many authors [13, 16, 5, 18, 17] had studied spacetimes in different

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ways on different manifolds and different curvature tensors.
Let (M, g) be an n-dimensional pseudo-Riemannian manifold and $\mathfrak{X}(M)$ be the Lie algebra of vector fields in M. We consider $X, Y, Z, W \in \mathfrak{X}(M)$ throughout the entire study.

Definition 1.1. A pseudo-Riemannian manifold (M, g) is a differentiable manifold M equipped with an everywhere non-degenerate, smooth, symmetric metric tensor g.

Tripathi and Gupta [12] had developed the notion of $\mathcal{T}$- curvature tensor in pseudo-Riemannian manifolds. They defined $\mathcal{T}$- curvature tensor as follows.

Definition 1.2. In an n-dimensional pseudo-Riemannian manifold (M, g), a $\mathcal{T}$- curvature tensor is a tensor of type $(1, 3)$ defined by

$$
\begin{aligned}
\mathcal{T}(X,Y)Z &= c_0R(X,Y)Z \\
&\quad + c_1S(Y,Z)X + c_2S(X,Z)Y + c_3S(X,Y)Z \\
&\quad + c_4g(Y,Z)QX + c_5g(X,Z)QY + c_6g(X,Y)QZ \\
&\quad + rc_7\left[g(Y,Z)X - g(X,Z)Y\right],
\end{aligned} \tag{1.1}
$$

where $X, Y, Z \in \mathfrak{X}(M)$; $c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7$ are smooth functions on M; S, Q, R, r, g are respectively the Ricci tensor, Ricci operator, curvature tensor, scalar curvature and pseudo-Riemannian metric tensor.

Definition 1.3. The Riemannian curvature tensor R of type $(0, 4)$ on M is a quadrilinear mapping $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ defined by $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ for any $X, Y, Z, W \in \mathfrak{X}(M)$.

$\mathcal{T}$-curvature tensor reduces to many other curvature tensors for different values of $c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7$.

Definition 1.4. A $\mathcal{T}$-curvature tensor of type $(0,4)$ is defined by

$$
\begin{aligned}
\mathcal{T}(X,Y,Z,W) &= c_0R(X,Y,Z,W) \\
&+c_1S(Y,Z)g(X,W)+c_2S(X,Z)g(Y,W) \\
&+c_3S(X,Y)g(Z,W)+c_4g(Y,Z)S(X,W) \\
&+c_5g(X,Z)S(Y,W)+c_6g(X,Y)S(Z,W) \\
&+rc_7\left[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\right],
\end{aligned}
\tag{1.2}
$$

where $X,Y,Z,W \in \mathfrak{X}(M)$, R is the Riemannian curvature tensor, S is the Ricci tensor, g is the pseudo-Riemannian metric tensor and $\mathcal{T}(X,Y,Z,W) = g\left(\mathcal{T}(X,Y)Z,W\right)$.

Definition 1.5. A spacetime is called an Einstein spacetime if the Ricci tensor S of type $(0,2)$ satisfies the relation $S = \frac{r}{n}$, $n > 2$ on M where r is the scalar curvature of (M^n, g).

Definition 1.6. A spacetime is called $\mathcal{T}$-flat if the $\mathcal{T}$-curvature tensor of type $(0,4)$ satisfies the relation $\mathcal{T}(X,Y,Z,W) = 0$ on M for any $X,Y,Z,W \in \mathfrak{X}(M)$.

Definition 1.7. A spacetime is called a spacetime with constant curvature if the curvature tensor satisfies the relation $R(X,Y,Z,W) = g(X,Z)g(Y,W)-g(X,W)g(Y,Z)$ on M for any $X,Y,Z,W \in \mathfrak{X}(M)$.

Definition 1.8. If a spacetime M admits a symmetry then it is said to be a curvature collineation (CC) [8, 9, 6] if

$$
\left(\pounds_\xi R\right)(X,Y)Z = 0, \tag{1.3}
$$

where R is the Riemannian curvature tensor.

Definition 1.9. The vector field ξ is said to be a Killing vector field if it satisfies the relation $\left(\pounds_\xi g\right)(X,Y) = 0$ where $X,Y \in \mathfrak{X}(M)$.

Definition 1.10. The vector field ξ is said to be a conformal Killing vector field if it satisfies the relation $\left(\pounds_\xi g\right)(X,Y) = 2\phi g(X,Y)$ where $X,Y \in \mathfrak{X}(M)$ and ϕ is being a scalar.

Definition 1.11. A spacetime is called $\mathcal{T}$-conservative if $(div\ \mathcal{T})(X,Y,Z) = 0$.

Definition 1.12. A $(0,2)$-type symmetric tensor field F in a pseudo-Riemannian manifold (M^n, g) is called Codazzi type if $(\nabla_X F)(Y,Z) = (\nabla_Y F)(X,Z)$ for $X, Y, Z \in \mathfrak{X}(M)$.

This paper has been arranged in the following manner. In the first unit we give introduction. In Section 2 we study spacetime admitting vanishing $\mathcal{T}$-curvature tensor and some geometric properties have been derived. Section 3 is devoted to the general relativistic viscous fluid spacetime admitting vanishing $\mathcal{T}$-curvature tensor. In Section 4 we discuss the general relativistic viscous fluid spacetime admitting divergence-free $\mathcal{T}$-curvature tensor.

2. A spacetime admitting vanishing $\mathcal{T}$-curvature tensor

In this unit we consider V_4 as a spacetime of dimension 4 in general relativity for our entire study. We obtain the following results.

Theorem 2.1. *If* $(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6) \neq 0$ *then a* $\mathcal{T}$*-flat spacetime is an Einstein spacetime.*

Proof. For a $\mathcal{T}$-flat spacetime $\mathcal{T}(X,Y,Z,W) = 0$. Then from the equation (1.2), we obtain

$$(2.1) \quad 0 = c_0 R(X,Y,Z,W) + c_1 S(Y,Z)g(X,W) + c_2 S(X,Z)g(Y,W) + c_3 S(X,Y)g(Z,W) + c_4 g(Y,Z)S(X,W)$$

$$+ c_5 g(X,Z)S(Y,W) + c_6 g(X,Y)S(Z,W) + r c_7 \left[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right].$$

Taking contraction on both sides over X and W, we derive

$$(2.2) \quad S(Y,Z) = -\left[\frac{r(c_4 + 3c_7)}{(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6)}\right] g(Y,Z).$$

Let $\alpha = -\left[\frac{r(c_4+3c_7)}{c_0+4c_1+c_2+c_3+c_5+c_6}\right]$. Then the equation (2.2) becomes

$$(2.3) \quad S(Y,Z) = \alpha g(Y,Z).$$

Clearly, if $(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6) \neq 0$ then this is an Einstein spacetime. □

Theorem 2.2. *If* $c_0 \neq 0$, $c_3 + c_6 = 0$, $(c_1 + c_2 + c_4 + c_5) = 0$ *and* $(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6) \neq 0$ *then a* $\mathcal{T}$*-flat spacetime is a spacetime with constant curvature.*

Proof. In view of the equation (2.3), the equation (2.1) implies that

$$R(X,Y,Z,W) = -\left[\frac{(c_1+c_4)\,\alpha + rc_7}{c_0}\right][g(Y,Z)g(X,W) + \left[\frac{rc_7 - (c_2+c_5)\alpha}{c_0}\right]g(X,Z)g(Y,W)] - \frac{\alpha(c_3+c_6)}{c_0}g(X,Y)g(Z,W). \tag{2.4}$$

It clearly follows that if $c_0 \neq 0$, $c_3 + c_6 = 0, (c_1 + c_2 + c_4 + c_5) = 0$ and $(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6) \neq 0$ then

$$R(X,Y,Z,W) = \left[\frac{(c_1+c_4)\,\alpha + rc_7}{c_0}\right][g(X,Z)g(Y,W) - g(Y,Z)g(X,W)].$$

That is, a $\mathcal{T}$-flat spacetime is a spacetime with constant curvature with respect to the above conditions. □

Theorem 2.3. *The energy momentum tensor is covariant constant in* $\mathcal{T}$*-flat spacetime satisfying the Einstein's field equation with the cosmological constant.*

Proof. We consider a spacetime satisfying the Einstein's field equation with the cosmological constant

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y), \tag{2.5}$$

where S, λ, r, k and $T(X,Y)$ are being the Ricci tensor, cosmological constant, scalar curvature, gravitational constant and energy momentum tensor respectively.
In view of the equations (2.3) and (2.5), we derive

$$T(X,Y) = \frac{1}{k}\left(\alpha - \frac{r}{2} + \lambda\right)g(X,Y). \tag{2.6}$$

By taking the covariant derivative with respect to Z on both sides, we gain

$$(\nabla_Z T)(X,Y) = -\frac{1}{k}\left[\frac{(c_4+3c_7)}{(c_0+4c_1+c_2+c_3+c_5+c_6)} + \frac{1}{2}\right]dr(Z)g(X,Y). \tag{2.7}$$

As a $\mathcal{T}$-flat spacetime is an Einstein spacetime with the condition $(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6) \neq 0$, hence the scalar curvature r is a constant. Therefore,

$$dr(Z) = 0,\ \forall Z. \tag{2.8}$$

The equations (2.7) and (2.8) jointly imply that

$$(\nabla_Z T)(X,Y) \;=\; 0.$$

Thus the energy momentum tensor $T(X,Y)$ is covariant constant. □

Theorem 2.4. *If a spacetime M with $\mathcal{T}$-curvature tensor with respect to a Killing vector field ξ is curvature collineation then the Lie derivative of $\mathcal{T}$-curvature tensor vanishes along ξ.*

Proof. The geometrical symmetries of a spacetime can be written as

$$\pounds_\xi A - 2\Omega A \;=\; 0, \tag{2.9}$$

where A is the physical or geometrical quantity, Ω is a scalar and $\pounds_\xi$ represents the Lie derivative with respect to ξ.

For the metric inheritance symmetry we put $A = g$ in the equation (2.9). Thus

$$(\pounds_\xi g)(X,Y) - 2\Omega g(X,Y) \;=\; 0. \tag{2.10}$$

Clearly, in this case if $\Omega = 0$ then ξ becomes a Killing vector field. Let a spacetime M with $\mathcal{T}$-curvature tensor with respect to a Killing vector field ξ be curvature collineation. Thus we gain

$$(\pounds_\xi g)(X,Y) \;=\; 0. \tag{2.11}$$

As M is admitting a curvature collineation, hence we derive from the equation (1.3) that

$$(\pounds_\xi S)(X,Y) \;=\; 0, \tag{2.12}$$

where S denotes the Ricci tensor.

We take the Lie derivative of the equation (1.1) and then with the help of the equations (1.3), (2.11) and (2.12), we derive $(\pounds_\xi \mathcal{T})(X,Y)Z = 0$. □

Theorem 2.5. *Let a spacetime satisfying the Einstein's field equation be of zero $\mathcal{T}$-curvature tensor. The spacetime admits the matter collineation with respect to ξ if and only if ξ is a Killing vector field.*

Proof. The symmetry of energy momentum tensor T is called matter collineation and it is defined by

$$(\pounds_\xi T)(X,Y) = 0,$$

where ξ is the symmetry generating vector field and $\pounds_\xi$ is the operator of Lie derivative along ξ.
Let ξ be a Killing vector field of vanishing $\mathcal{T}$-curvature tensor. Therefore

$$(\pounds_\xi g)(X,Y) = 0. \tag{2.13}$$

Taking the Lie derivative on both the sides of the equation (2.6) with respect to ξ, we have

$$\frac{1}{k}\left(\alpha - \frac{r}{2} + \lambda\right)(\pounds_\xi g)(X,Y) = (\pounds_\xi T)(X,Y). \tag{2.14}$$

Using the equation (2.13) in the equation (2.14), we have

$$(\pounds_\xi T)(X,Y) = 0. \tag{2.15}$$

This proves that the spacetime admits the matter collineation.
For the converse part, let $(\pounds_\xi T)(X,Y) = 0$. Therefore from the equation (2.14), we find

$$(\pounds_\xi g)(X,Y) = 0.$$

This shows that ξ is a Killing vector field. □

Theorem 2.6. *Let a spacetime satisfying the Einstein's field equation be of vanishing $\mathcal{T}$-curvature tensor. The vector field ξ is a conformal Killing vector field if and only if the energy momentum tensor has the Lie inheritance property with respect to ξ.*

Proof. Let ξ be a conformal Killing vector field. Therefore,

$$(\pounds_\xi g)(X,Y) = 2\phi g(X,Y), \tag{2.16}$$

where ϕ is being a scalar.
Now, from the equation (2.14), it follows that

$$\left(\alpha - \frac{r}{2} + \lambda\right) 2\phi g(X,Y) = k\,(\pounds_\xi T)(X,Y). \tag{2.17}$$

With the help of the equation (2.6) in the equation (2.17), we have

$$(\pounds_\xi T)(X,Y) = 2\phi T(X,Y). \tag{2.18}$$

This shows that the energy momentum tensor has the Lie inheritance property with respect to ξ.

For the converse part, let the energy momentum tensor have the Lie inheritance property with respect to ξ. Therefore,

$$(\pounds_\xi T)(X,Y) = 2\phi T(X,Y).$$

Clearly, the equation (2.16) holds good. This proves that ξ is a conformal Killing vector field. □

3. General relativistic viscous fluid spacetime admitting vanishing $\mathcal{T}$-curvature tensor

In this unit we consider the general relativistic viscous fluid spacetime admitting vanishing $\mathcal{T}$-curvature tensor satisfying the Einstein's field equation without cosmological constant with the condition $\sigma + p = 0$ where p, σ are respectively the isotropic pressure and the energy density. Furthermore, $\sigma + p = 0$ implies that the fluid behaves like a cosmological constant [7] and it is also called the phantom barrier [15]. The choice $\sigma = -p$ leads to the rapid expansion of this spacetime in cosmology and it is called inflation [10]. We obtain the following theorems.

Theorem 3.1. *If a $\mathcal{T}$-flat general relativistic viscous fluid spacetime with the condition $\sigma + p = 0$ where p, σ are respectively the isotropic pressure and the energy density satisfies the Einstein's field equation without cosmological constant, then*

$$\|Q\|^2 = \frac{4k^2p^2(c_4 + 3c_7)^2}{(c_0 + 4c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + 6c_7)^2},$$

where Q is the Ricci operator.

Proof. In a general relativistic viscous fluid spacetime with the condition $\sigma + p = 0$, the energy momentum tensor T takes the form [3]

$$T(X,Y) = pg(X,Y), \tag{3.1}$$

where p is the isotropic pressure, σ denotes the energy density and $g(U,U) = -1$, U is the velocity vector field of this flow.

The field equation of Einstein without cosmological constant takes the form

$$S(X,Y) - \frac{r}{2}g(X,Y) = kT(X,Y), \tag{3.2}$$

where r denotes the scalar curvature and $k \neq 0$.

Using the equations (2.3) and (3.1) in the equation (3.2), we have

$$\left(\alpha - \frac{r}{2} - kp\right) g(X,Y) = 0. \tag{3.3}$$

Taking contraction on both sides over X and Y, we derive

$$r = -\frac{2pk(c_0 + 4c_1 + c_2 + c_3 + c_5 + c_6)}{(c_0 + 4c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + 6c_7)}. \tag{3.4}$$

From the equations (2.3) and (3.4), it implies that

$$S(X,Y) = \frac{2pk(c_4 + 3c_7)}{(c_0 + 4c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + 6c_7)}g(X,Y). \tag{3.5}$$

If Q is the Ricci operator then $g(QX,Y) = S(X,Y)$ and $S(QX,Y) = S^2(X,Y)$. From the equation (3.5), we have

$$S(QX,Y) = \frac{4p^2k^2(c_4 + 3c_7)^2}{(c_0 + 4c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + 6c_7)^2}g(X,Y). \tag{3.6}$$

Taking contraction on both sides over X and Y, we get

$$\|Q\|^2 = \frac{4p^2k^2(c_4 + 3c_7)^2}{(c_0 + 4c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + 6c_7)^2}. \tag{3.7}$$

□

Theorem 3.2. *If a $\mathcal{T}$-flat general relativistic viscous fluid spacetime with the condition $\sigma + p = 0$ where p, σ are respectively the isotropic pressure and the energy density obeying the Einstein's field equation without cosmological constant satisfies the condition of timelike convergence then this spacetime also satisfies the relation*

$$\frac{p(c_4 + 3c_7)}{(c_0 + 4c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + 6c_7)} < 0.$$

Proof. The condition of timelike convergence [14] is given by

$$S(X,X) > 0, \tag{3.8}$$

for any timelike vector field X.

From the equations (3.1) and (3.2), it follows that

$$S(X,Y) - \frac{r}{2}g(X,Y) = kpg(X,Y). \tag{3.9}$$

Setting $X = Y = U$ in the equation (3.9) and with the help of the equation (3.4), we have

$$S(U,U) = -\frac{2pk(c_4 + 3c_7)}{(c_0 + 4c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + 6c_7)}. \tag{3.10}$$

Since $k > 0$ and $S(U,U) > 0$, so we obtain

$$\frac{p(c_4 + 3c_7)}{(c_0 + 4c_1 + c_2 + c_3 + 2c_4 + c_5 + c_6 + 6c_7)} < 0. \tag{3.11}$$

□

Theorem 3.3. *For a purely electromagnetic distribution the scalar curvature of a $\mathcal{T}$-flat spacetime with the condition $\sigma + p = 0$ where p, σ are respectively the isotropic pressure and the energy density satisfying the Einstein's field equation without cosmological constant is zero.*

Proof. Taking contraction on both sides of the equation (3.2) over X and Y, we gain

$$r = -kt, \tag{3.12}$$

where t is the trace of T.

Using the equation (3.12) in the equation (3.2), we derive

$$S(X,Y) = kT(X,Y) - \frac{kt}{2}g(X,Y). \tag{3.13}$$

For a purely electromagnetic distribution the Einstein's field equation without cosmological constant is given by

$$S(X,Y) = kT(X,Y). \tag{3.14}$$

From the equations (3.13) and (3.14), it implies that $t = 0$. Hence, we obtain $r = 0$ from the equation (3.12). □

4. General relativistic viscous fluid spacetime admitting divergence-free $\mathcal{T}$-curvature tensor

This part is devoted to the study of the general relativistic viscous fluid spacetime admitting the divergence-free $\mathcal{T}$-curvature tensor. We have the following theorems in this regard.

Theorem 4.1. *In a general relativistic viscous fluid spacetime admitting divergence-free $\mathcal{T}$-curvature tensor, if $c_1 + c_2 = 0$ and $c_3 = 0$ then the energy momentum tensor is of Codazzi type.*

Proof. From the equation (1.1), we have

$$(div\ \mathcal{T})(X,Y,Z) = (c_0 + c_1)(\nabla_X S)(Y,Z) + (c_2 - c_0)(\nabla_Y S)(X,Z) + c_3(\nabla_Z S)(X,Y) + \left(\frac{c_4}{2} + c_7\right) g(Y,Z)dr(X) + \left(\frac{c_5}{2} - c_7\right) g(X,Z)dr(Y) + \frac{c_6}{2} g(X,Y)dr(Z). \tag{4.1}$$

Putting $(div\ \mathcal{T})(X,Y,Z) = 0$ and $dr(X) = 0$ in the equation (4.1), we have

$$0 = (c_0 + c_1)(\nabla_X S)(Y,Z) + (c_2 - c_0)(\nabla_Y S)(X,Z) + c_3(\nabla_Z S)(X,Y). \tag{4.2}$$

Clearly, if $c_1 + c_2 = 0$ and $c_3 = 0$, then we derive from the equation (4.2) that

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{4.3}$$

From the equations (3.2) and (4.3), it implies that

$$(\nabla_X T)(Y,Z) = (\nabla_Y T)(X,Z).$$

Therefore, the energy momentum tensor is of Codazzi type. □

Theorem 4.2. *In a general relativistic viscous fluid spacetime admitting divergence-free $\mathcal{T}$-curvature tensor, if $c_1 + c_2 = 0$ and $c_3 = 0$ then the velocity vector field of the fluid is proportional to the gradient vector field of the energy density.*

Proof. It is already proved that the energy momentum tensor in the general relativistic viscous fluid spacetime is of Codazzi type. This implies that both the vorticity and shear of the fluid vanish and the velocity vector field is hyper-surface orthogonal. That is, the velocity vector field of the fluid is proportional to the gradient vector field of the energy density [4, 2]. □

Theorem 4.3. *For a general relativistic viscous fluid spacetime admitting divergence-free $\mathcal{T}$-curvature tensor, if $c_1 + c_2 = 0$ and $c_3 = 0$ then the possible local cosmological structure of this spacetime is of Petrov type I, D or O.*

Proof. Barnes [1] proved that if the shear and vorticity of a perfect fluid spacetime vanish then the velocity vector field U is hyper-surface orthogonal and the energy density is constant over the hyper-surface which is orthogonal to U. Hence, the local cosmological structure of this spacetime is of Petrov type I, D or O. □

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