Jordan Journal of Mathematics and Statistics (JJMS), 15(3B), 2022, pp 661 - 681
DOI: https://doi.org/10.47013/15.3.17

# CERTAIN SUBORDINATION RESULTS ON THE CLASS OF STRONGLY STARLIKE $p$-VALENT ANALYTIC FUNCTIONS.

RAJESH KUMAR MAURYA

Abstract. In this paper we define and study a class $\mathcal{LS}_p^*(\alpha)$ of $p$-valent analytic functions associated with the right half of the lemniscate of Bernoulli. This study is an attempt to find some symmetry or pattern when function $f \in \mathcal{A}_p$. Here we determine Hankel determinant of some initial coefficients of the Taylor series expansion. Sharp bounds of the Hankel determinant of order 2, bounds of the initial coefficients, Fekete-Szegö type problem and a radius result for this class are obtained.

## 1. Introduction

Let $\mathcal{H}[a,n]$ denotes a class of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...,$$

which are analytic in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Function $f \in \mathcal{H}[a,n]$ normalized if $f(0) = 0$ and $f'(0) = 1$.

**Definition 1.** *(See [8]) Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ by*

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k & (\lambda = n \in \mathbb{N}) \end{cases}$$

2020 *Mathematics Subject Classification.* Primary: 30C3xx; Secondary: 30C80; 30C20; 30C55; 30C45.

*Key words and phrases.* p-valent analytic functions, subordination, Hankel determinant.

Received: Feb. 20, 2021 Accepted: Dec. 7, 2021 .

**Definition 2.** *(See [5, 6]) The q-Derivatives $D_q$ of a function $f$ is defined in a given subset of $\mathbb{C}$ by*

$$(D_q f)(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z} & (z \neq 0) \\ f'(0) & (z = 0) \end{cases}$$

provided that $f'(0)$ exists, from definition 2 observe that

$$\lim_{q \to 1-} (D_q f)(z) = \lim_{q \to 1-} \frac{f(qz) - f(z)}{(q-1)z} = f'(z)$$

for a differentiable function $f$ in a given subset of $\mathbb{C}$

$$\lim_{q \to 1-} (D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

Let $\mathcal{A}_p$ denotes a subclass of functions in $\mathcal{H}[0,p]$ whose members are of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} \; z^{p+n} \quad (z \in \mathbb{U}) . \tag{1.1}$$

Denote the class $\mathcal{A}_1$ as $\mathcal{A}$.

In *Geometric Function Theory*, various classes based on geometric consideration of the image domain of $f$ have been defined, few of them are as follows:

(i)

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \operatorname{Re}\left( \frac{zf'(z)}{f(z)} \right) > 0 \;\; (z \in \mathbb{U}) \right\}$$

is well known the class of starlike functions associated with the positive half plane $\{w \in \mathbb{C} : \operatorname{Re}(w) > 0\}$.

(ii)

$$\mathcal{SP} = \left\{ f \in \mathcal{A} : \operatorname{Re}\left( \frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right| \;\; (z \in \mathbb{U}) \right\}$$

is associated with the parabolic region $\{w \in \mathbb{C} : \operatorname{Re}(w) > |w-1|\}$ in the positive half plane and is defined by Rønning [7].

(iii)

$$k\text{-}ST = \left\{ f \in \mathcal{A} : \operatorname{Re}\left( \frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \;\; (0 \leq k < \infty; z \in \mathbb{U}) \right\}.$$

is connected with a conic section symmetric about the real axis in the positive half plane $\{w \in \mathbb{C} : \operatorname{Re}(w) > k\,|w-1|\,, 0 \leq k < \infty\}$ introduced by Kanas and Wiśniowska [21]

(iv)

$$UCV_p = \left\{ f \in \mathcal{A}_p : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|1 + \frac{zf''(z)}{f'(z)} - p\right| \quad (z \in \mathbb{U}) \right\}$$

is defined by Al-Khasani and Al-Hajiry [9, 10] and is connected with the parabolic region $\{w \in \mathbb{C} : \operatorname{Re}(w) > |w - p|\}$ in the positive half plane.

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^*(\alpha)$ of $p$-valent starlike of order $\alpha\,(0 \leq \alpha < p)$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}),$$

where $f(z) \neq 0$ for any $z \in \mathbb{U}/\{0\}$.

**Definition 3.** *Let $\mathcal{P}$ denotes a class of functions $\phi \in \mathcal{H}[1,n]$ with $\operatorname{Re}(\phi(z)) > 0$ in $\mathbb{U}$. For $A, B, -1 < A \leq 1, -1 \leq B < A$, denote by $\mathcal{P}(A,B)$ the family of functions*

$$P(z) = 1 + b_1 z + ...$$

regular in $\mathbb{U}$, and such that $P(z)$ is in $\mathcal{P}(A,B)$ iff

$$P(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \ (z \in \mathbb{U})$$

where $w(z)$ is a schwarz class function i.e.$w(0) = 0, |w(z)| < 1$ for all $z \in \mathbb{U}$.

(v) $P(z)$ maps $\mathbb{U}$ onto a slit region on the right half of complex plane, based on this geometric consideration Janowski [28] defined a subclass of starlike functions $\mathcal{S}^*[A,B]$ as

$$\mathcal{S}^*[A,B] = \left\{ f \in \mathcal{A}\!: \ \frac{zf'(z)}{f(z)} = P(z), \ P(z) \ \in \mathcal{P}(A,B) \right\}$$

Srivastava et al. [12] combine the concept of Janowski [28] with the above mention $q-$calculus and defined $\mathcal{S}_q^*[A,B]$

$$\mathcal{S}_q^*[A,B] = \left\{ f \in \mathcal{A} : \left| \frac{(B-1)(\frac{(D_q f)(z)}{f(z)}) - (A-1)}{(B+1)(\frac{(D_q f)(z)}{f(z)}) - (A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q} \quad (z \in \mathbb{U}) \right\}.$$

Observe that $\lim\limits_{q \to 1-} \mathcal{S}_q^*[A,B] = \mathcal{S}^*[A,B]$ is class introduced by Janowski in [28]. Mahmood et. al. in [22] combine the concept of Srivastava et al. [12] and defined a meromorphically $q$-starlike functions associated with Janowski functions for $p$-valent analytic functions as

$$\mathcal{MS}_q^*[A,B] = \left\{ f \in \mathcal{A} : \left| \frac{(B-1)(-\frac{(D_qf)(z)}{f(z)}) - (A-1)}{(B+1)(-\frac{(D_qf)(z)}{f(z)}) - (A+1)} - \frac{1}{1-q} \right| < \frac{1}{1-q} \quad (z \in \mathbb{U}) \right\}.$$

A sufficiency condition based on coefficient estimates, and distortion inequalities has been studied in [22].

(vi) A function $f \in \mathcal{A}(\mathbb{U})$ is said to belong to class $\mathcal{S}_q^*$ if $f(0) = 0 = f'(0) - 1$ and

$$\left| \frac{z}{f(z)}(D_qf)(z) - \frac{1}{1-q} \right| < \frac{1}{1-q} \quad (z \in \mathbb{U}).$$

The notation $\mathcal{S}_q^*$ was first used by Sahoo et. al.[24]. Coefficient inequalities for $q$-starlike function has been studied in [26]. Combining concept of Sahoo et. al.[24] and using Ruscheweyh-type $q$-derivative operator Sahid Mahmood et. al. [23] define the following subclass of $q$-starlike functions as

$$\mathcal{RS}_q^*(\delta) = \left\{ f \in \mathcal{A} : \left| \frac{zD_q\mathcal{R}_q^\delta f(z)}{f(z)} - \frac{1}{1-q} \right| < \frac{1}{1-q} \quad (z \in \mathbb{U}; \delta > -1) \right\}.$$

upper bound of third Hankel determinants and sharp bounds for some coefficients has been determine in [23]. With the help of concepts introduced in the above mentioned articles we shall study a class of strongly starlike $p$-valent analytic functions.

We say that an analytic function $f$ is subordinate to the analytic function $g$ in $\mathbb{U}$ and write $f \prec g$ in $\mathbb{U}$, if and only if there exists a Schwarz class function $w$ analytic in $\mathbb{U}$ such that $f(z) = g(w(z)), z \in \mathbb{U}$. In particular, if $g$ is univalent in $\mathbb{U}$, we have the following equivalence:

$$f \prec g \text{ in } \mathbb{U} \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

Following Ma and Minda [27], we consider $\phi \in \mathcal{P}$ (see Definition 3), analytic univalent in $\mathbb{U}$, with $\phi(\mathbb{U})$ symmetrical with respect to the real axis and starlike with respect to $\phi(0) = 1$, and $\phi'(0) > 0$, for such function $\phi$, we define a class $\mathcal{S}_p^*(\alpha, [\phi])$ of functions $f \in \mathcal{A}_p$ ($f(z) \neq 0$ for any $z \in \mathbb{U} \setminus \{0\}$) satisfying the condition

$$\frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)} \prec \phi(z) \quad (0 \leq \alpha < p; z \in \mathbb{U}). \tag{1.2}$$

Taking $\phi(z)=\left(\frac{1+z}{1-z}\right)^{\beta}$ $(0<\beta\leq 1)$, we denote the class $\mathcal{S}_p^*(\alpha,[\phi])$ by $\mathcal{SS}_p^*(\alpha,\beta)$ and functions therein satisfy the condition:

$$\left|\arg\left(\frac{zf'(z)}{f(z)}-\alpha\right)\right|<(p-\alpha)\beta\frac{\pi}{2}\quad(0\leq\alpha<p,0<\beta\leq 1;z\in\mathbb{U}).$$

Note that for $p=1$ and for $\alpha=0$, the class $\mathcal{SS}_p^*(\alpha,\beta)=\mathcal{SS}_\beta^*$ was introduced earlier in [4] and [15] and is called a class of strongly starlike functions. Class $\mathcal{SS}_p^*(\alpha,1)$ is denoted by $\mathcal{S}_p^*(\alpha)$.

For the purpose of this paper, we denote in particular, the following classes:

$$\mathcal{S}_p^*\left(\alpha,\left[\frac{1+Az}{1+Bz}\right]\right)=\mathcal{S}_p^*(\alpha,A,B),\quad -1\leq B<A\leq 1$$

and

$$\mathcal{S}_p^*\left(\alpha,\left[\sqrt{1+z}\right]\right)=\mathcal{LS}_p^*(\alpha),$$

where

$$\mathcal{LS}_p^*(\alpha)=\left\{f\in\mathcal{S}_p^*(\alpha):\left|\left(\frac{zf'(z)-\alpha f(z)}{(p-\alpha)f(z)}\right)^2-1\right|<1\right\}.$$

Observe that

$$\mathcal{L}=\{w\in\mathbb{C}:\operatorname{Re}\{w\}>0,|w^2-1|<1\}$$

is the interior of the right half of the lemniscate of Bernoulli:

$$\partial\mathcal{L}:=\left\{w=u+iv\in\mathbb{C}:(u^2+v^2)^2-2(u^2-v^2)=0\right\}$$

and

$$\mathcal{L}\subset\{w\in\mathbb{C}:|\arg\, w|<\frac{\pi}{4}\}.$$

Therefore, we observe the inclusion:

$$\mathcal{LS}_p^*(\alpha)\subset S\mathcal{S}_p^*\left(\alpha,\frac{1}{2}\right)\subset\mathcal{S}_p^*(\alpha),$$

and that the class $\mathcal{LS}_p^*(\alpha)$ is a class of strongly starlike $p$-valent analytic functions associated with a positive region of lemniscate of Bernoulli, which is being studied in this paper. Results obtained include a representation formula and an inclusion with the class $\mathcal{S}_p^*(\alpha,A,B)$ which leads some examples for the class $\mathcal{LS}_p^*(\alpha)$. A radius result for certain functions of the class $\mathcal{LS}_p^*(\alpha)$, coefficients estimates for initial coefficients including a Fekete-Szegö problem and a Hankel determinant for the class $\mathcal{LS}_p^*(\alpha)$ are

also obtained. Further, a coefficient inequality for this class of functions is obtained. Based on the bounds of first three coefficients, a conjecture is proposed.
It is mentioned that the class $\mathcal{LS}_1^*(0) = \mathcal{SL}^*$ was introduced and studied in [14] see also [1, 2, 11, 13, 17].

## 2. Integral Representation

We first give an integral representation of the function $f \in \mathcal{LS}_p^*(\alpha)$.

**Theorem 1.** *Let $f \in \mathcal{LS}_p^*(\alpha)$. Then there exists a function $q \in \mathcal{H}[1,1]$ such that $q(\mathbb{U})$ is in the interior of the right half of the lemniscate of Bernoulli and the function $f$ is represented by*

$$f(z) = z^p \exp\left\{(p-\alpha)\int_0^z \frac{q(t)-1}{t}dt\right\} \quad (z \in \mathbb{U}). \tag{2.1}$$

*The extremal function of the class $\mathcal{LS}_p^*(\alpha)$ is given by*

(2.2)

$$f_1(z) = z^p \left(\frac{2}{1+\sqrt{1+z}}\right)^{2(p-\alpha)} \exp\left\{2(p-\alpha)\left(\sqrt{1+z}\right)-1\right)\right\} \;\; (0 \leq \alpha < p; z \in \mathbb{U}).$$

*Proof.* Let $f \in \mathcal{LS}_p^*(\alpha)$. Then, there is a function $q \in \mathcal{H}[1,1]$ such that

$$q(z) = \frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)} \prec \sqrt{1+z} \quad (z \in \mathbb{U}),$$

describes the interior of the right half of the Lemniscate of Bernoulli and it may be expressed as

$$\frac{zf'(z)}{f(z)} - \alpha = (p-\alpha)q(z). \tag{2.3}$$

On integrating (2.3), we get

$$\log\frac{f(z)}{z^p} = (p-\alpha)\int_0^z \frac{q(t)-1}{t}dt \;\; (z \in \mathbb{U})$$

and hence, the representation (2.1). If we take $q(z) = \sqrt{1+z}$ in (2.1) and then after simplifying we get the extremal function $f_1$ of the class $\mathcal{LS}_p^*(\alpha)$, which is given by (2.2). This proves Theorem 1. $\square$

In addition to the example (2.2) of the class $\mathcal{LS}_p^*(\alpha)$, we also have

$$f_n(z) = z^p \exp\left\{(p-\alpha)\int_0^z \frac{\sqrt{1+t^n}-1}{t}dt\right\} \in \mathcal{LS}_p^*(\alpha)$$

for any $n \in \mathbb{N}$.

We next find the condition on $A$ and $B$ so that $\mathcal{S}_p^*(\alpha, A, B) \subset \mathcal{LS}_p^*(\alpha)$.

**Theorem 2.** *Let* $-1 < B < A \leq 1$. *Then*

$$\mathcal{S}_p^*(\alpha, A, B) \subset \mathcal{LS}_p^*(\alpha).$$

*if and only if*

$$A \leq \frac{1+\sqrt{2}\ B}{\sqrt{2}+B}. \tag{2.4}$$

*Proof.* Let $f \in \mathcal{S}_p^*(\alpha, A, B)$ for $-1 < B < A \leq 1$.Then, we get

$$\left|\frac{zf'(z)-\alpha f(z)}{(p-\alpha)f(z)} - \frac{1-AB}{1-B^2}\right| \leq \frac{A-B}{1-B^2}$$

which shows that $w = \frac{zf'(z)-\alpha f(z)}{(p-\alpha)f(z)} \quad (z \in \mathbb{U})$ lies in the disc

$$D(c,r) := \{w \in \mathbb{C} : |w-c| \leq r\},$$

where

$$c = \frac{1-AB}{1-B^2}, \ \ r = \frac{A-B}{1-B^2}$$

and

$$\left|\arg\left(\frac{zf'(z)-\alpha f(z)}{(p-\alpha)f(z)}\right)\right| \leq \sin^{-1}\frac{r}{c}.$$

Now $\mathcal{S}_p^*(\alpha, A, B) \subset \mathcal{LS}_p^*(\alpha)$ if and only if the disc $D(c,r) \subset \mathcal{L}$ or,

$$\sin^{-1}\frac{r}{c} \leq \frac{\pi}{4}$$

which implies that

$$\frac{r}{c} \leq \frac{1}{\sqrt{2}},$$

or, if (2.4) holds. This proves Theorem 2. □

In view of the above Theorem 2 and the representation of $f$ given by (2.1) we get following examples for the class $\mathcal{LS}_p^*(\alpha)$:

(i) For $0 \leq \alpha < p$ and for $-\frac{1}{\sqrt{2}} < B < 0$,

$$g_1(z) = z^p \exp\left(-Bz\left(p-\alpha\right)\right) \in \mathcal{LS}_p^*(\alpha).$$

(ii) For $0 \leq \alpha < p$ and for $0 < A < \frac{1}{\sqrt{2}}$,

$$g_2(z) = z^p \exp\left(Az\left(p-\alpha\right)\right) \in \mathcal{LS}_p^*(\alpha).$$

(iii) For $0 \leq \alpha < p$ and for $0 < A < \sqrt{2} - 1$,

$$g_3(z) = z^p \left(1 - Az\right)^{-2(p-\alpha)} \in \mathcal{LS}_p^*(\alpha).$$

(iv) For $0 \leq \alpha < p$ and for $0 < B < 1$

$$g_3(z) = z^p \left(1 + Bz\right)^{\left(\frac{1}{B}-1\right)(p-\alpha)} \in \mathcal{LS}_p^*(\alpha).$$

## 3. A Radius Result

In this section, we find a radius result for some specific functions $f \in \mathcal{LS}_p^*(\alpha)$.

**Theorem 3.** *Let $f \in \mathcal{A}_p$ satisfy*

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right)^{\frac{1}{p-\alpha}} > 0 \quad \left(0 \leq \alpha < p;\ z \in \mathbb{U}\right). \tag{3.1}$$

*Then the radius $r_0\,(0 < r_0 < 1)$ for the function $f$ to be in the class $\mathcal{LS}_p^*(\alpha)$ is given by*

$$r_0 = \frac{\sqrt{2} - 1}{1 + \sqrt{1 + \left(\sqrt{2} - 1\right)^2}}. \tag{3.2}$$

*The radius is sharp.*

*Proof.* Let $h(z) = \left(\frac{f(z)}{z^p}\right)^{\frac{1}{p-\alpha}}$ Then

$$\frac{zh'(z)}{h(z)} = \frac{1}{p-\alpha}\left(\frac{zf'(z)}{f(z)} - p\right) = \frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)} - 1.$$

Since, in view of (3.1), $h \in \mathcal{P}$, we have [25]

$$\left|\frac{zh'(z)}{h(z)}\right| \leq \frac{2r}{1 - r^2} \quad \left(|z| = r < 1\right).$$

Thus, we have

$$\left|\frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)} - 1\right| \leq \frac{2r}{1 - r^2} \quad \left(|z| = r < 1\right) \tag{3.3}$$

and the function $f \in \mathcal{LS}_p^*(\alpha)$ if

$$\left|\frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)} - 1\right| \leq \sqrt{2} - 1. \tag{3.4}$$

Therefore, from (3.3) and (3.4), we get for $f \in \mathcal{LS}_p^*(\alpha)$, the radius $r$ satisfies

$$\frac{2r}{1 - r^2} \leq \sqrt{2} - 1$$

and $\max_{0<r<1} r = r_0$ is given by (3.2). Sharpness can be verified for the function

$$f(z) = z^p \left(\frac{1+z}{1-z}\right)^{p-\alpha} \quad (z \in \mathbb{U}).$$

Since, for this function

$$\frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)} - 1 = \frac{2z}{1-z^2}$$

and if $z = r_0$ is given by (3.2) that is if

$$\frac{2r_0}{1-r_0^2} = \sqrt{2} - 1,$$

we get

$$\left| \left(\frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)}\right)^2 - 1 \right| = 1.$$

This proves Theorem 3. □

## 4. Coefficient estimates and Hankel determinant

In 1976, Noonan and Thomas [16] defined the $q$th Hankel determinant of

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \mathbb{U}),$$

which is given for $q, n \in \mathbb{N}$, by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & a_{n+q} & & a_{n+2q-2} \end{vmatrix}.$$

The functional $H_2(1)$ is called Fekete-Szegö functional and the problem finding the upper bound of the generalized functional $|a_3 - \mu a_2^2|$ with real $\mu$ is called the Fekete-Szegö problem. The functional $H_2(2) = |a_2 a_4 - a_3^2|$ is known as 2nd Hankel determinant of $f$.

In this section, results on coefficient estimates for initial coefficients including a Fekete-Szegö problem, Hankel determinant, and a coefficient inequality for the class $\mathcal{LS}_p^*(\alpha)$ are obtained. To obtain the results, we apply following lemmas.

**Lemma 1.** *Let $p \in \mathcal{P}$ be of the form*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{U}) . \tag{4.1}$$

*Then*

$$|c_n| \leq 2 \quad (n \in \mathbb{N}) \tag{4.2}$$

*and*

$$\left|c_2 - \mu c_1^2\right| \leq 2 \max \{1, |2\mu - 1|\} . \tag{4.3}$$

*The result (4.2) may be found in* [18] *and result (4.3) in* [27].

**Lemma 2.** [19, 20] *If $p \in \mathcal{P}$ is given by (4.1), then*

$$2c_2 = c_1^2 + \left(4 - c_1^2\right) x \tag{4.4}$$

*and*

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{4.5}$$

*for some $x, z$ such that $|x| \leq 1$ and $|z| \leq 1$.*

**Theorem 4.** *Let $f \in \mathcal{LS}_p^*(\alpha)$ be of the form (1.1).Then*

$$|a_{p+1}| \leq \frac{p - \alpha}{2} \tag{4.6}$$

*and for $\mu \in \mathbb{C}$,*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p - \alpha}{4} \left[\max \left\{1, \left|\frac{1}{4} - \frac{p - \alpha}{2} (1 - 2\mu)\right|\right\}\right] \tag{4.7}$$

*In particular, for the range: $0 < (p - \alpha) \leq \frac{5}{2}$,*

$$|a_{p+2}| \leq \frac{p - \alpha}{4}. \tag{4.8}$$

*The results are sharp.*

*Proof.* Let $f \in \mathcal{LS}_p^*(\alpha)$, then for a Schwarz function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$, we have

$$\frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)} = \sqrt{1 + w(z)}, \ (z \in \mathbb{U}) . \tag{4.9}$$

Now, for this $w(z)$, there exists a function $p \in \mathcal{P}$ such that

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + ... \quad .$$

which implies that

$$\sqrt{1 + w(z)} = \left( \frac{2p(z)}{1 + p(z)} \right)^{1/2}$$

$$= 1 + \frac{c_1}{4} z + \frac{1}{4} \left( c_2 - \frac{5}{8} c_1^2 \right) z^2 + \frac{1}{4} \left( c_3 - \frac{5}{4} c_1 c_2 + \frac{13}{32} c_1^3 \right) z^3 + ... \ . \tag{4.10}$$

also, let

$$\frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...,$$

then on writing the series expressions of $f(z)$ and $f'(z)$ form (1.1), and then, on equating the coefficients of $z^{p+1}$, $z^{p+2}$ and $z^{p+2}$ on both the sides of the equation

$$zf'(z) - \alpha f(z) = (p - \alpha) f(z) \left( 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... \right),$$

we obtain on simplifying for $p_1, p_2, p_3$ that

$$\frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)} = [1 + \frac{a_{p+1}}{p-\alpha} z + \frac{2a_{p+2} - a_{p+1}^2}{p - \alpha} z^2 + \tag{4.11}$$

$$+ \frac{3a_{p+3} + a_{p+1}^3 - 3a_{p+1}a_{p+2}}{p-\alpha} z^3 + ... \quad .]$$

therefore, in view of (4.9), we obtain from (4.10) and (4.11)

$$a_{p+1} = \frac{p - \alpha}{4} c_1, \tag{4.12}$$

$$a_{p+2} = \frac{p-\alpha}{8} \left[ c_2 - \{5 - 2(p-\alpha)\} \frac{c_1^2}{8} \right], \tag{4.13}$$

$$a_{p+3} = \frac{p-\alpha}{12} \left[ c_3 - \{10 - 3(p - \alpha)\} \frac{c_1 c_2}{8} + \right.$$

$$\left. \left\{ 26 - 15 \, (p - \alpha) + 2(p-\alpha)^2 \right\} \frac{c_1^3}{64} \right] . \tag{4.14}$$

Applying (4.2) of Lemma 1, to (4.12), we obtain result (4.6), and applying (4.3) of Lemma 2, we get for some $\mu \in \mathbb{C}$,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| = \frac{p-\alpha}{8}\left|c_2 - \eta c_1^2\right| \leq \frac{p-\alpha}{4}\max\left(1, |2\eta - 1|\right), \tag{4.15}$$

where

$$\eta = \frac{5}{8} - \frac{p-\alpha}{4}(1-2\mu). \tag{4.16}$$

This proves inequality (4.7) and in particular, taking $\mu = 0$ in (4.7), we obtain

$$|a_{p+2}| \leq \frac{p-\alpha}{4}\left[\max\left\{1, \left|\frac{1}{4} - \frac{p-\alpha}{2}\right|\right\}\right] = \frac{p-\alpha}{4},$$

since, for $0 < (p-\alpha) \leq \frac{5}{2}$, we have

$$\left|\frac{1}{4} - \frac{p-\alpha}{2}\right| \leq 1$$

and this proves the estimate (4.8). Sharpness of the estimates (4.6) and (4.8) can be seen, respectively, for the functions $f_1$ and $f_2$ such that

$$\frac{zf_1'(z) - \alpha f_1(z)}{(p-\alpha)f_1(z)} = \sqrt{1+z} \quad (z \in U)$$

and

$$\frac{zf_2'(z) - \alpha f_2(z)}{(p-\alpha)f_2(z)} = \sqrt{1+z^2} \quad (z \in U),$$

and the estimate (4.7) is sharp for these $f_1$ and $f_2$. This completes the proof of Theorem 4. □

Taking $\mu$ to be real in Theorem 4, we get following result.

**Corollary 1.** *Let $f \in \mathcal{LS}_p^*(\alpha)$ be of the form (1.1). Then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} \frac{p-\alpha}{16}\left[2(p-\alpha)(1-2\mu) - 1\right] & \text{if } \mu \leq \kappa, \\ \frac{p-\alpha}{4} & \text{if } \kappa \leq \mu \leq \kappa + \frac{2}{p-\alpha}, \\ \frac{p-\alpha}{16}\left[1 - 2(p-\alpha)(1-2\mu)\right] & \text{if } \mu \geq \kappa + \frac{2}{p-\alpha}, \end{cases} \tag{4.17}$$

*where*

$$\kappa = \frac{1}{2} - \frac{5}{4(p-\alpha)}. \tag{4.18}$$

*The result is sharp.*

*Proof.* For real values of $\mu$, from (4.7), we get

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p-\alpha}{4} \tag{4.19}$$

if

$$\left|\frac{1}{4} - \frac{p-\alpha}{2}(1-2\mu)\right| \leq 1.$$

This proves the inequality(4.17)for $\kappa \leq \mu \leq \kappa + \frac{2}{p-\alpha}$, where $\kappa$ is given by (4.18). Also, from (4.7), we get

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p-\alpha}{4}\left|\frac{1}{4} - \frac{p-\alpha}{2}(1-2\mu)\right| \tag{4.20}$$

if

$$\left|\frac{1}{4} - \frac{p-\alpha}{2}(1-2\mu)\right| \geq 1$$

i.e. either

$$\frac{2(p-\alpha)(2\mu-1)+1}{4} \leq -1$$

or

$$\frac{2(p-\alpha)(2\mu-1)+1}{4} \geq 1.$$

and hence, (4.20) proves inequalities in (4.17) for $\mu \leq \kappa$ and $\mu \geq \kappa + \frac{2}{p-\alpha}$, where $\kappa$ is given by (4.18). Sharpness of (4.17) can be verified as follows: □

(i) For the extreme range of $\mu$, i.e. when $\mu < \kappa$ or $\mu > \kappa + \frac{2}{p-\alpha}$, the equality holds for the function $f_1(z)$ considered to show the sharpness in the proof of Theorem 4 and is given by (2.2).

(ii) For the middle range of $\mu$, i.e. when $\kappa < \mu < \kappa + \frac{2}{p-\alpha}$, the equality holds for the function $f_2(z)$ considered to show the sharpness in the proof of Theorem 4 and is given by

$$f_2(z) = z^p\left(\frac{2}{1+\sqrt{1+z^2}}\right)^{(p-\alpha)} \exp\left\{(p-\alpha)\left(\sqrt{1+z^2}) - 1\right)\right\} \ (0 \leq \alpha < p; z \in \mathbb{U}).$$

(iii) For $\mu = \kappa$, equality holds for the functions $f(z)$ given by

$$\frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)} = \sqrt{1 + \frac{z(z+\epsilon)}{1+\epsilon z}} \quad (0 \leq \epsilon \leq 1),$$

while for $\mu = \kappa + \frac{2}{p-\alpha}$, the equality holds for the functions $f(z)$ given by

$$\frac{zf'(z) - \alpha f(z)}{(p-\alpha)f(z)} = \sqrt{1 - \frac{z(z+\epsilon)}{1+\epsilon z}} \quad (0 \leq \epsilon \leq 1).$$

This completes the proof of Corollary 1.

For the range $\kappa \le \mu \le \kappa + \frac{2}{p-\alpha}$, although the above upper bound is sharp, it can be further improved in the next result.

**Theorem 5.** *Let $f \in \mathcal{A}$ of the form (1.1) belong to the class $\mathcal{LS}_p^*(\alpha)$. Then for a real $\mu$ ( $\kappa \le \mu \le \kappa + \frac{2}{p-\alpha}$) :*

$$(4.21) \qquad \left|a_{p+2} - \mu a_{p+1}^2\right| + (\mu - \kappa)\left|a_{p+1}\right|^2$$

$$\le \frac{p-\alpha}{4} \quad \left(\kappa \le \mu \le \kappa + \frac{1}{p-\alpha}\right)$$

*and*

$$(4.22) \qquad \left|a_{p+2} - \mu a_{p+1}^2\right| + \left(\kappa + \frac{2}{p-\alpha} - \mu\right)\left|a_{p+1}\right|^2$$

$$\le \frac{p-\alpha}{4} \quad \left(\kappa + \frac{1}{p-\alpha} \le \mu \le \kappa + \frac{2}{p-\alpha}\right).$$

*where $\kappa$ is given by (4.18).*

*Proof.* Observe from $\kappa$ and $\eta$ given, respectively, by (4.18) and (4.16) that

$$\mu - \kappa = \frac{2}{p-\alpha}\eta$$

and hence, using (4.12) and (4.13) and following (4.15), we get for $\kappa < \mu \le \kappa + \frac{1}{p-\alpha}$

$$\begin{aligned} a_{p+2} - \mu a_{p+1}^2 + (\mu - \kappa) a_{p+1}^2 &= \frac{p-\alpha}{8}\left(c_2 - \eta c_1^2\right) + \frac{p-\alpha}{8}\eta c_1^2 \\ &= \frac{p-\alpha}{8} c_2 \end{aligned}$$

which on using result (4.2) of Lemma 1, proves (4.21). Similarly, for $\kappa + \frac{1}{p-\alpha} \le \mu < \kappa + \frac{2}{p-\alpha}$

$$a_{p+2} - \mu a_{p+1}^2 + \left(\mu - \kappa - \frac{2}{p-\alpha}\right) a_{p+1}^2$$

$$= \frac{p-\alpha}{8}\left(c_2 - \eta c_1^2\right) + \frac{p-\alpha}{8}(\eta - 1) c_1^2$$

$$= \frac{p-\alpha}{8}\left(c_2 - c_1^2\right)$$

which on applying result (4.3 ) of Lemma 2, proves (4.22). □

**Theorem 6.** *If a function* $f \in \mathcal{LS}_p^*(\alpha)$ *be of the form (1.1), then for the range* $0<(p-\alpha)<\frac{\sqrt{41}}{2}$,

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| \le \frac{(p-\alpha)^2}{16}. \tag{4.23}$$

*The estimate is sharp.*

*Proof.* Putting the values of $a_{p+1}, a_{p+2}$ and $a_{p+3}$ from (4.12), (4.13) and (4.14), respectively, we get

$$a_{p+1}a_{p+3}-a_{p+2}^2$$

$$= \frac{(p-\alpha)^2}{48}\left[c_1c_3-\{10-3(p-\alpha)\}\frac{c_1^2c_2}{8}+\right.$$

$$\left.\left\{26-15(p-\alpha)+2(p-\alpha)^2\right\}\frac{c_1^4}{64}\right]-\frac{(p-\alpha)^2}{64}\left[c_2-\{5-2(p-\alpha)\}\frac{c_1^2}{8}\right]^2$$

$$= \frac{(p-\alpha)^2}{48}\left[c_1c_3-\frac{3}{4}c_2^2-\frac{5}{16}c_1^2c_2+\left\{29-4(p-\alpha)^2\right\}\frac{c_1^4}{256}\right]. \tag{4.24}$$

Putting the values of $c_2$ and $c_3$ from (4.4) and (4.5), respectively, and taking $c_1=c\in[0,2]$ in (4.24), by simple arrangement of terms get,

$$\begin{aligned} &\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| \\ &= \frac{(p-\alpha)^2}{3072}\left|\left\{\frac{5}{4}-(p-\alpha)^2\right\}c^4-4\left(4-c^2\right)\left(c^2+12\right)x^2-2c^2\left(4-c^2\right)x\right. \\ &\left.+32c(4-c^2)(1-|x|^2)z\right|. \end{aligned} \tag{4.25}$$

Therefore, on using the triangle inequality with non-negative coefficients and putting $|x|=\rho\ (\le 1)$, we get

$$\begin{aligned} \left|a_{p+1}a_{p+3}-a_{p+2}^2\right| &\le \frac{(p-\alpha)^2}{3072}\left[\left|\frac{5}{4}-(p-\alpha)^2\right|c^4+4\left(4-c^2\right)\left(c^2+12\right)\rho^2+\right. \\ &\quad\left. 2c^2\left(4-c^2\right)\rho+32c(4-c^2)(1-\rho^2)\right] \\ &=: \frac{(p-\alpha)^2}{3072}G(\rho,c). \end{aligned}$$

Observe that for $0<\rho<1$, and for fixed $c\in[0,2]$,

$$\frac{\partial G(\rho,c)}{\partial\rho} = \left|5-4(p-\alpha)^2\right|c^3+8\rho\left(4-c^2\right)(c-6)(c-2)+2c^2\left(4-c^2\right)>0$$

and hence, for $c \in [0,2]$,

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| \leq \frac{(p-\alpha)^2}{3072}\lim_{\rho\to 1}G(\rho,c)$$

$$=\frac{(p-\alpha)^2}{3072}\left[\left|\frac{5}{4}-(p-\alpha)^2\right|c^4+4\left(4-c^2\right)\left(c^2+12\right)+2c^2\left(4-c^2\right)\right]$$

$$=:\frac{(p-\alpha)^2}{3072}g(c).$$

further, observe that for $c \in [0,2]$, and for the given range of $(p-\alpha)$,

$$\begin{aligned} g'(c) &= \left|5-4(p-\alpha)^2\right|c^3+4\left[-2c\left(c^2+12\right)+2c\left(4-c^2\right)\right]+\left[4c\left(4-c^2\right)-4c^3\right] \\ &= \left[\left\{\left|5-4(p-\alpha)^2\right|-24\right\}c^2-48\right]c=0 \end{aligned}$$

only if $c=0$ and

$$g''(c)=3\left\{\left|5-4(p-\alpha)^2\right|-24\right\}c^2-48<0$$

at $c=0$. Thus, we obtain

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| \leq \frac{(p-\alpha)^2}{3072}g(0)=\frac{(p-\alpha)^2}{16}.$$

this proves the estimate (4.23). Sharpness may be verified for the function $f \in \mathcal{LS}_p^*(\alpha)$ given by

$$\frac{zf'(z)-\alpha f(z)}{(p-\alpha)f(z)}=\sqrt{1+z^2} \quad (z \in \mathbb{U}). \tag{4.26}$$

□

**Remark 1.** *From the results obtained in Theorem 4, Corollary 1 and Theorem 6, we get results of Raza and Malik* [17] *for the class* $\mathcal{SL}^*$.

Obtain following coefficient inequality for the class $\mathcal{LS}_p^*(\alpha)$:

**Theorem 7.** *Let* $f \in \mathcal{LS}_p^*(\alpha)$ *be of the form (1.1). Then*

$$\sum_{k=1}^{\infty}\left[\left(\frac{k-\alpha}{p-\alpha}\right)^2-2\right]|a_k|^2 \leq 1. \tag{4.27}$$

*Proof.* Let $f \in \mathcal{LS}_p^*(\alpha)$ be of the form (1.1). Then, for a Schwarz function $w(z)$ analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ in $\mathbb{U}$, from (4.9), we have

$$\left(\frac{zf'(z)-\alpha f(z)}{p-\alpha}\right)^2-(f(z))^2=(f(z))^2\,w(z)$$

and hence, on using the Parseval's identity for $|z|=r$ $(r<1)$,

$$\begin{aligned}2\pi\sum_{k=0}^{\infty}|a_{p+k}|^2 r^{2k} &= \int_0^{2\pi}\left|f(re^{i\theta})\right|^2 d\theta \quad (a_p=1)\\ &\geq \int_0^{2\pi}\left|f(re^{i\theta})\right|^2\left|w(re^{i\theta})\right| d\theta\\ &= \int_0^{2\pi}\left(\frac{re^{i\theta}f'(re^{i\theta})-\alpha f(re^{i\theta})}{p-\alpha}\right)^2 d\theta-\int_0^{2\pi}\left|f(re^{i\theta})\right|^2 d\theta\end{aligned}$$

which on writing the series expansions of $f$ and $f'$, proves that

$$4\pi\sum_{k=0}^{\infty}|a_{p+k}|^2 r^{2k}\geq 2\pi\sum_{k=0}^{\infty}\left(\frac{p+k-\alpha}{p-\alpha}\right)^2|a_{p+k}|^2 r^{2(p+k)}.$$

Taking limit $r\to 1^-$, we obtain

$$\sum_{k=1}^{\infty}\left[\left(1+\frac{k}{p-\alpha}\right)^2-2\right]|a_{p+k}|^2\leq 1$$

which is the inequality (4.27). □

**Corollary 2.** *Let $f \in \mathcal{LS}_p^*(\alpha)$ be of the form (1.1). Then for the range $0<(p-\alpha)<\frac{1}{\sqrt{2}-1}$,*

$$\sum_{k=1}^{\infty}|a_{p+k}|^2\leq\frac{1}{\left(1+\frac{1}{p-\alpha}\right)^2-2}. \tag{4.28}$$

**Theorem 8.** *Let $f \in \mathcal{LS}_p^*(\alpha)$ be of the form (1.1). Then for the range $0<(p-\alpha)<\frac{7}{2}$,*

$$|a_{p+3}|\leq\frac{p-\alpha}{6}\quad(k\in\mathbb{N}). \tag{4.29}$$

*The result is sharp.*

*Proof.* Putting the values of $c_2$ and $c_3$ from (4.4) and (4.5), respectively, and taking $c_1 = c \in [0,2]$ in (4.14), we obtain on simple arrangement of terms

$$a_{p+3} = \frac{p-\alpha}{192}\left[\left\{2-3\left(p-\alpha\right)+2(p-\alpha)^2\right\}\frac{c^3}{4} - 4c(4-c^2)x^2 + \right.$$
$$\left. 8(4-c^2)(1-|x|^2)z + \left\{3\left(p-\alpha\right)-2\right\}c\left(4-c^2\right)x\right].$$

Therefore, on using the triangle inequality with non-negative coefficients and putting $|x| = \rho\ \ (\leq 1)$, we get

$$|a_{p+3}| \leq \frac{p-\alpha}{192}\left[\left\{2-3\left(p-\alpha\right)+2(p-\alpha)^2\right\}\frac{c^3}{4} + 4c(4-c^2)\rho^2 + \right.$$
$$\left. 8(4-c^2)(1-\rho^2) + \left|3\left(p-\alpha\right)-2\right|c\left(4-c^2\right)\rho\right]$$
$$=: \frac{p-\alpha}{192}F\left(\rho, c\right).$$

Observe that for any $\rho \in [0,1]$, as $c \to 2$,

$$|a_{p+3}| \leq \frac{p-\alpha}{48}\left\{\left(p-\alpha-\frac{3}{4}\right)^2 + \frac{7}{16}\right\} \tag{4.30}$$

and for $0 < \rho < 1$ and $c \to 0$,

$$\frac{\partial F\left(\rho, c\right)}{\partial \rho} = \left[8c\rho - 16\rho + \left|3\left(p-\alpha\right)-2\right|c\right]\left(4-c^2\right) < 0.$$

Thus, for $c \to 0$,

$$|a_{p+3}| \leq \frac{p-\alpha}{192}\lim_{\rho\to 0}F\left(\rho, c\right) \tag{4.31}$$

$$= \frac{p-\alpha}{192}\left[\left\{2-3\left(p-\alpha\right)+2(p-\alpha)^2\right\}\frac{c^3}{4} + 8(4-c^2)\right] \leq \frac{p-\alpha}{6}$$

when $c \to 0$. But for the range $0 < (p-\alpha) \leq \frac{7}{2}$, we observe from (4.30) and $\in$(4.31) that

$$\frac{p-\alpha}{48}\left\{\left(p-\alpha-\frac{3}{4}\right)^2 + \frac{7}{16}\right\} \leq \frac{p-\alpha}{6}.$$

This proves the result (4.29). Sharpness of the estimate (4.29) can be seen for the function $f_3$ such that

$$\frac{zf_3'(z) - \alpha f_3(z)}{(p-\alpha)f_3(z)} = \sqrt{1+z^3}\quad (z \in \mathbb{U}).$$

This completes Theorem 8. □

In view of bounds given by (4.6), (4.8) and (4.29), we propose the following conjecture.

**Conjecture 1.** *Let* $f \in \mathcal{LS}_p^*(\alpha)$ *be of the form (1.1). Then for bounded value of* $(p-\alpha)$,

$$|a_{p+n}| \le \frac{p-\alpha}{2n} \quad (n \in \mathbb{N}) \tag{4.32}$$

*and hence,*

$$\sum_{n=1}^{\infty} |a_{p+n}|^2 \le \frac{\pi^2 (p-\alpha)^2}{12}.$$

**Remark 2.** *The bounds given by (4.32) in Conjecture 1 , improves the result given by (4.28) for the range* $0 < (p-\alpha) < \frac{1}{\sqrt{2}-1}$.

**Remark 3.** *Taking* $p = 1, \alpha = 0$, *our estimates given by (4.6), (4.8) and (4.29) coincides with the estimates obtained by Sokół in* [13], *where based on the estimates the conjecture:* $|a_n| \le \frac{1}{2n} \quad (n \in \mathbb{N})$ *was proposed for the class* $\mathcal{SL}^*$.

## Conclusion

In this article, we have introduced and studied a class of strongly starlike $p$-valent analytic functions associated with the positive region of lemniscate of Bernoulli,We have given an integral representation for this class, and determined radius of the circle which $p$-valent analytic function $f$ lies in this class. Based on coefficient estimates, Fekete-Szegö inequality and a Sharp bound for 2nd Hankel determinant have been found. It has been observed that for $p = 1$ the class considered in this article have analogous properties as of class of univalent function .

## Acknowledgement

Author is thankful to referees for their valuable suggestions.

## References

[1] R.M. Ali , Nak Eun Cho, V. Ravichandran and S. Sivprasad Kumar, Differential subordination for functions associated with the lemniscateof Bernoulli,*Taiwanese J. Math.*, **16(3)** (2012), 1017–1026.

[2] R. M. Ali , N. E. Cho, N. K. Jain and V. Ravichandran, Radii of starlikeness and convexity of functions defined by subordination with fixed second coefficient. *Filomat*, **26(3)**(2012), 553–561
.

[3] R.M. Ali and V. Ravichandran: Classes of meromorphic alpha-convex functions. *Taiwan. J. Math.* **14**(2010), 1479–1490 .

[4] D. A. Brannan and W. E. Kirwan, On some classes of bounded univalent functions, *J London Math. Soc.*, **1(2)** (1969), 431.

[5] F.H. Jackson, On $q$-definite integrals. *Q.S. Pure Appl. Math.* **41**(1910), 193-203 .

[6] F.H. Jackson, $q$-Difference equations, *Am. J. Math.* **32**(1910), 305-314 .

[7] F. R$o$nning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118** (1993), 189-196.

[8] G. Gasper and M.Rahman, Basic Hypergeometric Series Cambridge University Press, Cambridge (1990)

[9] H. A. Al-Khasani and S. S. Al-Hajiry, Subordination results for the family of uniformly convex $p$-valent functions, *J. Inequal. Pure and Appl. Math.*, **7(1)** (2006), Art. 20.

[10] H. A. Al-Khasani and S. S. Al-Hajiry, A note on certain inequalities for $p$-valent functions, *J. Inequal. Pure and Appl. Math.*, **9(3)** (2008), Art. 90.

[11] H. M. Srivastava, B. Khan, N. Khan, M. Tahir, S. Ahmad and N. Khan, Upper bound of the third Hankel determinant for a subclass of $q$-starlike functions associated with the $q$-exponential function, *Bulletin des Sciences Mathématiques*, **167(2)** (2021), Article ID 102942, 1–16.

[12] H. M. Srivastava, B. Khan, N. Khan and Q. Z. Ahmad, Coefficient inequalities for $q$-starlike functions associated with the Janowski functions,*Hokkaido Math. J.***48(2)** June (2019), 407–425.

[13] J. Sokol, Coefficient Estimate in a class of strongly starlike function, *Kyungpook Math. J.*, **49** (2009), 349–353.

[14] J. Sokół, J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike function, *Folia Scient. Univ. Tech. Ras.*, **19** (1996), 101–105.

[15] J. Stankiewicz, Quelques problems extrem aux dans les classes des functions $\alpha-$angulairement eto les Ann, *Univ. Marie Curie Sklodowska*, **sect. A. 20** (1966), 59–75.

[16] J. W. Noonan and D. K.Thomas, On the second order Hankel determinant of a really mean $p$-valent functions. *Trans. Amer. Math. Soc.*, **223(2)** (1976), 337–346.

[17] M. Raza and Sarfraz Nawaz Malik, Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli, *Journal of Inequality and Applications*, **2013** (412), (2013).

[18] P. L. Duren, Univalent Functions, Springer, New York, 1983.

[19] R. J. Libera and E. J. Zlotkiewicz, Early coefficient of the inverse of a regular convex function, *Proc. Amer. Math. Soc.*, **85(2)** (1982), 225–230.

[20] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in $\wp$, *Proc. Amer. Math. Soc.*, **87(2)**, (1983), 251–257.

[21] S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, *J. Comput Appl. Math.*, **105** (1999), 327–336.

[22] S. Mahmood, Q. Z. Ahmad, H. M. Srivastava, N. Khan, B. Khan and M. Tahir, A certain subclass of meromorphically $q$-starlike functions associated with the Janowski functions,*J. Inequal. Appl.* (2019), Article ID 88, 1–11.

[23] S. Mahmood, H. M. Srivastava, N. Khan, Q. Z. Ahmad, B. Khan and I. Ali, Upper bound of the third Hankel determinant for a subclass of $q$-starlike functions,*J. Inequal. Appl.* , **(11)**(2019), Article ID 347, 1–13.

[24] S.K. Sahoo and N.L. Sharma, On a generalization of close-to-convex functions. *Am. Pol. Math.*,**113** (2015), 93-108.

[25] T. H. MacGregor, The radius of univalence of ceratin analytic functions, *Proc. Amer. Math. Soc.*, **14** (1963), 514–520.

[26] Uçar, H.E.Ö. Coefficient inequality for $q$-starlike functions. *Appl. Math. Comput.*,**276**(2016), 122–126.

[27] W. C. Ma and D. Minda , A unified treatment of some special classes of univalent functions, *In Procidings of the conferences of complex analysis Tianjin*, 1992, *Conf. Proc. Lecture Notes Anal., Int. Press, Cambridge, MA,* (1994).

[28] W. Janowski, Some extremal problems for certain families of analytic functions, *Ann. Pol. Math.*, *28*(1973) , 297-326.

Department of Mathematics Govt. Post Graduate College Gopeshwar, Uttarakhand India

*Email address*: `rajeshkrmaurya@gmail.com`