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ON GENERALIZED $\mathcal{D}$-CONFORMAL DEFORMATIONS OF ALMOST CONTACT METRIC MANIFOLDS AND HARMONIC MAPS

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Abstract. The objective of this paper is to study and construct harmonic maps among almost contact metric manifolds by introducing the notion of generalized $\mathcal{D}$-conformal deformation.

1. Introduction

Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map among Riemannian manifolds. Then ϕ is said to be harmonic if it is a critical point of the functional energy:

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g,$$

with respect to compactly supported variations. Equivalently, ϕ is harmonic if it satisfies the associated Euler-Lagrange equations:

$$\tau(\phi) = Tr_g \nabla d\phi = 0,$$

where $\tau(\phi)$ is called the tension field of ϕ; one can refer to [1], [4], [5] and [6] for background on harmonic maps. There exist several type of deformations of almost contact metric structures. In this context, the notion of generalized $\mathcal{D}$-conformal deformation is studied in [9] where the authors gave the generalized $\mathcal{D}$-conformal deformation of some particular structures and they studied the scalar curvature associated with this type of structures. In the first section of this paper, we present some

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new results on the generalized $\mathcal{D}$-conformal deformation where we prove in Theorem 2.1 the relation between the Levi-Civita connections on $(M^{2m+1}, \varphi, \xi, \eta, g)$ and $\left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ and we treat some particular cases. The objective of the second and the third sections, where we present the two possible cases (Theorem 2.2 and Theorem 2.3), is to characterize the harmonicity of the identity map relative to this deformation. In each case, we construct several examples where we determine the functions α and β so that the identity map is harmonic.

2. The main results

In this section, we consider $(M^{2m+1}, \varphi, \xi, \eta, g)$ an almost contact metric manifold. A generalized $\mathcal{D}$-conformal deformation is defined as change of structure tensors of the form (see [9])

$$\overline{\varphi} = \varphi, \qquad \overline{\eta} = \alpha\eta, \qquad \overline{\xi} = \frac{1}{\alpha}\xi, \qquad \overline{g} = \beta g + \left(\alpha^2 - \beta\right)\eta \otimes \eta,$$

where α and β are positive functions on M; one can easily check that $\left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is an almost contact metric manifold, too. Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections on $(M^{2m+1}, \varphi, \xi, \eta, g)$ and $\left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ respectively.

2.1. **Some properties of the generalized $\mathcal{D}$-conformal deformation.**

Proposition 2.1. *Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and let $\left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ be a generalized $\mathcal{D}$-conformal deformation of $(M^{2m+1}, \varphi, \xi, \eta, g)$. Then, we have*

$$\begin{aligned}
\overline{g}\left(\overline{\nabla}_X Y, Z\right) &= \beta g\left(\nabla_X Y, Z\right) + \left(\alpha^2 - \beta\right)\eta(Z)\,\eta\left(\nabla_X Y\right) + \alpha\eta(Y)\,\eta(Z)\,X(\alpha) \\
&\quad + \alpha\eta(X)\,\eta(Z)\,Y(\alpha) - \alpha\eta(X)\,\eta(Y)\,Z(\alpha) - \frac{1}{2}\eta(Y)\,\eta(Z)\,X(\beta) \\
&\quad + \frac{1}{2}\left(\alpha^2 - \beta\right)\eta(X)\,g\left(\nabla_Y\xi, Z\right) - \frac{1}{2}\left(\alpha^2 - \beta\right)\eta(X)\,g\left(\nabla_Z\xi, Y\right) \\
&\quad + \frac{1}{2}\left(\alpha^2 - \beta\right)\eta(Y)\,g\left(\nabla_X\xi, Z\right) - \frac{1}{2}\left(\alpha^2 - \beta\right)\eta(Y)\,g\left(\nabla_Z\xi, X\right) \\
&\quad + \frac{1}{2}\left(\alpha^2 - \beta\right)\eta(Z)\,g\left(\nabla_X\xi, Y\right) + \frac{1}{2}\left(\alpha^2 - \beta\right)\eta(Z)\,g\left(\nabla_Y\xi, X\right) \\
&\quad - \frac{1}{2}\eta(X)\,\eta(Z)\,Y(\beta) + \frac{1}{2}\eta(X)\,\eta(Y)\,Z(\beta) + \frac{1}{2}g(Y, Z)\,X(\beta) \\
&\quad + \frac{1}{2}g(X, Z)\,Y(\beta) - \frac{1}{2}g(X, Y)\,Z(\beta).
\end{aligned}$$

Proof. By using the Koszul formula, for all $X, Y, Z \in \Gamma(TM)$, we have

$$2\overline{g}\left(\overline{\nabla}_X Y, Z\right) = X\left(\overline{g}\left(Y, Z\right)\right) + Y\left(\overline{g}\left(X, Z\right)\right) - Z\left(\overline{g}\left(X, Y\right)\right)$$
$$+ \overline{g}\left(\left[X, Y\right], Z\right) + \overline{g}\left(\left[Z, X\right], Y\right) - \overline{g}\left(X, \left[Y, Z\right]\right).$$

As $\overline{g} = \beta g + \left(\alpha^2 - \beta\right)\eta \otimes \eta$, we get

$$\begin{aligned}
2\overline{g}\left(\overline{\nabla}_X Y, Z\right) = {} & X\left(\beta g\left(Y, Z\right) + \left(\alpha^2 - \beta\right)\eta\left(Y\right)\eta\left(Z\right)\right) \\
& + Y\left(\beta g\left(X, Z\right) + \left(\alpha^2 - \beta\right)\eta\left(X\right)\eta\left(Z\right)\right) \\
& - Z\left(\beta g\left(X, Y\right) + \left(\alpha^2 - \beta\right)\eta\left(X\right)\eta\left(Y\right)\right) \\
& + \left(\alpha^2 - \beta\right)\eta\left(\left[X, Y\right]\right)\eta\left(Z\right) + \beta g\left(\left[Z, X\right], Y\right) \\
& + \left(\alpha^2 - \beta\right)\eta\left(\left[Z, X\right]\right)\eta\left(Y\right) - \beta g\left(X, \left[Y, Z\right]\right) \\
& - \left(\alpha^2 - \beta\right)\eta\left(X\right)\eta\left(\left[Y, Z\right]\right) + \beta g\left(\left[X, Y\right], Z\right).
\end{aligned}$$

For the term $X\left(\beta g\left(Y, Z\right) + \left(\alpha^2 - \beta\right)\eta\left(Y\right)\eta\left(Z\right)\right)$, a long calculation gives

$$\begin{aligned}
X\left(\beta g\left(Y, Z\right) + \left(\alpha^2 - \beta\right)\eta\left(Y\right)\eta\left(Z\right)\right) = {} & X\left(\beta g\left(Y, Z\right)\right) + X\left(\left(\alpha^2 - \beta\right)\eta\left(Y\right)\eta\left(Z\right)\right) \\
= {} & \beta g\left(\nabla_X Y, Z\right) + \beta g\left(Y, \nabla_X Z\right) + g\left(Y, Z\right)X\left(\beta\right) \\
& + \left(\alpha^2 - \beta\right)\eta\left(Z\right)\left\{g\left(\nabla_X \xi, Y\right) + \eta\left(\nabla_X Y\right)\right\} \\
& + \left(\alpha^2 - \beta\right)\eta\left(Y\right)\left\{g\left(\nabla_X \xi, Z\right) + \eta\left(\nabla_X Z\right)\right\} \\
& + 2\alpha\eta\left(Y\right)\eta\left(Z\right)X\left(\alpha\right) - \eta\left(Y\right)\eta\left(Z\right)X\left(\beta\right).
\end{aligned}$$

By a similar calculation, we obtain

$$\begin{aligned}
Y\left(\beta g\left(X, Z\right) + \left(\alpha^2 - \beta\right)\eta\left(X\right)\eta\left(Z\right)\right) = {} & \beta g\left(\nabla_Y X, Z\right) + \beta g\left(X, \nabla_Y Z\right) \\
& + \left(\alpha^2 - \beta\right)\eta\left(Z\right)\left\{g\left(\nabla_Y \xi, X\right) + \eta\left(\nabla_Y X\right)\right\} \\
& + \left(\alpha^2 - \beta\right)\eta\left(X\right)\left\{g\left(\nabla_Y \xi, Z\right) + \eta\left(\nabla_Y Z\right)\right\} \\
& + 2\alpha\eta\left(X\right)\eta\left(Z\right)Y\left(\alpha\right) - \eta\left(X\right)\eta\left(Z\right)Y\left(\beta\right) \\
& + g\left(X, Z\right)Y\left(\beta\right)
\end{aligned}$$

and

$$\begin{aligned} Z\left(\beta g\left(X,Y\right)+\left(\alpha^{2}-\beta\right)\eta\left(X\right)\eta\left(Y\right)\right) &= \beta g\left(\nabla_{Z}X,Y\right)+\beta g\left(X,\nabla_{Z}Y\right) \\ &+\left(\alpha^{2}-\beta\right)\eta\left(Y\right)\left\{g\left(\nabla_{Z}\xi,X\right)+\eta\left(\nabla_{Z}X\right)\right\} \\ &+\left(\alpha^{2}-\beta\right)\eta\left(X\right)\left\{g\left(\nabla_{Z}\xi,Y\right)+\eta\left(\nabla_{Z}Y\right)\right\} \\ &+2\alpha\eta\left(X\right)\eta\left(Y\right)Z\left(\alpha\right)-\eta\left(X\right)\eta\left(Y\right)Z\left(\beta\right) \\ &+g\left(X,Y\right)Z\left(\beta\right). \end{aligned}$$

Finally, it is clear that

$$\left(\alpha^{2}-\beta\right)\eta\left([X,Y]\right)\eta\left(Z\right)=\left(\alpha^{2}-\beta\right)\eta\left(Z\right)\eta\left(\nabla_{X}Y\right)-\left(\alpha^{2}-\beta\right)\eta\left(Z\right)\eta\left(\nabla_{Y}X\right),$$

$$\left(\alpha^{2}-\beta\right)\eta\left([Z,X]\right)\eta\left(Y\right)=\left(\alpha^{2}-\beta\right)\eta\left(Y\right)\eta\left(\nabla_{Z}X\right)-\left(\alpha^{2}-\beta\right)\eta\left(Y\right)\eta\left(\nabla_{X}Z\right)$$

and

$$\left(\alpha^{2}-\beta\right)\eta\left(X\right)\eta\left([Y,Z]\right)=\left(\alpha^{2}-\beta\right)\eta\left(X\right)\eta\left(\nabla_{Y}Z\right)-\left(\alpha^{2}-\beta\right)\eta\left(X\right)\eta\left(\nabla_{Z}Y\right).$$

It follows that

$$\begin{aligned} \overline{g}\left(\overline{\nabla}_{X}Y,Z\right) &= \beta g\left(\nabla_{X}Y,Z\right)+\left(\alpha^{2}-\beta\right)\eta\left(Z\right)\eta\left(\nabla_{X}Y\right)+\alpha\eta\left(Y\right)\eta\left(Z\right)X\left(\alpha\right) \\ &+\alpha\eta\left(X\right)\eta\left(Z\right)Y\left(\alpha\right)-\alpha\eta\left(X\right)\eta\left(Y\right)Z\left(\alpha\right)-\frac{1}{2}\eta\left(Y\right)\eta\left(Z\right)X\left(\beta\right) \\ &+\frac{1}{2}\left(\alpha^{2}-\beta\right)\eta\left(X\right)g\left(\nabla_{Y}\xi,Z\right)-\frac{1}{2}\left(\alpha^{2}-\beta\right)\eta\left(X\right)g\left(\nabla_{Z}\xi,Y\right) \\ &+\frac{1}{2}\left(\alpha^{2}-\beta\right)\eta\left(Y\right)g\left(\nabla_{X}\xi,Z\right)-\frac{1}{2}\left(\alpha^{2}-\beta\right)\eta\left(Y\right)g\left(\nabla_{Z}\xi,X\right) \\ &+\frac{1}{2}\left(\alpha^{2}-\beta\right)\eta\left(Z\right)g\left(\nabla_{X}\xi,Y\right)+\frac{1}{2}\left(\alpha^{2}-\beta\right)\eta\left(Z\right)g\left(\nabla_{Y}\xi,X\right) \\ &-\frac{1}{2}\eta\left(X\right)\eta\left(Z\right)Y\left(\beta\right)+\frac{1}{2}\eta\left(X\right)\eta\left(Y\right)Z\left(\beta\right)+\frac{1}{2}g\left(Y,Z\right)X\left(\beta\right) \\ &+\frac{1}{2}g\left(X,Z\right)Y\left(\beta\right)-\frac{1}{2}g\left(X,Y\right)Z\left(\beta\right). \end{aligned}$$

□

By applying Proposition 2.1, we obtain the following result:

Theorem 2.1. For any $X, Y \in \Gamma(TM)$, the relation between $\overline{\nabla}_X Y$ and $\nabla_X Y$ is given by

(2.1)
$$\begin{aligned}
\overline{\nabla}_X Y &= \nabla_X Y - \frac{\alpha}{\beta}\eta(X)\eta(Y)\,grad\alpha + \frac{1}{2\beta}\eta(X)\eta(Y)\,grad\beta - \frac{1}{2\beta}g(X,Y)\,grad\beta \\
&+ \frac{\alpha^2-\beta}{\alpha\beta}\eta(X)\eta(Y)\xi(\alpha)\xi - \frac{\alpha^2-\beta}{2\alpha^2\beta}\eta(X)\eta(Y)\xi(\beta)\xi + \frac{\alpha^2-\beta}{2\alpha^2\beta}g(X,Y)\xi(\beta)\xi \\
&+ \frac{\alpha^2-\beta}{2\beta}\eta(Y)\nabla_X\xi + \frac{\alpha^2-\beta}{2\beta}\eta(X)\nabla_Y\xi + \frac{1}{2\beta}X(\beta)Y + \frac{1}{2\beta}Y(\beta)X \\
&- \frac{1}{2\beta}\eta(X)Y(\beta)\xi - \frac{1}{2\beta}\eta(Y)X(\beta)\xi - \frac{\alpha^2-\beta}{2\beta}\eta(X)Tr_g g(\nabla_\bullet\xi, Y)\bullet \\
&+ \frac{(\alpha^2-\beta)^2}{2\alpha^2\beta}\eta(X)g(\nabla_\xi\xi, Y)\xi - \frac{\alpha^2-\beta}{2\beta}\eta(Y)Tr_g g(\nabla_\bullet\xi, X)\bullet \\
&+ \frac{(\alpha^2-\beta)^2}{2\alpha^2\beta}\eta(Y)g(\nabla_\xi\xi, X)\xi + \frac{1}{\alpha}\eta(Y)X(\alpha)\xi + \frac{1}{\alpha}\eta(X)Y(\alpha)\xi \\
&+ \frac{\alpha^2-\beta}{2\alpha^2}g(\nabla_X\xi, Y)\xi + \frac{\alpha^2-\beta}{2\alpha^2}g(\nabla_Y\xi, X)\xi,
\end{aligned}$$

where

$$Tr_g g(X, \nabla_\bullet\xi)\bullet = g(X, \nabla_{e_i}\xi)e_i + g(X, \nabla_{\varphi e_i}\xi)\varphi e_i + g(X, \nabla_\xi\xi)\xi.$$

Proof. If we consider an orthonormal frame $\{e_i, \varphi e_i, \xi\}_{i=1}^m$ on the almost contact metric manifold $(M^{2m+1}, \varphi, \xi, \eta, g)$, then an orthonormal frame on $\left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is given by

$$\left\{\overline{e}_i = \frac{1}{\sqrt{\beta}}e_i, \overline{\varphi e}_i = \frac{1}{\sqrt{\beta}}\varphi e_i, \overline{\xi} = \frac{1}{\alpha}\xi\right\}_{i=1}^m.$$

For all $X, Y \in \Gamma(TM)$, we have

$$\overline{\nabla}_X Y = \overline{g}\left(\overline{\nabla}_X Y, \overline{e}_i\right)\overline{e}_i + \overline{g}\left(\overline{\nabla}_X Y, \overline{\varphi e}_i\right)\overline{\varphi e}_i + \overline{g}\left(\overline{\nabla}_X Y, \overline{\xi}\right)\overline{\xi}.$$

By Proposition 2.1, we obtain

$$\begin{aligned}\overline{g}\left(\overline{\nabla}_X Y, \overline{e_i}\right)\overline{e_i} &= g\left(\nabla_X Y, e_i\right)e_i - \frac{\alpha}{\beta}\eta\left(X\right)\eta\left(Y\right)e_i\left(\alpha\right)e_i + \frac{1}{2\beta}\eta\left(X\right)\eta\left(Y\right)e_i\left(\beta\right)e_i \\ &- \frac{1}{2\beta}g\left(X,Y\right)e_i\left(\beta\right)e_i + \frac{\left(\alpha^2-\beta\right)}{2\beta}\eta\left(Y\right)g\left(\nabla_X\xi, e_i\right)e_i \\ &+ \frac{\left(\alpha^2-\beta\right)}{2\beta}\eta\left(X\right)g\left(\nabla_Y\xi, e_i\right)e_i + \frac{1}{2\beta}X\left(\beta\right)g\left(Y, e_i\right)e_i \\ &+ \frac{1}{2\beta}Y\left(\beta\right)g\left(X, e_i\right)e_i - \frac{\left(\alpha^2-\beta\right)}{2\beta}\eta\left(X\right)g\left(\nabla_{e_i}\xi, Y\right)e_i \\ &- \frac{\left(\alpha^2-\beta\right)}{2\beta}\eta\left(Y\right)g\left(\nabla_{e_i}\xi, X\right)e_i.\end{aligned}$$

Similar calculation gives us

$$\begin{aligned}\overline{g}\left(\overline{\nabla}_X Y, \overline{\varphi e_i}\right)\overline{\varphi e_i} &= g\left(\nabla_X Y, \varphi e_i\right)\varphi e_i - \frac{\alpha}{\beta}\eta\left(X\right)\eta\left(Y\right)\left(\varphi e_i\right)\left(\alpha\right)\varphi e_i \\ &+ \frac{1}{2\beta}\eta\left(X\right)\eta\left(Y\right)\left(\varphi e_i\right)\left(\beta\right)\varphi e_i - \frac{1}{2\beta}g\left(X,Y\right)\left(\varphi e_i\right)\left(\beta\right)\varphi e_i \\ &+ \frac{\left(\alpha^2-\beta\right)}{2\beta}\eta\left(Y\right)g\left(\nabla_X\xi, \varphi e_i\right)\varphi e_i + \frac{\left(\alpha^2-\beta\right)}{2\beta}\eta\left(X\right)g\left(\nabla_Y\xi, \varphi e_i\right)\varphi e_i \\ &+ \frac{1}{2\beta}X\left(\beta\right)g\left(Y, \varphi e_i\right)\varphi e_i + \frac{1}{2\beta}Y\left(\beta\right)g\left(X, \varphi e_i\right)\varphi e_i \\ &- \frac{\left(\alpha^2-\beta\right)}{2\beta}\eta\left(X\right)g\left(\nabla_{\varphi e_i}\xi, Y\right)\varphi e_i - \frac{\left(\alpha^2-\beta\right)}{2\beta}\eta\left(Y\right)g\left(\nabla_{\varphi e_i}\xi, X\right)\varphi e_i\end{aligned}$$

and

$$\begin{aligned}\overline{g}\left(\overline{\nabla}_X Y, \overline{\xi}\right)\overline{\xi} &= g\left(\nabla_X Y, \xi\right)\xi + \frac{1}{\alpha}\eta\left(Y\right)X\left(\alpha\right)\xi + \frac{1}{\alpha}\eta\left(X\right)Y\left(\alpha\right)\xi \\ &- \frac{\left(\alpha^2-\beta\right)}{2\alpha^2}\eta\left(X\right)g\left(\nabla_\xi\xi, Y\right)\xi - \frac{\left(\alpha^2-\beta\right)}{2\alpha^2}\eta\left(Y\right)g\left(\nabla_\xi\xi, X\right)\xi \\ &+ \frac{\left(\alpha^2-\beta\right)}{2\alpha^2}g\left(\nabla_X\xi, Y\right)\xi + \frac{\left(\alpha^2-\beta\right)}{2\alpha^2}g\left(\nabla_Y\xi, X\right)\xi \\ &- \frac{1}{\alpha}\eta\left(X\right)\eta\left(Y\right)\xi\left(\alpha\right)\xi + \frac{1}{2\alpha^2}\left\{\eta\left(X\right)\eta\left(Y\right) - g\left(X,Y\right)\right\}\xi\left(\beta\right)\xi,\end{aligned}$$

which gives us

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla_X Y - \frac{\alpha}{\beta}\eta(X)\eta(Y)\,grad\alpha + \frac{1}{2\beta}\eta(X)\eta(Y)\,grad\beta - \frac{1}{2\beta}g(X,Y)\,grad\beta \\
&+ \frac{\alpha^2-\beta}{\alpha\beta}\eta(X)\eta(Y)\xi(\alpha)\xi - \frac{\alpha^2-\beta}{2\alpha^2\beta}\eta(X)\eta(Y)\xi(\beta)\xi + \frac{\alpha^2-\beta}{2\alpha^2\beta}g(X,Y)\xi(\beta)\xi \\
&+ \frac{\alpha^2-\beta}{2\beta}\eta(Y)\nabla_X\xi + \frac{\alpha^2-\beta}{2\beta}\eta(X)\nabla_Y\xi + \frac{1}{2\beta}X(\beta)Y + \frac{1}{2\beta}Y(\beta)X \\
&- \frac{1}{2\beta}\eta(X)Y(\beta)\xi - \frac{1}{2\beta}\eta(Y)X(\beta)\xi - \frac{\alpha^2-\beta}{2\beta}\eta(X)Tr_g g(\nabla_\bullet\xi,Y)\bullet \\
&- \frac{\alpha^2-\beta}{2\beta}\eta(Y)Tr_g g(\nabla_\bullet\xi,X)\bullet + \frac{(\alpha^2-\beta)^2}{2\alpha^2\beta}\eta(X)g(\nabla_\xi\xi,Y)\xi \\
&+ \frac{(\alpha^2-\beta)^2}{2\alpha^2\beta}\eta(Y)g(\nabla_\xi\xi,X)\xi + \frac{1}{\alpha}\eta(Y)X(\alpha)\xi + \frac{1}{\alpha}\eta(X)Y(\alpha)\xi \\
&+ \frac{\alpha^2-\beta}{2\alpha^2}g(\nabla_X\xi,Y)\xi + \frac{\alpha^2-\beta}{2\alpha^2}g(\nabla_Y\xi,X)\xi.\end{aligned}$$

□

By considering the elements of the orthonormal frames of $(M^{2m+1},\varphi,\xi,\eta,g)$ and $\left(M,\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)$, Theorem 2.1 allows us to deduce the following corollary.

Corollary 2.1. *For the orthonormal frame $\{e_i,\varphi e_i,\xi\}_{i=1}^m$ of the almost contact metric manifold $(M^{2m+1},\varphi,\xi,\eta,g)$, we have*

$$\begin{aligned}\overline{\nabla}_{e_i}e_i &= \nabla_{e_i}e_i - \frac{m}{2\beta}grad\beta + \frac{m(\alpha^2-\beta)}{2\alpha^2\beta}\xi(\beta)\xi \\
&+ \frac{1}{\beta}e_i(\beta)e_i - \frac{\alpha^2-\beta}{\alpha^2}\eta(\nabla_{e_i}e_i)\xi,\end{aligned}$$

$$\begin{aligned}\overline{\nabla}_{\varphi e_i}\varphi e_i &= \nabla_{\varphi e_i}\varphi e_i - \frac{m}{2\beta}grad\beta + \frac{m(\alpha^2-\beta)}{2\alpha^2\beta}\xi(\beta)\xi \\
&+ \frac{1}{\beta}(\varphi e_i)(\beta)\varphi e_i - \frac{\alpha^2-\beta}{\alpha^2}\eta(\nabla_{\varphi e_i}\varphi e_i)\xi\end{aligned}$$

and

$$\overline{\nabla}_\xi\xi = \frac{\alpha^2}{\beta}\nabla_\xi\xi - \frac{\alpha}{\beta}grad\alpha + \frac{\alpha^2+\beta}{\alpha\beta}\xi(\alpha)\xi.$$

Similarly, for the orthonormal frame $\left\{\overline{e_i}=\frac{1}{\sqrt{\beta}}e_i,\overline{\varphi e_i}=\frac{1}{\sqrt{\beta}}\varphi e_i,\overline{\xi}=\frac{1}{\alpha}\xi\right\}_{i=1}^m$, we obtain

$$\begin{aligned}\overline{\nabla}_{\overline{e_i}}\overline{e_i} &= \frac{1}{\beta}\nabla_{e_i}e_i - \frac{m}{2\beta^2}grad\beta + \frac{m(\alpha^2-\beta)}{2\alpha^2\beta^2}\xi(\beta)\xi \\
&+ \frac{1}{2\beta^2}e_i(\beta)e_i - \frac{\alpha^2-\beta}{\alpha^2\beta}\eta(\nabla_{e_i}e_i)\xi,\end{aligned}$$

$$\overline{\nabla}_{\overline{\varphi e_i}}\overline{\varphi e_i} = \frac{1}{\beta}\nabla_{\varphi e_i}\varphi e_i - \frac{m}{2\beta^2}grad\beta + \frac{m\left(\alpha^2-\beta\right)}{2\alpha^2\beta^2}\xi\left(\beta\right)\xi$$
$$+\frac{1}{2\beta^2}\left(\varphi e_i\right)\left(\beta\right)\varphi e_i - \frac{\alpha^2-\beta}{\alpha^2\beta}\eta\left(\nabla_{\varphi e_i}\varphi e_i\right)\xi$$

and

$$\overline{\nabla}_{\overline{\xi}}\overline{\xi} = \frac{1}{\beta}\nabla_\xi\xi - \frac{1}{\alpha\beta}grad\alpha + \frac{1}{\alpha\beta}\xi\left(\alpha\right)\xi.$$

From Theorem 2.1, if $(M^{2m+1},\varphi,\xi,\eta,g)$ is a Kenmotsu manifold, we obtain the following Corollary.

Corollary 2.2. *Let $(M^{2m+1},\varphi,\xi,\eta,g)$ be a Kenmotsu manifold, by using the fact that*

$$\nabla_X\xi = X-\eta\left(X\right)\xi, \qquad \nabla_Y\xi = Y-\eta\left(Y\right)\xi, \qquad \nabla_\xi\xi = 0$$

and

$$Tr_g g\left(X,\nabla_\bullet\xi\right)\bullet = X-\eta\left(X\right)\xi, \qquad Tr_g g\left(Y,\nabla_\bullet\xi\right)\bullet = Y-\eta\left(Y\right)\xi,$$

the equation (2.1) becomes

$$\begin{aligned}\overline{\nabla}_X Y = {} & \nabla_X Y - \frac{\alpha}{\beta}\eta\left(X\right)\eta\left(Y\right)grad\alpha + \frac{1}{2\beta}\eta\left(X\right)\eta\left(Y\right)grad\beta\\ & - \frac{1}{2\beta}g\left(X,Y\right)grad\beta + \frac{\alpha^2-\beta}{\alpha\beta}\eta\left(X\right)\eta\left(Y\right)\xi\left(\alpha\right)\xi\\ & - \frac{\alpha^2-\beta}{2\alpha^2\beta}\eta\left(X\right)\eta\left(Y\right)\xi\left(\beta\right)\xi + \frac{\alpha^2-\beta}{2\alpha^2\beta}g\left(X,Y\right)\xi\left(\beta\right)\xi\\ & - \frac{1}{2\beta}\eta\left(X\right)Y\left(\beta\right)\xi - \frac{1}{2\beta}\eta\left(Y\right)X\left(\beta\right)\xi + \frac{1}{\alpha}\eta\left(Y\right)X\left(\alpha\right)\xi\\ & + \frac{1}{\alpha}\eta\left(X\right)Y\left(\alpha\right)\xi + \frac{1}{2\beta}X\left(\beta\right)Y + \frac{1}{2\beta}Y\left(\beta\right)X\\ & - \frac{\alpha^2-\beta}{\alpha^2}\eta\left(X\right)\eta\left(Y\right)\xi + \frac{\alpha^2-\beta}{\alpha^2}g\left(X,Y\right)\xi.\end{aligned} \tag{2.2}$$

In particular, we deduce that

$$\overline{\nabla}_{e_i}e_i = \nabla_{e_i}e_i - \frac{m}{2\beta}grad\beta + \frac{m\left(\alpha^2-\beta\right)}{2\alpha^2\beta}\xi\left(\beta\right)\xi$$
$$+\frac{1}{\beta}e_i\left(\beta\right)e_i + \frac{m\left(\alpha^2-\beta\right)}{\alpha^2}\xi,$$

$$\overline{\nabla}_{\overline{e_i}}\overline{e_i} = \frac{1}{\beta}\nabla_{e_i}e_i - \frac{m}{2\beta^2}grad\beta + \frac{m\left(\alpha^2-\beta\right)}{2\alpha^2\beta^2}\xi\left(\beta\right)\xi$$
$$+\frac{1}{2\beta^2}e_i\left(\beta\right)e_i + \frac{m\left(\alpha^2-\beta\right)}{\alpha^2\beta}\xi,$$

$$\overline{\nabla}_{\varphi e_i}\varphi e_i = \nabla_{\varphi e_i}\varphi e_i - \frac{m}{2\beta}grad\beta + \frac{m\left(\alpha^2-\beta\right)}{2\alpha^2\beta}\xi\left(\beta\right)\xi + \frac{1}{\beta}\left(\varphi e_i\right)\left(\beta\right)\varphi e_i + \frac{m\left(\alpha^2-\beta\right)}{\alpha^2}\xi,$$

$$\overline{\nabla}_{\overline{\varphi e_i}}\overline{\varphi e_i} = \frac{1}{\beta}\nabla_{\varphi e_i}\varphi e_i - \frac{m}{2\beta^2}grad\beta + \frac{m\left(\alpha^2-\beta\right)}{2\alpha^2\beta^2}\xi\left(\beta\right)\xi + \frac{1}{2\beta^2}\left(\varphi e_i\right)\left(\beta\right)\varphi e_i + \frac{m\left(\alpha^2-\beta\right)}{\alpha^2\beta}\xi,$$

$$\overline{\nabla}_{\xi}\xi = -\frac{\alpha}{\beta}grad\alpha + \frac{\alpha^2+\beta}{\alpha\beta}\xi\left(\alpha\right)\xi$$

and

$$\overline{\nabla}_{\overline{\xi}}\overline{\xi} = -\frac{1}{\alpha\beta}grad\alpha + \frac{1}{\alpha\beta}\xi\left(\alpha\right)\xi.$$

As a first result, we study the harmonicity of the identity map $Id_M : (M^{2m+1},\varphi,\xi,\eta,g) \longrightarrow \left(M^{2m+1},\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)$.

2.2. The harmonicity of $Id_M : (M^{2m+1},\varphi,\xi,\eta,g) \longrightarrow \left(M^{2m+1},\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)$.

Theorem 2.2. Let $(M^{2m+1},\varphi,\xi,\eta,g)$ be an almost contact manifold and let $\left(\overline{\varphi}=\varphi,\overline{\xi}=\frac{1}{\alpha}\xi,\overline{\eta}=\alpha\eta,\overline{g}\right)$ be a generalized $\mathcal{D}$-conformal deformation defined on M, where

$$\overline{g} = \beta g + \left(\alpha^2-\beta\right)\eta\otimes\eta.$$

The tension field of $Id_M : (M^{2m+1},\varphi,\xi,\eta,g) \longrightarrow \left(M^{2m+1},\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)$ is given by

$$\tau\left(Id\right) = -\frac{\alpha}{\beta}grad\alpha - \frac{m-1}{\beta}grad\beta + \frac{\alpha^2+\beta}{\alpha\beta}\xi\left(\alpha\right)\xi + \frac{(m-1)\,\alpha^2-m\beta}{\alpha^2\beta}\xi\left(\beta\right)\xi + \frac{\alpha^2-\beta}{\alpha^2}\left(div\xi\right)\xi + \frac{\alpha^2-\beta}{\beta}\nabla_{\xi}\xi.$$

In particular, if $(M^{2m+1},\varphi,\xi,\eta,g)$ is a Kenmotsu manifold, the tension field of the identity map $Id_M : (M^{2m+1},\varphi,\xi,\eta,g) \longrightarrow \left(M^{2m+1},\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)$ becomes

$$\tau\left(Id\right) = -\frac{\alpha}{\beta}grad\alpha - \frac{m-1}{\beta}grad\beta + \frac{\alpha^2+\beta}{\alpha\beta}\xi\left(\alpha\right)\xi + \frac{(m-1)\,\alpha^2-m\beta}{\alpha^2\beta}\xi\left(\beta\right)\xi + \frac{2m\left(\alpha^2-\beta\right)}{\alpha^2}\xi.$$

Proof. By definition, we have

$$\tau(Id) = \overline{\nabla}_{e_i}e_i - \nabla_{e_i}e_i + \overline{\nabla}_{\varphi e_i}\varphi e_i - \nabla_{\varphi e_i}\varphi e_i + \overline{\nabla}_\xi \xi - \nabla_\xi \xi.$$

Using Corollary 2.1, we obtain

$$\begin{aligned}\overline{\nabla}_{e_i}e_i - \nabla_{e_i}e_i &= -\frac{m}{2\beta}grad\beta + \frac{m(\alpha^2-\beta)}{2\alpha^2\beta}\xi(\beta)\xi \\ &\quad + \frac{1}{\beta}e_i(\beta)e_i + \frac{\alpha^2-\beta}{\alpha^2}g(\nabla_{e_i}\xi, e_i)\xi, \\ \overline{\nabla}_{\varphi e_i}\varphi e_i - \nabla_{\varphi e_i}\varphi e_i &= -\frac{m}{2\beta}grad\beta + \frac{m(\alpha^2-\beta)}{2\alpha^2\beta}\xi(\beta)\xi \\ &\quad + \frac{1}{\beta}(\varphi e_i)(\beta)\varphi e_i + \frac{\alpha^2-\beta}{\alpha^2}g(\nabla_{\varphi e_i}\xi, \varphi e_i)\xi\end{aligned}$$

and

$$\overline{\nabla}_\xi \xi - \nabla_\xi \xi = \frac{\alpha^2-\beta}{\beta}\nabla_\xi \xi - \frac{\alpha}{\beta}grad\alpha + \frac{\alpha^2+\beta}{\alpha\beta}\xi(\alpha)\xi,$$

which gives

$$\begin{aligned}\tau(Id) = &-\frac{\alpha}{\beta}grad\alpha - \frac{m}{\beta}grad\beta + \frac{\alpha^2+\beta}{\alpha\beta}\xi(\alpha)\xi + \frac{m(\alpha^2-\beta)}{\alpha^2\beta}\xi(\beta)\xi \\ &+ \frac{1}{\beta}e_i(\beta)e_i + \frac{1}{\beta}(\varphi e_i)(\beta)\varphi e_i + \frac{\alpha^2-\beta}{\alpha^2}g(\nabla_{e_i}\xi, e_i)\xi \\ &+ \frac{\alpha^2-\beta}{\alpha^2}g(\nabla_{\varphi e_i}\xi, \varphi e_i)\xi + \frac{\alpha^2-\beta}{\beta}\nabla_\xi \xi\end{aligned}$$

Using the fact that

$$div\xi = Tr_g g(\nabla_\bullet \xi, \bullet) = g(\nabla_{e_i}\xi, e_i) + g(\nabla_{\varphi e_i}\xi, \varphi e_i),$$

we deduce that

$$\begin{aligned}\tau(Id) = &-\frac{\alpha}{\beta}grad\alpha - \frac{m-1}{\beta}grad\beta + \frac{\alpha^2+\beta}{\alpha\beta}\xi(\alpha)\xi \\ &+ \frac{(m-1)\alpha^2 - m\beta}{\alpha^2\beta}\xi(\beta)\xi + \frac{\alpha^2-\beta}{\alpha^2}(div\xi)\xi + \frac{\alpha^2-\beta}{\beta}\nabla_\xi \xi\end{aligned}$$

If $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a Kenmotsu manifold, then $div\xi = 2m$ and $\nabla_\xi \xi = 0$ and in this case, the tension field of the identity map Id_M becomes

$$\begin{aligned}\tau(Id) = &-\frac{\alpha}{\beta}grad\alpha - \frac{m-1}{\beta}grad\beta + \frac{\alpha^2+\beta}{\alpha\beta}\xi(\alpha)\xi \\ &+ \frac{(m-1)\alpha^2 - m\beta}{\alpha^2\beta}\xi(\beta)\xi + \frac{2m(\alpha^2-\beta)}{\alpha^2}\xi.\end{aligned}$$

□

Corollary 2.3. *The identity map* $Id_M : (M, \varphi, \xi, \eta, g) \longrightarrow \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ *is harmonic if and only if*

$$\alpha^3 grad\alpha + (m-1)\,\alpha^2 grad\beta - \alpha\left(\alpha^2+\beta\right)\xi\left(\alpha\right)\xi - \left((m-1)\,\alpha^2 - m\beta\right)\xi\left(\beta\right)\xi$$
$$-\left(\alpha^2-\beta\right)\beta\left(div\xi\right)\xi - \alpha^2\left(\alpha^2-\beta\right)\nabla_\xi\xi = 0.$$

Moreover, if $(M^{2m+1}, \varphi, \xi, \eta, g)$ *is a Kenmotsu manifold, we conclude that* $Id_M : (M, \varphi, \xi, \eta, g) \longrightarrow \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ *is harmonic if and only if*

$$\alpha^3 grad\alpha + (m-1)\,\alpha^2 grad\beta - \alpha\left(\alpha^2+\beta\right)\xi\left(\alpha\right)\xi$$
$$-\left((m-1)\,\alpha^2 - m\beta\right)\xi\left(\beta\right)\xi - 2m\left(\alpha^2-\beta\right)\beta\xi = 0.$$

Remark 1. *If the functions* α *and* β *depend only on the direction of* ξ*, we conclude that the identity map* $Id_M : (M, \varphi, \xi, \eta, g) \longrightarrow \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ *is harmonic if and only if*

$$\alpha\beta\xi\left(\alpha\right)\xi - m\beta\xi\left(\beta\right)\xi + \left(\alpha^2-\beta\right)\beta\left(div\xi\right)\xi + \alpha^2\left(\alpha^2-\beta\right)\nabla_\xi\xi = 0.$$

In the case where $(M^{2m+1}, \varphi, \xi, \eta, g)$ *is a Kenmotsu manifold, the harmonicity condition of* $Id_M : (M, \varphi, \xi, \eta, g) \longrightarrow \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ *is given by the following equation :*

$$\alpha\xi\left(\alpha\right) - m\xi\left(\beta\right) + 2m\left(\alpha^2-\beta\right) = 0.$$

Example 2.1. [11] We consider the manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$. The Riemannian metric on M is defined by

$$g = \frac{1}{z^2}dx^2 + \frac{1}{z^2}dy^2 + \frac{1}{z^2}dz^2,$$

and the orthonormal frame is given by $e_1 = z\frac{\partial}{\partial x}$, $e_2 = z\frac{\partial}{\partial y}$ and $e_3 = z\frac{\partial}{\partial z}$. The vector fields e_1, e_2 and e_3 satisfy

$$\begin{aligned}
&\nabla_{e_1}e_1 = e_3, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = -e_1,\\
&\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_2}e_3 = -e_2,\\
&\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0.
\end{aligned}$$

If we suppose that the functions α and β depend only on z, we deduce that the tension field of the identity map $Id_M : (M, \varphi, \xi, \eta, g) \longrightarrow \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is given by

$$\tau\left(Id\right) = \frac{1}{\alpha^2}\left(z\alpha\alpha' - z\beta' - 2\left(\alpha^2-\beta\right)\right)\xi.$$

Then the identity map Id_M is harmonic if and only if the functions α and β are solutions of the following differential equation

$$z\alpha\alpha' - z\beta' - 2\left(\alpha^2 - \beta\right) = 0.$$

To solve this last equation, we will give some special solutions :

(1) Looking for particular solutions of type $\alpha = k_1 z^a$ and $\beta = k_2 z^{2a}$ $(a, k_1, k_2 \in \mathbb{R}^*, k_1, k_2 > 0\ )$. The identity map Id_M is harmonic if and only if

$$a = \frac{2\left(k_1^2 - k_2\right)}{k_1^2 - 2k_2}.$$

For example, if $k_1 = 2$ and $k_2 = 1$, we find $a = 3$, then $\alpha = 2z^3$ and $\beta = z^6$.

(2) Other particular solutions are given by $\alpha = C_1 z^2$ and $\beta = C_2 z^2$, where $C_1, C_2 > 0$.

Example 2.2. [8] We consider the manifold $M = \{(x, y, z) \in \mathbb{R}^3, \}$. The Riemannian metric on M is defined by

$$g = \frac{1}{e^{2z}} dx^2 + \frac{1}{e^{2z}} dy^2 + dz^2,$$

and the orthonormal frame is given by $e_1 = e^z \frac{\partial}{\partial x}$, $e_2 = e^z \frac{\partial}{\partial y}$ and $e_3 = \frac{\partial}{\partial z}$. The vector fields e_1, e_2 and e_3 satisfy

$$\begin{array}{lll} \nabla_{e_1} e_1 = e_3, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = -e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = e_3, & \nabla_{e_2} e_3 = -e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = 0. \end{array}$$

If we suppose that the functions α and β depend only on z, we deduce that the identity map $Id_M : (M, \varphi, \xi, \eta, g) \longrightarrow \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is harmonic if and only if

$$\alpha\alpha' - \beta' + 2\left(\alpha^2 - \beta\right) = 0. \tag{2.3}$$

To solve this last equation, we will give some special solutions:

(1) Particular solutions are given by : $\alpha = C_1 e^{-2z}$ and $\beta = C_2 e^{-2z}$ $(C_1, C_2 \in \mathbb{R}^*, C_1, C_2 > 0\ )$.

(2) Looking for particular solutions of type $\alpha = C_1 e^{az}$ and $\beta = C_2 e^{bz}$, we obtain $b = 2a$ and $(a + 2)\, C_1^2 - 2\,(a + 1)\, C_2 = 0$.
Then $\alpha(z) = C_1 e^{az}$ and $\beta(z) = \frac{(a+2)}{2(a+1)} C_1^2 e^{2az}$.

Example 2.3. [7] We consider the manifold $M = \mathbb{R}^5$. An orthonormal frame is given by $e_1 = e^{-v}\frac{\partial}{\partial x}$, $e_2 = e^{-v}\frac{\partial}{\partial y}$ and $e_3 = e^{-v}\frac{\partial}{\partial z}$, $e_4 = e^{-v}\frac{\partial}{\partial u}$ and $e_5 = e^{-v}\frac{\partial}{\partial v}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. Taking $e_5 = \xi$ and using Koszul's formula, we get the following

$$\begin{array}{lllll}
\nabla_{e_1}e_1 = -e_5, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = 0, & \nabla_{e_1}e_5 = e_1 \\
\nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = -e_5, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = 0, & \nabla_{e_2}e_5 = e_2 \\
\nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = -e_5, & \nabla_{e_3}e_4 = 0, & \nabla_{e_3}e_5 = e_3 \\
\nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = -e_5, & \nabla_{e_4}e_5 = e_4 \\
\nabla_{e_5}e_1 = 0, & \nabla_{e_5}e_2 = 0, & \nabla_{e_5}e_3 = 0, & \nabla_{e_5}e_4 = 0, & \nabla_{e_5}e_5 = 0.
\end{array}$$

If we suppose that the functions α and β depend only on v, the identity map Id_M is harmonic if and only if the functions α and β are solutions of the following differential equation

$$e^{-v}\alpha\alpha' + 4\alpha^2 - 2e^{-v}\beta' - 4\beta = 0. \tag{2.4}$$

To solve this last equation, we will give some special solutions:

(1) A simple calculation proves that the functions $\alpha = k_1 e^{-4e^v}$ and $\beta = k_2 e^{-2e^v}$ $(k_1, k_2 > 0)$ are solutions of (2.4).
(2) Looking for particular solutions of type $\alpha = k_1 e^{ae^v}$ and $\beta = k_2 e^{be^v}$ $(a, b, k_1, k_2 \in \mathbb{R}^*, k_1, k_2 > 0\ )$, we obtain $\alpha = k_1 e^{ae^v}$ and $\beta = k_2 e^{2ae^v}$, where $a = \frac{4\left(k_2 - k_1^2\right)}{k_1^2 - 4k_2}$. For example, if $k_1 = k_2 = 2$, we find $a = 2$ and $b = 4$, then $\alpha = 2e^{2e^v}$ and $\beta = 2e^{4e^v}$.

Example 2.4. [10] We consider $m \in \mathbb{N}^*$ and the manifold $M = \mathbb{R}^{2m+1}$. The Riemannian metric on M is defined by

$$g = e^{2(z+e^z)}dx_i^2 + e^{-2(z-e^z)}dy_i^2 + e^{2z}dz^2,$$

and the orthonormal frame is given by

$$X_i = e^{-(z+e^z)}\frac{\partial}{\partial x_i}, \qquad Y_i = e^{z-e^z}\frac{\partial}{\partial y_i}, \qquad \xi = e^{-z}\frac{\partial}{\partial z}, i = 1, \dots, m.$$

The vector fields X_i, Y_i and ξ satisfy

$$\begin{aligned}
&\nabla_{X_i}X_j = -\left(1+e^{-z}\right)\delta_{ij}\xi, && \nabla_{X_i}Y_j = 0, && \nabla_{X_i}\xi = \left(1+e^{-z}\right)X_i,\\
&\nabla_{Y_i}Y_j = -\left(1-e^{-z}\right)\delta_{ij}\xi, && \nabla_{Y_i}X_j = 0, && \nabla_{Y_i}\xi = \left(1-e^{-z}\right)Y_i,\\
&\nabla_{\xi}X_i = 0, && \nabla_{\xi}Y_i = 0, && \nabla_{\xi}\xi = 0.
\end{aligned}$$

If we suppose that the functions α and β depend only on z, we deduce that the identity map $Id_M : (M = \mathbb{R}^{2m+1}, \varphi, \xi, \eta, g) \longrightarrow \left(M = \mathbb{R}^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is harmonic if and only if the functions α and β are solutions of the following differential equation

$$e^{-z}\alpha\alpha' + 2\alpha^2 - me^{-z}\beta' - 2\beta = 0.$$

Particular solutions of this last equation are given by the following system

$$\begin{cases} e^{-z}\alpha' + 2\alpha = 0 \\ me^{-z}\beta' + 2\beta = 0 \end{cases}$$

which leads us to $\alpha = C_1 e^{2e^{-z}}$ and $\beta = C_2 e^{\frac{2}{m}e^{-z}}$, where $C_1, C_2 > 0$.

2.3. **The harmonicity of** $Id : \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$.

Theorem 2.3. Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be an almost contact manifold and let $\left(\overline{\varphi} = \varphi, \overline{\xi} = \frac{1}{\alpha}\xi, \overline{\eta} = \alpha\eta, \overline{g}\right)$ be a generalized $\mathcal{D}$-conformal deformation defined on M, where

$$\overline{g} = \beta g + \left(\alpha^2 - \beta\right)\eta \otimes \eta.$$

The tension field of $Id : \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$ is given by

$$\begin{aligned}
\tau(Id) = {}& \frac{1}{\alpha\beta} grad\alpha + \frac{m-1}{\beta^2} grad\beta - \frac{\alpha^2+\beta}{\alpha^3\beta}\xi(\alpha)\xi \\
& - \frac{(m-1)\alpha^2 - m\beta}{\alpha^2\beta^2}\xi(\beta)\xi - \frac{\alpha^2-\beta}{\alpha^2\beta}(div\xi)\xi - \frac{\alpha^2-\beta}{\alpha^2\beta}\nabla_{\xi}\xi.
\end{aligned}$$

In particular, if $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a Kenmotsu manifold, the tension field of the identity map becomes

$$\begin{aligned}
\tau(Id) = {}& \frac{1}{\alpha\beta} grad\alpha + \frac{m-1}{\beta^2} grad\beta - \frac{\alpha^2+\beta}{\alpha^3\beta}\xi(\alpha)\xi \\
& - \frac{(m-1)\alpha^2 - m\beta}{\alpha^2\beta^2}\xi(\beta)\xi - \frac{2m\left(\alpha^2-\beta\right)}{\alpha^2\beta}\xi.
\end{aligned}$$

Proof. By definition, we have

$$\tau(Id) = \nabla_{\overline{e_i}}\overline{e_i} - \overline{\nabla}_{\overline{e_i}}\overline{e_i} + \nabla_{\overline{\varphi e_i}}\overline{\varphi e_i} - \overline{\nabla}_{\overline{\varphi e_i}}\overline{\varphi e_i} + \nabla_{\overline{\xi}}\overline{\xi} - \overline{\nabla}_{\overline{\xi}}\overline{\xi}.$$

A simple calculation gives us

$$\nabla_{\overline{e_i}}\overline{e_i} = \frac{1}{\beta}\nabla_{e_i}e_i - \frac{1}{2\beta^2}e_i(\beta)\,e_i,$$

$$\nabla_{\overline{\varphi e_i}}\overline{\varphi e_i} = \frac{1}{\beta}\nabla_{\varphi e_i}\varphi e_i - \frac{1}{2\beta^2}(\varphi e_i)(\beta)\,\varphi e_i$$

and

$$\nabla_{\overline{\xi}}\overline{\xi} = \frac{1}{\alpha}\nabla_{\xi}\frac{1}{\alpha}\xi = \frac{1}{\alpha^2}\nabla_{\xi}\xi - \frac{1}{\alpha^3}\xi(\alpha)\,\xi.$$

Then

$$\begin{aligned}\nabla_{\overline{e_i}}\overline{e_i} - \overline{\nabla}_{\overline{e_i}}\overline{e_i} &= \frac{m}{2\beta^2}grad\beta - \frac{1}{\beta^2}e_i(\beta)\,e_i - \frac{m(\alpha^2-\beta)}{2\alpha^2\beta^2}\xi(\beta)\,\xi \\ &+ \frac{\alpha^2-\beta}{\alpha^2\beta}\eta(\nabla_{e_i}e_i)\,\xi,\end{aligned}$$

$$\begin{aligned}\nabla_{\overline{\varphi e_i}}\overline{\varphi e_i} - \overline{\nabla}_{\overline{\varphi e_i}}\overline{\varphi e_i} &= \frac{m}{2\beta^2}grad\beta - \frac{1}{\beta^2}(\varphi e_i)(\beta)\,\varphi e_i - \frac{m(\alpha^2-\beta)}{2\alpha^2\beta^2}\xi(\beta)\,\xi \\ &+ \frac{\alpha^2-\beta}{\alpha^2\beta}\eta(\nabla_{\varphi e_i}\varphi e_i)\,\xi\end{aligned}$$

and

$$\nabla_{\overline{\xi}}\overline{\xi} - \overline{\nabla}_{\overline{\xi}}\overline{\xi} = \frac{1}{\alpha\beta}grad\alpha - \frac{\alpha^2-\beta}{\alpha^2\beta}\nabla_{\xi}\xi - \frac{\alpha^2+\beta}{\alpha^3\beta}\xi(\alpha)\,\xi.$$

It follows that

$$\begin{aligned}\tau(Id) &= \frac{1}{\alpha\beta}grad\alpha + \frac{m-1}{\beta^2}grad\beta - \frac{\alpha^2+\beta}{\alpha^3\beta}\xi(\alpha)\,\xi \\ &- \frac{(m-1)\,\alpha^2 - m\beta}{\alpha^2\beta^2}\xi(\beta)\,\xi - \frac{\alpha^2-\beta}{\alpha^2\beta}(div\xi)\,\xi - \frac{\alpha^2-\beta}{\alpha^2\beta}\nabla_{\xi}\xi.\end{aligned}$$

If $(M^{2m+1},\varphi,\xi,\eta,g)$ is a Kenmotsu manifold, then $div\xi = 2m$ and $\nabla_\xi\xi = 0$ and the tension field of the identity map Id becomes

$$\begin{aligned}\tau(Id) &= \frac{1}{\alpha\beta}grad\alpha + \frac{m-1}{\beta^2}grad\beta - \frac{\alpha^2+\beta}{\alpha^3\beta}\xi(\alpha)\,\xi \\ &- \frac{(m-1)\,\alpha^2 - m\beta}{\alpha^2\beta^2}\xi(\beta)\,\xi - \frac{2m(\alpha^2-\beta)}{\alpha^2\beta}\xi.\end{aligned}$$

□

Remark 2. *If the functions* α *and* β *depend only on the direction of* ξ*, then the tension field of the identity map* $Id : \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$ *takes the following form*

$$\tau(Id) = -\frac{1}{\alpha^3}\xi(\alpha)\xi + \frac{m}{\alpha^2\beta}\xi(\beta)\xi - \frac{\alpha^2-\beta}{\alpha^2\beta}(div\xi)\xi - \frac{\alpha^2-\beta}{\alpha^2\beta}\nabla_\xi\xi,$$

and in the case where $(M^{2m+1}, \varphi, \xi, \eta, g)$ *is a Kenmotsu manifold, the tension field of the identity map is given by the following formula*

$$\tau(Id) = -\frac{1}{\alpha^3}\xi(\alpha)\xi + \frac{m}{\alpha^2\beta}\xi(\beta)\xi - \frac{2m(\alpha^2-\beta)}{\alpha^2\beta}\xi.$$

Example 2.5. [3] We consider the manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$. The Riemannian metric on M is defined by

$$g = \frac{1}{z^2}dx^2 + \frac{1}{z^2}dy^2 + \frac{1}{z^2}dz^2,$$

and the orthonormal frame is given by $e_1 = z\frac{\partial}{\partial x}$, $e_2 = z\frac{\partial}{\partial y}$ and $e_3 = -z\frac{\partial}{\partial z}$. The vector fields e_1, e_2 and e_3 satisfy

$$\begin{aligned} &\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = e_1, \\ &\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_3 = e_2, \\ &\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0. \end{aligned}$$

If we suppose that the functions α and β depend only on z, we deduce that the identity map $Id : \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow (M, \varphi, \xi, \eta, g)$ is harmonic if and only if the functions α and β are solutions of the following differential equation

$$z\beta\alpha' - z\alpha\beta' - 2\alpha^3 + 2\alpha\beta = 0.$$

Let us look for particular solutions of type $\alpha = k_1z^a$ and $\beta = k_2z^{2a}$ where $(k_1, k_2 > 0)$. The identity map Id is harmonic if and only if $k_2 = \frac{2}{2-a}k_1^2, a < 2$, which gives us $\alpha = k_1z^a$ and $\beta = \frac{2}{2-a}k_1^2z^{2a}$ where $a < 2$. For example, if we take $a = -2$, we get $\alpha = \frac{k_1}{z^2}$ and $\beta = \frac{k_1^2}{2z^4}$.

Example 2.6. [10] We consider the manifold

$$M = \left\{(x_1, \cdots, x_m, y_1, \cdots, y_m, t) \in \mathbb{R}^{2m+1}, z > 0\right\}$$

with standard coordinates $(x_1, \cdots, x_m, y_1, \cdots, y_m, t)$. Let

$$\xi = -t\frac{\partial}{\partial t}, \quad X_i = t\left(1+t^2\right)\frac{\partial}{\partial x_i} + t^3\frac{\partial}{\partial y_i}, \quad Y_i = -t^3\frac{\partial}{\partial x_i} + t\left(1-t^2\right)\frac{\partial}{\partial y_i}$$

be a global basis of M. If we suppose that the functions α and β depend only on t, we deduce that the identity map $Id : \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$ is harmonic if and only if the functions α and β are solutions of the following differential equation

$$t\beta\alpha' - mt\alpha\beta' - 2m\alpha^3 + 2m\alpha\beta = 0. \tag{2.5}$$

For this equation we can give some particular solutions:

(1) As first particular solutions, we obtain: $\alpha = \sqrt{\frac{C}{m}}t$ and $\beta = Ct^2$ $(C > 0)$.
(2) Other particular solutions are $\alpha = Ct^{-2m}$ and $\beta = \frac{C^2}{2m}t^{-4m}$ $(C > 0)$.

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