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# CERTAIN SEMIPRIME MODULES

H. KHABAZIAN

Abstract. In this work, we introduce a certain semiprime modules called "semi-vital" and show that a ring $R$ is semiprime iff $R$ is a semi-vital $R$-module. Then, we collect some basic properties concerning semi-vital modules.

## 1. Introduction

One of the important properties of a semiprime ring is that for any ideal $J$, $ann_l(J)$ is the unique largest right ideal having zero intersection with $J$. We show that this property exists in every semi-vital module for any annihilator submodule. This property is very helpful in the decomposition theory. In semiprime and semi-vital modules, the class of submodules are concerned and in $l\mathbb{AI}$-semiprime rings [4], the class of left annihilator ideals are concerned. We also introduce $\mathbb{AM}$-semi-vital modules in which the attention are focused on the class of annihilator submodules. Furthermore, we introduce $\mathbb{EM}$-semi-oltimate modules in which the attention are focused on the class of eliminator submodules.

In this paper, these modules are investigated and various facts are obtained.

Through the paper we apply the notations introduced in [5], [6] and [7]. Some of these notations are as follow.

For any set $S$ of subgroups of an additive group, we set $\Sigma(S) = \sum_{I \in S} I$. For any class $\mathcal{C}$ of subgroups, a $\mathcal{C}$-subgroup means a subgroup from the class $\mathcal{C}$, the class of minimal $\mathcal{C}$-subgroups is denoted by $\mathcal{C}^{mn}$ and the set of $\mathcal{C}$-subgroups of $M$ is denoted by $\langle \mathcal{C} \colon M \rangle$, also for a subgroup $K \subseteq M$, the set of $\mathcal{C}$-subgroups of $M$ having zero intersection

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with $K$ is shown by $\langle \mathcal{C} \,|\, K\rangle$ and we set $C_{\mathcal{C}}(K) = \Sigma\langle \mathcal{C} \,|\, K\rangle$.
Moreover, the class of eliminator submodules is shown by $\mathbb{EM}$, the class of submodules is shown by $\mathbb{M}$, the class of annihilator submodules is shown by $\mathbb{AM}$, and the class of submodules $P$ for which $P \cap P^\circ = 0$ and $P^{\circ\circ} = P$ is shown by $\mathbb{C}$, the class of subgroups $P$ for which $P \cap P^\bullet = 0$ is shown by $I\mathbb{D}$, and the class of submodules $P$ for which $P \cap P^\bullet = 0$ and $P^{\bullet\bullet} = P$ is shown by $\mathbb{B}$. Furthermore, we set $P^\circ = ann_M((P\!:\!M))$ and $Ecl_M(M/P)$ is shown by $P^\bullet$ [6].
For more information about prime submodules, semiprime submodules, prime modules and semiprime modules see [2], [9], [10] and [11].

## 2. $I\mathbb{F}$-submodules and $\mathbb{C}$-submodules

**Definition 2.1.** *The class of subgroups $P$ for which $P \cap P^\circ = 0$ is denoted by $I\mathbb{F}$.*

**Lemma 2.1.** *Let $M$ be an $R$-module. If $K$ is an annihilator submodule of $L$ and $L$ is an annihilator submodule of $M$, then $K$ is an annihilator submodule of $M$.*

*Proof.* We have $L = ann_M(ann_R(L))$, so

$$K = ann_L(ann_R(K)) = L \cap ann_M(ann_R(K)) =$$

$$ann_M(ann_R(L)) \cap ann_M(ann_R(K)) = ann_M(ann_R(L) + ann_R(K))$$

Thus, $K$ is an annihilator submodule of $M$.

**Lemma 2.2.** *Let $M$ be an $R$-module.*

(1) *For every submodules $L$ and $N$, $L \subseteq N$ implies $N^\circ \subseteq L^\circ$.*
(2) *For every submodule $K$, $C_{\mathbb{AM}}(K) \subseteq C_{\mathbb{M}}(K) \subseteq K^\circ$.*
(3) *For every $I\mathbb{F}$-submodule $K$, $C_{\mathbb{AM}}(K) = C_{\mathbb{M}}(K) = K^\circ$ and $C_{\mathbb{AM}}(K) \cap K = 0$.*

*Proof.* (1) We have $ann_R(M/L) \subseteq ann_R(M/N)$. Thus, $N^\circ \subseteq L^\circ$.
(2) Follows from [5, (3-3)].
(3) $K^\circ$ is an annihilator submodule, so $K^\circ \subseteq C_{\mathbb{AM}}(K)$. Applying (2) completes the proof.

**Lemma 2.3.** *Let $M$ be a cofaithful $R$-module and $K$ be a submodule.*

(1) $C_{\mathbb{AM}}(K) \subseteq K^\times \subseteq C_{\mathbb{M}}(K)$.

(2) *$K$ is an $I\mathbb{F}$-submodule iff $K^\circ = K^\times$. In this case $K$ is an $I\mathbb{D}$-submodule, $K^\circ = K^\bullet$, $K \subseteq K^{\circ\circ}$ and $K^\circ$ is closed.*

*Proof.* (1) Every annihilator submodule is an eliminator submodule, so applying [6, (1-11)] completes the proof.

(2⇒) $K^\circ$ is an annihilator submodule, so is an eliminator submodule, thus $K^\circ \subseteq K^\times$ by [6, (1-11)]. On the other hand $K^\times \subseteq K^\bullet \subseteq K^\circ$ by [6, (2-9)] implying $K^\circ = K^\bullet = K^\times$. Thus, $K$ is an $I\mathbb{D}$-submodule and $K^\circ$ is closed by [6, (2-10)]. Finally $K \subseteq K^{\circ\circ}$ by [5, (3-3)].

(2⇐) Follows from [6, (1-11)].

**Lemma 2.4.** *Let $M$ be an $R$-module and $K$ be a submodule.*

(1) *For any annihilator submodule $J$, $M = K \oplus J$ implies that $K^\circ = J$.*
(2) *For any $I\mathbb{F}$-submodule $J$, $M = K \oplus J$ implies that $K = J^\circ$.*

*Proof.* (1) We have $K^\circ = ann_M(ann_R(M/K)) = ann_M(ann_R(J)) = J$.

(2) $K \subseteq J^\circ$ by [5, (3-3)], so $K = J^\circ$.

**Lemma 2.5.** *Let $M$ be an $R$-module and $K$ be an $I\mathbb{F}$-submodule.*

(1) *For any submodule $J$, $K \subseteq_e J$ implies that $J^\circ = K^\circ$ and $J$ is an $I\mathbb{F}$-submodule.*
(2) *If $K^\circ$ is also an $I\mathbb{F}$-submodule, then $K^\circ$ is a $\mathbb{C}$-submodule and $K \subseteq_e K^{\circ\circ}$.*

*Proof.* (1) We have $K^\circ \cap J = 0$, implying $K^\circ \subseteq J^\circ$ by [5, (3-3)]. On the other hand $J^\circ \subseteq K^\circ$ by (1-3). Thus, $J^\circ = K^\circ$. Finally, we have $J^\circ \cap K = 0$, implying $J^\circ \cap J = 0$.

(2) We have $K \subseteq K^{\circ\circ}$ by (1-4), so $K^{\circ\circ\circ} \subseteq K^\circ$ by (1-3). Also we have $K^\circ \subseteq K^{\circ\circ\circ}$ by (1-4). Thus, $K^{\circ\circ\circ} = K^\circ$. Finally let $J \subseteq K^{\circ\circ}$ be a submodule with $K \cap J = 0$. Then, $J \subseteq K^\circ$ by [5, (3-3)], implying $J = 0$.

**Lemma 2.6.** *Let $M$ be an $R$-module.*

(1) *For every $\mathbb{C}$-submodule $K$, $K^\circ$ is also a $\mathbb{C}$-submodule.*
(2) *Every $\mathbb{C}$-submodule is an annihilator submodule.*
(3) *If $M$ is cofaithful, then every $\mathbb{C}$-submodule is a $\mathbb{B}$-submodule.*
(4) *Any two $I\mathbb{F}$-submodule with zero intersection are block orthogonal.*

*Proof.* (1) Set $L = K^\circ$. Since $L^\circ = K^{\circ\circ} = K$, $L^{\circ\circ} = K^\circ = L$. Also, $K^\circ \cap K^{\circ\circ} = 0$, because $L \cap L^\circ = 0$.

(2) Let $K$ be a $\mathbb{C}$-submodule. Set $L = K^\circ$. Then, $K = L^\circ$, on the other hand, $L^\circ$ is an annihilator submodule.

(3) Follows from (1-4).

(4) Follows from [5, (3-3)].

**Proposition 2.1.** *Let $M$ be an $R$-module.*

(1) *Every $\mathbb{B}$-submodule is a completely $\mathbb{EM}$-closed eliminator submodule.*
(2) *Every $\mathbb{M} \cap \mathbb{IF}$-summand submodule is an annihilator submodule.*
(3) *Every $\mathbb{AM}$-summand submodule $K$ is an $I\mathbb{F}$-submodule and $M = K \oplus K^\circ$.*
(4) *Every $\mathbb{AM}$-summand annihilator submodule is a $\mathbb{C}$-summand $\mathbb{C}$-submodule.*
(5) *Every $\mathbb{AM} \cap \mathbb{IF}$-summand submodule is a $\mathbb{C}$-summand $\mathbb{C}$-submodule.*
(6) *Every $\mathbb{C}$-summand submodule is a $\mathbb{C}$-summand $\mathbb{C}$-submodule.*
(7) *Every direct summand annihilator $I\mathbb{F}$-submodule is a $\mathbb{C}$-summand $\mathbb{C}$-submodule.*
(8) *Every direct summand $\mathbb{C}$-submodule is a $\mathbb{C}$-summand.*
(9) *If $M$ is cofaithful, then every $\mathbb{C}$-submodule is closed.*
(10) *For every $\mathbb{C}$-summand $\mathbb{C}$-submodule $K$, $K^\circ$ is also a $\mathbb{C}$-summand $\mathbb{C}$-submodule.*
(11) *Every $\mathbb{C}$-submodule is a completely $\mathbb{AM}$-closed annihilator submodule.*

*Proof.* (1) Let $K$ be a $\mathbb{B}$-submodule and $L$ be an eliminator submodule with $K \sqsubseteq_e^{\mathbb{EM}} L$. Then, $L \cap K^\bullet = 0$, implying $L \subseteq K^{\bullet\bullet} = K$ by [6, (2-9)].

(2) Let $K$ be an $\mathbb{M}\cap\mathbb{IF}$-summand submodule. There exists an $I\mathbb{F}$-submodule $J$ with $M = K \oplus J$. Then, $J^\circ = K$ by (1-5), implying that $K$ is an annihilator submodule.

(3) There exists an annihilator submodule $J$ with $M = K \oplus J$. Then, $K^\circ = J$ by (1-5), implying $K \cap K^\circ = 0$.

(4) Let $K$ be an $\mathbb{AM}$-summand annihilator submodule. There exists an annihilator submodule $J$ with $M = K \oplus J$. Then, $K^\circ = J$ and $J^\circ = K$ by (1-5), implying $K^{\circ\circ} = K$ and $J^{\circ\circ} = J$. Also, $K^\circ \cap K = 0$ and $J^\circ \cap J = 0$.

(5) Let $K$ be an $\mathbb{AM} \cap \mathbb{IF}$-summand submodule. There exists an annihilator $I\mathbb{F}$-submodule $J$ with $M = K \oplus J$. Then, $K^\circ = J$ and $J^\circ = K$ by (1-5), implying $K^{\circ\circ} = K$ and $J^{\circ\circ} = J$. Also, $K^\circ \cap K = 0$ and $J^\circ \cap J = 0$.

(6) Follows from (5).

(7) Let $K$ be a direct summand annihilator $I\mathbb{F}$-submodule. There exists a submodule $J$ with $M = K \oplus J$. Then, $K^\circ = J$ and $J^\circ = K$ by (1-5), implying $K^{\circ\circ} = K$ and $J^{\circ\circ} = J$. Also, $K^\circ \cap K = 0$ and $J^\circ \cap J = 0$.

(8) Follows from (7).

(9) Let $K$ be a $\mathbb{C}$-submodule. Set $L = K^\circ$. $L$ is a $\mathbb{C}$-submodule by (1-7), so $L^\circ$ is closed by (1-4). On the other hnd, $L^\circ = K$.

(10) We have $M = K \oplus K^\circ$ by (3). Also $K^\circ$ is a $\mathbb{C}$-submodule by (1-7).

(11) Let $K$ be a $\mathbb{C}$-submodule and $L$ be an annihilator submodule with $K \sqsubseteq_e^{\mathbb{AM}} L$. Then, $L \cap K^\circ = 0$, implying $L \subseteq K^{\circ\circ} = K$ by [5, (3-3)]. Notice that (1-8) shows that the classes $\mathbb{AM} \cap \mathbb{AM}^\oplus$ and $\mathbb{C} \cap \mathbb{C}^\oplus$ are identical. It means that in every module $M$, $\langle \mathbb{C} \cap \mathbb{C}^\oplus : \mathbb{M} \rangle = \langle \mathbb{AM} \cap \mathbb{AM}^\oplus : \mathbb{M} \rangle$, thus $\langle (\mathbb{C} \cap \mathbb{C}^\oplus)^{\gtrdot\ltimes} : \mathbb{M} \rangle = \langle (\mathbb{AM} \cap \mathbb{AM}^\oplus)^{\gtrdot\ltimes} : \mathbb{M} \rangle$. Also every $\mathbb{AM}$-summand annihilator submodule is a $\mathbb{EM}$-summand eliminator submodule. It means that in every module $M$, $\langle \mathbb{AM} \cap \mathbb{AM}^\oplus : \mathbb{M} \rangle \subseteq \langle \mathbb{EM} \cap \mathbb{EM}^\oplus : \mathbb{M} \rangle$. Moreover, $\langle \mathbb{C} : \mathbb{M} \rangle \subseteq \langle \mathbb{B} : \mathbb{M} \rangle$.

**Lemma 2.7.** *Let $M$ be an $R$-module and $K$ be an $I\mathbb{F}$-submodule. The following are equivalent:*

1. *$K$ is essential in a $\mathbb{C}$-summand $\mathbb{C}$-submodules.*
2. *$K^\circ$ is $\mathbb{C}$-summand $\mathbb{C}$-submodules and $K \subseteq_e K^{\circ\circ}$.*
3. *$K^\circ$ is an $I\mathbb{F}$-submodule and $K^{\circ\circ}$ is a direct summand.*
4. *$K^\circ$ is a direct summand $I\mathbb{F}$-submodule.*
5. *$K$ is essential in a direct summand annihilator submodule.*

*Proof.* (5⇒1 and 2) $K$ is essential in a direct summand annihilator submodule $J$. Then, $K^\circ = J^\circ$ and $J$ is an $I\mathbb{F}$-submodule by (1-6). So, $J$ is $\mathbb{C}$-summand $\mathbb{C}$-submodule by (1-8), implying $K^{\circ\circ} = J$. Thus, $K^\circ$ and $K^{\circ\circ}$ are $\mathbb{C}$-summand $\mathbb{C}$-submodules by (1-8) and $K \subseteq_e K^{\circ\circ}$.

(2⇒3 and 4) We have $M = K^\circ \oplus K^{\circ\circ}$ by (1-8).

(3⇒5) $K^\circ$ is a $\mathbb{C}$-submodule and $K \subseteq_e K^{\circ\circ}$ by (1-6).

(4⇒5) $K^\circ$ is a $\mathbb{C}$-summand $\mathbb{C}$-submodule by (1-8) and $K \subseteq_e K^{\circ\circ}$ by (1-6). Thus, $K^{\circ\circ}$ is a $\mathbb{C}$-summand $\mathbb{C}$-submodule by (1-8).

**Lemma 2.8.** *Let $M$ be an $R$-module. Every annihilator $I\mathbb{F}$-submodule is essential in a direct summand annihilator $I\mathbb{F}$-submodule iff*

(1) *For any annihilator $I\mathbb{F}$-submodule $K$, $K^\circ$ is an $I\mathbb{F}$-submodule.*
(2) *Any $\mathbb{C}$-submodules is a direct summand.*

In this case, every $\mathbb{C}$-submodules is a $\mathbb{C}$-summand. *Proof.* ($\Rightarrow$) (1) $K$ is essential in a direct summand annihilator $I\mathbb{F}$-submodule. Then, $K^\circ$ is an $I\mathbb{F}$-submodule by (1-9).
(2) Let $K$ be a $\mathbb{C}$-submodule. $K$ is essential in a direct summand annihilator $I\mathbb{F}$-submodule $J$. Then, $K = J$ because $K$ is closed by (1-8). Thus, $K$ is a direct summand. Therefore, $K$ is a $\mathbb{C}$-summand by (1-8).
($\Leftarrow$) Let $K$ be an annihilator $I\mathbb{F}$-submodule. $K^\circ$ is an $I\mathbb{F}$-submodule, so $K^{\circ\circ}$ is a $\mathbb{C}$-submodule and $K \subseteq_e K^{\circ\circ}$ by (1-6). On the other hand, $K^{\circ\circ}$ is a direct summand.

**Proposition 2.2.** *For every module, the map given by $I \longrightarrow I^\circ$ is a $\mathbb{C}$-organizer map.*

*Proof.* The map is well defined by (1-7). The rest is obvious by [5, (3-3)].

**Proposition 2.3.** *Every module is $\mathbb{AM}$-intersection, $\mathbb{M} \cap \mathbb{IF}$-intersection and $\mathbb{C}$-intersection.*

*Proof.* Let $M$ be an $R$-module. It is easy to see that $M$ is $\mathbb{AM}$-intersection. Also, $M$ is $\mathbb{M} \cap \mathbb{IF}$-intersection by [5, (3-3)]. Let $I$ and $J$ be be a $\mathbb{C}$-submodules. $I \cap J$ is a $I\mathbb{F}$-submodule, so $(I \cap J) \cap (I \cap J)^\circ = 0$, implying $I \cap J \subseteq (I \cap J)^{\circ\circ}$ by [5, (3-3)]. On the other hand, $(I \cap J)^{\circ\circ} \subseteq I^{\circ\circ} = I$ and similarly, $(I \cap J)^{\circ\circ} \subseteq J$, implying $(I \cap J)^{\circ\circ} \subseteq I \cap J$. Thus, $(I \cap J)^{\circ\circ} = I \cap J$.

**Lemma 2.9.** *Let $M$ be an $R$-module and $I_1$, $I_2$, ..., $I_n$ be annihilator submodules. For any submodule $N$, if $I_j \cap N = 0$ for all $1 \leq j \leq n$, then $ann_M(ann_R(I_1)ann_R(I_2) \cdots ann_R(I_n)) \cap N = 0$.*

*Proof.* By the induction on $n$. Set $J = ann_R(I_1)ann_R(I_2) \cdots ann_R(I_{n-1})$ and $K = ann_M(Jann_R(I_n)) \cap N$. Then, $KJann_R(I_n) = 0$, so $KJ \subseteq I_n$, thus $KJ \subseteq I_n \cap N = 0$, implying $K \subseteq ann_M(J)$. On the other hand $ann_M(J) \cap N = 0$ by the induction. Therefore, $K = 0$.

**Lemma 2.10.** *In any* $\mathbb{AM}$*-Noetherian module, every submodule is* $\mathbb{AM}$*-separable and* $\mathbb{AM}$*-perfect.*

*Proof.* Let $M$ be a $\mathbb{AM}$-Noetherian module and $K$ be a submodule. Consider an $\mathbb{AM}$-complement $J$ to $K$. It is enough to show that for every annihilatoe submodule $I$, $K \cap I = 0$ implies $I \subseteq J$. We have $ann_M(ann_R(I)ann_R(J)) \cap K = 0$ by (1-13). On the other hand, $J \subseteq ann_M(ann_R(I)ann_R(J))$, so $I \subseteq ann_M(ann_R(I)ann_R(J)) = J$.

**Proposition 2.4.** *Every* $\mathbb{AM}$*-Noetherian module is* $\mathbb{AM}$*-cute.*

*Proof.* Follows from (1-14).

**Proposition 2.5.** *Every* $\mathbb{AM}$*-Noetherian module is generalized* $\mathbb{AM}\cap\mathbb{AM}$*-intersection and the map given by* $I \longrightarrow C_{\mathbb{AM}}(I)$ *is a* $\mathbb{AM} \cap \mathbb{AM}$*-organizer map.*

*Proof.* Follows from [7, (1-12) and (1-13)].

**Lemma 2.11.** *Let* $M$ *be an* $R$*-module. If every* $\mathbb{C}$*-submodule is a direct summand, then*

(1) *Every* $\mathbb{C}$*-submodule is a* $\mathbb{C}$*-summand.*
(2) $\langle \mathbb{AM} \cap \mathbb{AM}^{\oplus} \colon \mathbb{M} \rangle = \langle \mathbb{C} \colon \mathbb{M} \rangle$.
(3) $\langle (\mathbb{AM} \cap \mathbb{AM}^{\oplus})^{\gtrdot\ltimes} \colon \mathbb{M} \rangle = \langle \mathbb{C}^{\gtrdot\ltimes} \colon \mathbb{M} \rangle$.

*Proof.* (1) Follows from (1-8).
(2) We have $\langle \mathbb{C} \cap \mathbb{C}^{\oplus} \colon \mathbb{M} \rangle = \langle \mathbb{AM} \cap \mathbb{AM}^{\oplus} \colon \mathbb{M} \rangle$ by (1-8). Applying (1) completes the proof.
(3) Follows from (2).

**Lemma 2.12.** *Let* $M$ *be an* $R$*-module. The following are equivalent:*

(1) *Every* $\mathbb{C}$*-submodule is a direct summand and* $M$ *is* $\mathbb{AM} \cap \mathbb{AM}^{\oplus}$*-semisimple.*
(2) $M$ *is* $\mathbb{C}$*-semisimple.*
(3) $\langle (\mathbb{AM} \cap \mathbb{AM}^{\oplus})^{\gtrdot\ltimes} \colon \mathbb{M} \rangle = \langle \mathbb{C}^{\gtrdot\ltimes} \colon \mathbb{M} \rangle$ *and* $M$ *is* $\mathbb{AM} \cap \mathbb{AM}^{\oplus}$*-semisimple.*

*Proof.* (1⇒2) Follows from (1-17).
(2⇒1) Follows from (1-11), (1-12), [5, (2-9)] and (1-17).
(1⇒3) Follows from (1-17).
(3⇒2) It is obvious.

**Lemma 2.13.** *Let $M$ be a module. The following conditions are equivalent.*

(1) *$M$ is cofaithful.*

(2) *$M \in \langle \mathbb{B}{:}\mathbb{M} \rangle$.*

(3) *$M \in \langle \mathbb{C}{:}\mathbb{M} \rangle$.*

*Proof.* (1⇔2) and (1⇔3) Straightforward.

## 3. $\mathbb{AM}$-semi-vital modules.

**Definition 3.1.** *Let $M$ be an $R$-module. ($\mathcal{F}$ is an arbitrary class of subgroups).*

(1) *$M$ is said to be* **neat** *if $ann_R(M)$ is a prime ideal.*

(2) *$M$ is said to be* **semi-neat** *if $ann_R(M)$ is a semiprime ideal.*

(3) *$M$ is said to be* **symetric** *if for every ideals $I$ and $J$, $IJ \subseteq ann_R(M)$ implies $JI \subseteq ann_R(M)$.*

(4) *$M$ is said to be* **$\mathcal{F}$-semiprime** *if every $\mathcal{F}$-subgroup $N$ of $M$ is a semi-neat $R$-module ($ann_R(N)$ is a semiprime ideal).*

(5) *$M$ is said to be* **$\mathcal{F}$-vital** *if for every nonzero $\mathcal{F}$-subgroup $N$ of $M$, $N^\circ = 0$.*

(6) *$M$ is said to be* **$\mathcal{F}$-semi-vital** *if for every nonzero $\mathcal{F}$-subgroup $N$ of $M$, $N \not\subseteq N^\circ$.*

(7) *$M$ is said to be* **$\mathcal{F}$-multiplication** *if for every $\mathcal{F}$-subgroup $N$ of $M$, $M(N{:}M) = N$.*

(8) *$M$ is said to be* **$\mathcal{F}$-firm** *if for every $\mathcal{F}$-subgroup $N$ of $M$, $ann_R(M(N{:}M)) = ann_R(N)$.*

(9) *$M$ is said to be* **$\mathcal{F}$-bounded** *if for every $\mathcal{F}$-subgroup $N$ of $M$, $M(N{:}M) = 0$ implies $N = 0$, in other words for every nonzero $\mathcal{F}$-subgroup $N$ of $M$, there exists $a \in R$ with $0 \neq Ma \subseteq N$.*

It is clear that if every $\mathcal{F}$-subgroup is an $I\mathbb{F}$-subgroup, then $M$ is $\mathcal{F}$-semi-vital.

**Definition 3.2.** (1) *The class of semiprime submodules is denoted by $\mathbb{SM}$.*

(2) *A $\mathbb{M}$-semi-vital module is also called semi-vital.*

(3) *A $\mathbb{M}$-vital module is also called vital.*

(4) *A $\mathbb{M}$-firm module is also called firm.*

(5) *A $\mathbb{M}$-bounded module is also called bounded.*

It is clear that $\mathbb{M}$-semiprime module means semiprime module [2] and [11, INTRODUCTION], $\mathbb{M}$-multiplication module means multiplication module [12] and [1], $\mathbb{SM}$-multiplication module means semiprime multiplication module in [3, Definition 3.2] and $N$ being a semiprime submodule in [2] and [11, INTRODUCTION] means that $M/N$ is a semiprime module. Also, "multiplication" implies "firm" and "bounded", and for any submodule $N$, $N \subseteq N^\circ$ iff $N(N\!:\!M) = 0$.

**Definition 3.3.** *Let $R$ be a ring.*

1. *$R$ is called* **middle-faithful** *if for any $a, b \in R$, $aRb = 0$ implies $ab = 0$.*
2. *$R$ is called* **right $\mathcal{F}$-firm** *if $R_R$ is $\mathcal{F}$-firm, in other words, if for any $\mathcal{F}$-subgroup $N$, $ann_r(R(N\!:\!R)_r) = ann_r(N)$.*
3. *$R$ is called* **right $\mathcal{F}$-bounded** *if $R_R$ is $\mathcal{F}$-bounded, in other words, if for any nonzero $\mathcal{F}$-subgroup $N$, there exists $a \in R$ with $0 \neq Ra \subseteq N$.*

**Lemma 3.1.** *Let $M$ be a right $R$-module. For any $K \subseteq M$ and $N = K \cap K^\circ$ we have $N \subseteq N^\circ$.*

*Proof.* We have $K^\circ ann_R(M/K) = 0$, $N \subseteq K^\circ$ and $ann_R(M/N) \subseteq ann_R(M/K)$. Thus, $N ann_R(M/N) = 0$.

**Lemma 3.2.** *Let $M$ be a right $R$-module. For any annihilator submodule $N$, $N \cap N^\circ \subseteq \mathcal{Z}(M)$.*

*Proof.* Since $N^\circ$ is an annihilator submodule, we may assume that $N \subseteq N^\circ$ by (2-4). So, it is enough to show that $ann_R(N) \subseteq_e^{r\mathbb{I}} R$. Let $J$ be a right ideal with $ann_R(N) \cap J = 0$. We have $J ann_R(N) = 0$, so $MJ ann_R(N) = 0$, thus $MJ \subseteq ann_M(ann_R(N)) = N$, then $J \subseteq ann_R(M/N) \subseteq ann_R(N)$, implying $J = 0$.

**Proposition 3.1.** *Let $M$ be a module. $M$ is $\mathbb{AM}$-semi-vital iff every annihilator submodule is an $I\mathbb{F}$-submodule.*

*Proof.* ($\Rightarrow$) Let $K$ be an annihilator submodule. Set $N = K \cap K^\circ$. $N$ is an annihilator submodule, and $N \subseteq N^\circ$ by (2-4), so $N = 0$. Thus, $K$ is an $I\mathbb{F}$-submodule.
($\Leftarrow$) Let $N$ be an annihilator submodule with $N \subseteq N^\circ$. $N$ is an an $I\mathbb{F}$-submodule, so $N = N \cap N^\circ = 0$.

**Lemma 3.3.** *Let $M$ be a $\mathbb{AM}$-mini $R$-module. If every minimal annihilator submodule is a $I\mathbb{F}$-submodules, then $M$ is $\mathbb{AM}$-semi-vital.*

*Proof.* Temporarily suppose it is not so. There exist a nonzero annihilator submodule $K$ with $K \subseteq K^\circ$ (2-4). $K$ contains a minimal annihilator submodule $I$. $I$ is an $I\mathbb{F}$-submodules, also $I \subseteq K \subseteq K^\circ \subseteq I^\circ$ by (1-3), implying $I = 0$ which is a contradiction.

**Lemma 3.4.** *Let $M$ be an $\mathbb{AM}$-ind.finite $R$-module. If every $\mathbb{AM}$-uniform annihilator submodule is an $I\mathbb{F}$-submodules, then $M$ is $\mathbb{AM}$-semi-vital.*

*Proof.* Temporarily suppose it is not so. There exist a nonzero annihilator submodule $K$ with $K \subseteq K^\circ$ by (2-4). $K$ contains an $\mathbb{AM}$-uniform annihilator submodule $I$ by [5, (2-3)]. $I$ is an $I\mathbb{F}$-submodules, also $I \subseteq K \subseteq K^\circ \subseteq I^\circ$ by (1-3), implying $I = 0$ which is a contradiction.

**Proposition 3.2.** (1) *Every $\mathcal{F}$-bounded and semi-neat module is $\mathcal{F}$-semi-vital.*

(2) *Every $\mathcal{F}$-semi-vital module is $\mathcal{F}$-bounded.*

(3) *Every $\mathcal{F}$-bounded and prime module is $\mathcal{F}$-vital.*

(4) *Every $\mathcal{F}$-firm semi-neat module is $\mathcal{F}$-semiprime.*

(5) *Every cofaithful $\mathcal{F}$-firm module is $\mathcal{F}$-bounded.*

*Proof.* (1) Let $M$ be a $\mathcal{F}$-bounded and semi-neat right $R$- module. Now let $N$ be a $\mathcal{F}$-subgroup with $N \subseteq N^\circ$. Then $N(N{:}M) = 0$, on the other hand $M(N{:}M) \subseteq N$, so $M(N{:}M)^2 = 0$, thus $M(N{:}M) = 0$, implying $N = 0$.

(2) Let $M$ be a $\mathcal{F}$-semi-vital right $R$-module and $N$ be a $\mathcal{F}$-subgroup with $M(N : M) = 0$. Then, $N^\circ = M$, thus $N \subseteq N^\circ$, implying $N = 0$.

(3) Let $M$ be a $\mathcal{F}$-bounded and neat right $R$- module. Now let $N$ be a $\mathcal{F}$-subgroup with $N^\circ \neq 0$. Since $N^\circ(N{:}M) = 0$, $M(N{:}M) = 0$, implying $N = 0$.

(4) Let $M$ be a $\mathcal{F}$-firm semi-neat right $R$-module. Now let $N$ be a $\mathcal{F}$-subgroup and $I$ be an ideal with $NI^2 = 0$. Since $M(N{:}M) \subseteq N$, $M(N{:}M)I^2 = 0$, so $M(N{:}M)I = 0$, implying $NI = 0$.

(5) Let $M$ be a cofaithful $\mathcal{F}$-firm right $R$-module and $N$ be a $\mathcal{F}$-subgroup with $M(N{:}M) = 0$. Then, $ann_R(N) = R$, implying $N = 0$.

**Lemma 3.5.** (1) *Every bounded and semi-neat module is semi-vital.*

(2) *Every semi-vital module is bounded.*

(3) *Every bounded and prime module is vital.*

(4) *Every firm semi-neat module is semiprime.*

(5) *Every cofaithful firm module is bounded.*

(6) *Every semi-vital symetric module is semiprime.*

*Proof.* (1 to 5) Follows from (2-9).

(6) Let $M$ be a semi-vital symetric right $R$-module, $I$ be an ideal and $N$ be submodule with $NI^2 = 0$. We have $M(NI:M) \subseteq NI$, so $M(NI:M)I = 0$, thus $MI(NI:M) = 0$, implying $NI(NI:M) = 0$. Thus, $NI = 0$.

**Lemma 3.6.** *Let $R$ be a ring. The following conditions are equivalent.*

(1) *$R$ is semiprime.*

(2) *$R_R$ is semiprime and faithful.*

(3) *$R_R$ is $\mathbb{I}$-semiprime and faithful.*

(4) *$R_R$ is semi-neat and faithful.*

*Proof.* (1⇒2) Set $M = R$. Let $I$ be an ideal and $N \subseteq M$ be a submodule with $NI^2 = 0$. Then, $(NI)^2 \subseteq NI^2 = 0$, implying $NI = 0$.

(2⇒3) It is obvious.

(3⇒1) Let $I$ be an ideal with $I^2 = 0$. Then, $MI^2 = 0$, so $MI = 0$, implying $I = 0$.

(1⇔4) It is obvious.

**Lemma 3.7.** *Let $R$ be a ring. Considering $R$ as a right $R$-module,*

(1) *For any left ideal $I$, $I^\circ \subseteq ann_l(I)$.*

(2) *For any left annihilator $I$, $I^\circ = ann_l(I)$.*

*Proof.* (1) Set $M = R$. It is clear that $I \subseteq (I:M)$. Thus, $I^\circ = ann_M((I:M)) = ann_l((I:M)) \subseteq ann_l(I)$.

(2) We have $M(I:M) \subseteq I$, so $(I:M)ann_r(I) = 0$, implying $(I:M) \subseteq ann_l(ann_r(I)) = I$. On the other hand, it is clear that $I \subseteq (I:M)$. Thus, $I^\circ = ann_l(I)$.

**Lemma 3.8.** *Let $R$ be a middle-faithful ring. Considering $R$ as a right $R$-module, for any left ideal $I$, $I^\circ = ann_l(I)$.*

*Proof.* Set $M = R$. We have $M(I:M) \subseteq I$, , so $ann_l(I)R(I:M) = 0$, thus $ann_l(I)(I:M) = 0$, implying $ann_l(I) \subseteq ann_l((I:M)) = ann_M((I:M)) = I^\circ$. Applying (2-12) completes the proof.

**Lemma 3.9.** *Let $R$ be a right faithful ring. $R_R$ is a $l\mathbb{I}$-firm and $l\mathbb{I}$-bounded.*

*Proof.* Set $M = R_R$. Let $N$ be a left ideal. Then, $N \subseteq (N:M)$, so $RN \subseteq M(N:M) \subseteq N$, thus $ann_R(N) \subseteq ann_R(M(N:M)) \subseteq ann_R(RN) = ann_R(N)$. Therefore, $ann_R(M(N:M)) = ann_R(N)$. Now let $N$ be a nonzero left ideal. Consider $0 \neq a \in N$. Then, $0 \neq Ma = Ra \subseteq N$.

**Lemma 3.10.** *A ring $R$ is $l\mathbb{A}\mathbb{I}$-semiprime* [4] *iff $R_R$ is $\mathbb{A}\mathbb{M}$-semi-vital. Proof.* (⇒) *Let $I$ be an annihilator submodule with $I \subseteq I^\circ$. $I$ is a left annihilator ideal and $I \subseteq ann_l(I)$ by (2-12), so $I^2 = 0$, implying $I = 0$.*
*(⇐) Let $I$ be a left annihilator ideal with $I^2 = 0$. $I$ is an annihilator submodule and $I \subseteq ann_l(I) = I^\circ$ by (2-12). Thus, $I = 0$. Applying* [4, Proposition 2.7] *completes the proof.*

**Lemma 3.11.** *A ring $R$ is semiprime iff $R_R$ is $\mathbb{I}$-semi-vital and $R$ is middle-faithful. Proof.* (⇒) *Let $a, b \in R$ with $aRb = 0$. Set $I = RaR + aR$ and $J = RbR + Rb$. $I$ and $J$ are ideals and $IJ = 0$, so $I \cap J = 0$. On the other hand $ab \in I \cap J$, so $ab = 0$. Thus, $R$ is middle-faithful. Let $I$ be an ideal with $I \subseteq I^\circ$. Then, $I \subseteq ann_l(I)$ by (2-13), so $I^2 = 0$, implying $I = 0$.*
*(⇐) Let $I$ be an ideal with $I^2 = 0$. Then, $I \subseteq ann_l(I) = I^\circ$ by (2-13). Thus, $I = 0$.*

**Lemma 3.12.** *Let $R$ be a right $r\mathbb{I}$-bounded ring. For any right ideal $N$, $R(N:R)_r \subseteq_e^{r\mathbb{I}} N$.*

*Proof. Let $L \subseteq N$ be a nonzero right ideal. There exists $a \in R$ with $0 \neq Ra \subseteq L$. Then, $a \in (N:R)_r$, so $Ra \in R(N:R)_r$, thus $L \cap R(N:R)_r \neq 0$.*

**Lemma 3.13.** *Any semiprime right $r\mathbb{I}$-bounded ring $R$ is right $r\mathbb{I}$-firm and $R_R$ is semi-vital.*

*Proof. Let $N$ be a nonzero right ideal. Set $J = R(N:R)_r$. $J$ is an ideal and $J \subseteq_e^{r\mathbb{I}} N$ by (2-17). On the other hand, $J \cap ann_r(J) = 0$, so $N \cap ann_r(J) = 0$, thus,*

*$Nann_r(J) = 0$, then $ann_r(J) \subseteq ann_r(N)$, implying $ann_r(J) = ann_r(N)$. Thus, $R$ is right r$\mathbb{I}$-firm. This means that $R_R$ is bounded and firm. On the other hand, $R_R$ is semipriem. Therefore, $R_R$ is semi-vital by (2-9).*

## 4. $\mathbb{EM}$-SEMI-OLTIMATE MODULES.

**Definition 4.1.** *Let $M$ be an $R$-module.*

(1) *$M$ is said to be* $\mathcal{F}$**-semi-oltimate** *if for every $\mathcal{F}$-subgroup $N$, $N \subseteq N^\bullet$ implies $N = 0$.*
(2) *$M$ is said to be* $\mathcal{F}$**-oltimate** *if for every $\mathcal{F}$-subgroup $N$, $N^\bullet \neq 0$ implies $N = 0$.*

*It is clear that if every $\mathcal{F}$-subgroup is an $I\mathbb{D}$-subgroup, then $M$ is $\mathcal{F}$-semi-oltimate.*

**Lemma 4.1.** *For any $K \subseteq M$ and $N = K \cap K^\bullet$ we have $N \subseteq N^\bullet$.*

*Proof.* *Let $c \in N$. There exists a nonempty finite set $A \subseteq M$ with $cann_R(A/K) = 0$. On the other had, $ann_R(A/N) \subseteq ann_R(A/K)$, so $cann_R(A/N) = 0$, implying $c \in N^\bullet$.*

**Lemma 4.2.** *Let $M$ be a cofaithful weakly polyform right $R$-module. If $N$ is an annihilator-like submodule $N$, then for any $0 \neq n \in N \cap N^\bullet$, there exists $r \in R$ with $0 \neq nr \in \mathcal{Z}(M)$.*

*Proof.* *Since $N^\bullet$ is an eliminator submodule, we may assume that $N \subseteq N^\bullet$ by (3-2). There exists a finite set $A \subseteq M$ with $n(N : A) = 0$, then there exists $r \in R$ and $a \in M$ with $nr \neq 0$ and $nr(N : a) = 0$ by* [6, (2-1)]. *Let $J$ be a r$\mathbb{I}$-complement to $(N : a)$. Then $N \cap aJ = 0$. On the other hand, for each $b \in J$, $nrb(N : ab) = 0$, so $ann_R(ab) = (N : ab) \subseteq ann_R(nrb)$, implying $nrb = 0$ by* [8, (1-3)] *and* [7, (3-8)]. *Thus, $(N : a) + J \subseteq ann_R(nr)$, implying $0 \neq nr \in \mathcal{Z}(M)$.*

**Proposition 4.1.** *Let $M$ be a module. $M$ is $\mathbb{EM}$-semi-oltimate iff every eliminator submodule is an $I\mathbb{D}$-submodule.*

*Proof.* *($\Rightarrow$) Let $K$ be an eliminator submodule. Set $N = K \cap K^\bullet$. $N$ is an eliminator submodule, and $N \subseteq N^\bullet$ by (3-2), so $N = 0$. Thus, $K$ is an $I\mathbb{D}$-submodule.*
*($\Leftarrow$) Let $N$ be an eliminator submodule with $N \subseteq N^\bullet$. $N$ is an an $I\mathbb{D}$-submodule, so $N = N \cap N^\bullet = 0$.*

**Lemma 4.3.** *Let $M$ be a $\mathbb{EM}$-mini $R$-module. If every minimal eliminator submodule is a $I\mathbb{D}$-submodules, then $M$ is $\mathbb{EM}$-semi-oltimate.*

*Proof.* Temporarily suppose it is not so. There exist a nonzero eliminator submodule $K$ with $K \subseteq K^{\bullet}$ by (3-2). $K$ contains a minimal eliminator submodule $I$. $I$ is an $I\mathbb{D}$-submodules, also $I \subseteq K \subseteq K^{\bullet} \subseteq I^{\bullet}$ by [6, (2-11)], implying $I = 0$ which is a contradiction.

**Lemma 4.4.** *Let $M$ be an $\mathbb{EM}$-ind.finite $R$-module. If every $\mathbb{EM}$-uniform eliminator submodule is an $I\mathbb{D}$-submodules, then $M$ is $\mathbb{EM}$-semi-oltimate.*

*Proof.* Temporarily suppose it is not so. There exist a nonzero eliminator submodule $K$ with $K \subseteq K^{\bullet}$ (3-2). $K$ contains an $\mathbb{EM}$-uniform eliminator submodule $I$ by [5, (2-3)]. $I$ is an $I\mathbb{D}$-submodules, also $I \subseteq K \subseteq K^{\bullet} \subseteq I^{\circ}$ by [6, (2-11)], implying $I = 0$ which is a contradiction.

**Lemma 4.5.** *Any $\mathcal{F}$-semi-vital module is $\mathcal{F}$-semi-oltimate.*

*Proof.* Let $M$ be a $\mathcal{F}$-semi-vital module and $N$ be a $\mathcal{F}$-subgroup with $N \subseteq N^{\bullet}$. Since $N^{\bullet} \subseteq N^{\circ}$, $N \subseteq N^{\circ}$, implying $N = 0$.

**Proposition 4.2.** *Every eliminator submodule of an $\mathbb{EM}$-semi-oltimate module is a $\mathbb{EM}$-semi-oltimate module.*

*Proof.* Let $M$ be a $\mathbb{EM}$-semi-oltimate module and $L$ be an eliminator submodule of $M$. Now let $N$ be a nonzero eliminator submodule of $L$. $N$ is a eliminator submodule of $M$ by [6, (2-8)], so there exists $n \in N$ such that for every finite set $A \subseteq M$ , $n(N\!:\!A) \neq 0$. Thus, for every finite set $A \subseteq L$, $n(N\!:\!A) \neq 0$.

**Lemma 4.6.** *Let $M$ be a module and $J$ be an $\mathbb{EM}$-summand submodule. If $J$ is a $\mathbb{EM}$-semi-oltimate module, then for every eliminator submodule $I \subseteq M$, $I^{\bullet} \cap I \cap J = 0$.*

*Proof.* There exists an eliminator submodules $K$ such that $M = J \oplus K$. Let $b \in I^{\bullet} \cap I \cap J$. There exists a finite set $A \subseteq M$ with $b(I : A) = 0$. For each $a \in A$, there exist unique $a_J \in J$ and $a_K \in K$ with $a = a_J + a_K$. Set $B = \{a_J \mid a \in A\}$ and $C = \{a_K \mid a \in A\}$. Let $r \in ann_R(C)$. Then, $br(I : Ar) = 0$, on the *other hand,* $(I : Ar) = (I : Br) = (I \cap J : Br)$*, so* $br(I \cap J : Br) = 0$*, implying* $br \in (I \cap J)^{\bullet} \cap (I \cap J) = 0$ *because* $I \cap J$ *is an eliminator submodule of* $J$*. Thus,* $bann_R(C) = 0$*, implying* $b \in J \cap K = 0$*.*

**Proposition 4.3.** *Let* $M$ *be a cofaithful module and* $A$ *be an independent set of eliminator submodules such that* $M = \bigoplus(A)$*. Then,* $M$ *is* $\mathbb{EM}$*-semi-oltimate iff each element of* $A$ *is* $\mathbb{EM}$*-semi-oltimate.*

*Proof.* ($\Rightarrow$) *Follows from (3-8).*
($\Leftarrow$) *Let* $I$ *be aneliminator submodule with* $I \subseteq I^{\bullet}$*. Then,* $I = \bigoplus\{I \cap J \mid J \in A\}$ *bay* [7, (3-5)]*. On the other, for any* $J \in A$*,* $M = J \oplus J^{\times}$*, so* $J$ *is an* $\mathbb{EM}$*-summand eliminator submodule, implying* $I \cap J = 0$ *by (3-9). Thus,* $I = 0$*.*

**Lemma 4.7.** $\mathbb{EM}$-*CS* $\Rightarrow$ $\mathbb{B}$-*CS* $\Rightarrow$ $\mathbb{C}$-*CS and* $\mathbb{AM}$-*CS* $\Rightarrow$ $\mathbb{C}$-*CS*

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Department of Mathematical Science, Isfahan University of Technology, Isfahan.Iran

*Email address*: khabaz@iut.ac.ir