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A NEW EXTENSION OF THE INVERSE POWER LOMAX DISTRIBUTION

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Abstract. In this article, we propose a new extension of the inverse power Lomax distribution that takes advantage of the functionalities of the sine transformation. It is called the *sine inverse power Lomax (SIPL)* distribution. In the first part, its primary characteristics are first identified. The heavy-tailed nature of the SIPL distribution, as well as the versatility of its distribution functions, are emphasized. Also, among other things, we prove some first-order stochastic dominance structures and derive expressions for the quantile function, diverse moments, and income curves. Subsequently, the predictive ability of the SIPL model is investigated. A maximum likelihood calculation technique is used to estimate the parameters of the model, and simulations are run to verify its effectiveness. Then, two actual data sets are considered for analysis. When the SIPL model is compared to other Lomax-type models, it comes first according to standard statistical metrics.

1. Introduction

Data may have an observational distribution with more weighted tails than classical probability distributions. For modeling purposes, the so-called heavy-tailed distributions are ideal candidates in this situation. The Lomax distribution, developed by [15], has piqued the interest of many researchers due to its flexible, heavy-tailed nature with only two parameters and the simplicity of its corresponding functions. It is applied in a wide range of fields, including income and wealth measurement, engineering, industry, and many others. Concerning the Lomax distribution, we may refer to

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[6] for discussions on its heavy-tailed nature, [14] for various estimation methods of its parameters, [5] for some developments on its record values, and [7] for its application to lifetime modeling. Researchers have suggested various novel distributions that cope with heavy-tailed data in recent decades, including generalizations, extensions, or modifications of the Lomax distribution, in the spirit of variety and optimality. One can mention the exponentiated lomax distribution by [1], five-parameter beta Lomax distribution by [23], power Lomax distribution by [22], inverse power Lomax (IPL) distribution by [11], type II Topp-Leone power Lomax distribution by [4], Kumaraswamy generalized power Lomax distribution by [17] and sine power Lomax distribution by [18].

A review of the IPL distribution is needed to comprehend the intent of this paper. First, it was introduced and studied in detail from both a mathematical and an applied perspective in [11]. The IPL distribution is mathematically identical to the distribution of the random variable $X^{-1/\beta}$, where X has the Lomax distribution with parameters α and λ, or equivalently, the distribution of Y^{-1}, where Y has the power Lomax distribution with parameters α, β, and λ, thus explaining the term *inverse* in SIPL, or equivalently, the distribution of $Z^{1/\beta}$, where Z is the inverse Lomax distribution with parameters α and λ. More concretely, the IPL distribution is a three-parameter heavy-tailed distribution defined in terms of functionality by the following cumulative distribution function (cdf):

$$G_{IPL}(x;\jmath) = \left(1 + \frac{x^{-\beta}}{\lambda}\right)^{-\alpha}, \quad x > 0, \tag{1.1}$$

and $G_{IPL}(x;\jmath) = 0$ for $x \leq 0$, where $\jmath = (\alpha, \beta, \lambda)$, α is a strictly positive shape parameter, and β and λ are strictly positive scale parameters. The probability density function (pdf) of the IPL distribution is

$$g_{IPL}(x;\jmath) = \frac{\alpha\beta}{\lambda} x^{-\beta-1} \left(1 + \frac{x^{-\beta}}{\lambda}\right)^{-\alpha-1}, \quad x > 0,$$

and $g_{IPL}(x;\jmath) = 0$ for $x \leq 0$, and related hazard rate function (hrf) is expressed as

$$h_{IPL}(x;\jmath) = \frac{\alpha\beta}{\lambda} \frac{x^{-\beta-1}\left(1 + x^{-\beta}/\lambda\right)^{-\alpha-1}}{1 - \left(1 + x^{-\beta}/\lambda\right)^{-\alpha}}, \quad x > 0,$$

and $h_{IPL}(x;\jmath) = 0$ for $x \leq 0$. These functions have a modest level of complexity and can be manipulated analytically for various integral calculus problems. The figures in [11] show that the pdf and hrf of the IPL distribution have similar curve behaviors: they are mostly decreasing or upside-down shaped with a heavy right tail. These characteristics are essential for the statistical modeling of a wide range of heavy-tailed data. In addition, the IPL distribution has well-defined quantile and moment characteristics, which is clearly advantageous for further probability or statistical developments. As described in [11], the inference on the parameters of the IPL distribution is quite manageable. In particular, the maximum likelihood method performs quite efficiently under different censoring schemes. In addition, a comparison study shows that the IPL model fits essential data better than other models, including those nested within the IPL model, such as the Lomax, power Lomax, and inverse Lomax models, as well as more distinct models such as the inverse Weibull, generalized inverse Weibull, and exponentiated Lomax models. Despite its obvious advantages, the IPL distribution has some drawbacks, such as the lack of versatility of its left tail, which prevents the capture of some characteristics for small values in data, and the low diversity of shapes of its hrf, which prevents optimal modeling of some phenomena with complex attributes.

The aim of this paper is to propose an extended IPL distribution that enhances the modeling capacities of the IPL distribution while maintaining its mathematical simplicity and three-parameter dependence. To achieve this aim, we use the sinusoidal transformation as formerly described in [12] and [25, 26, 27]. Hence, the so-called sine formed class of distributions is applied to the IPL distribution to introduce a new extended version, that we name the sine inverse power Lomax (SIPL) distribution. The work of [18], which shows that the sinusoidal transformation of the power Lomax distribution is stronger on many aspects of the power Lomax distribution, is a motivational justification for the study of this extended distribution. According to the findings of the investigations, the SIPL is victorious on the following points: (i) the associated pdf possesses various curves including unimodal, symmetrical, asymmetrical on right and left, reversed J-shaped curves, (ii) the associated hrf exhibits decreasing, increasing, bathtub, reversed bathtub and decreasing-increasing-decreasing-shaped curves, (iii) the quantile and (ordinary, inverse and incomplete) moments properties of the SIPL distribution are rather manageable, (iv) in order to estimate the parameters of the SIPL model, the standard maximum likelihood estimation technique can be used effectively, and (v) for most of the data sets, the SIPL model outperforms other Lomax-structured models in terms of fit, including the IPL model.

These points are developed through the following plan. In Section 2, the primary functions of the SIPL distribution, as well as their main analytical properties, are discussed. Some technical properties and useful statistical functions are developed in Section 3. The theoretical and computational aspects of the maximum likelihood estimation are developed in Section 4. The SIPL modeling strategy is applied to referenced data in Section 5. A final discussion is offered in Section 6.

2. The SIPL distribution

The SIPL distribution, as previously mentioned, is a member of the sine formed class of distributions created by [12] and [25, 26, 27] using the IPL distribution as a parent. This section focuses on the main relevant functions of the SIPL distribution.

2.1. **Cumulative distribution and probability density functions.** The following definition determines the cdf.

Definition 2.1. The SIPL distribution is defined by the following cdf:

$$F_{SIPL}(x;\jmath) = \sin\left[\frac{\pi}{2}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}\right], \quad x>0$$

and $F_{SIPL}(x;\jmath) = 0$ for $x \le 0$, where the notations of (1.1) have been used, that is, $\jmath = (\alpha,\beta,\lambda)$, α is a strictly positive shape parameter, and β and λ are strictly positive scale parameters.

Differentiating $F_{SIPL}(x;\jmath)$ yields the pdf of the SIPL distribution. As a result, it is given as

$$f_{SIPL}(x;\jmath) = \frac{\pi}{2}\frac{\alpha\beta}{\lambda}x^{-\beta-1}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha-1}\cos\left[\frac{\pi}{2}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}\right], \quad x>0,$$

and $f_{SIPL}(x;\jmath) = 0$ for $x \le 0$.

With different values of the parameters, different curvature forms of this pdf are obtained as shown in Figure 1.

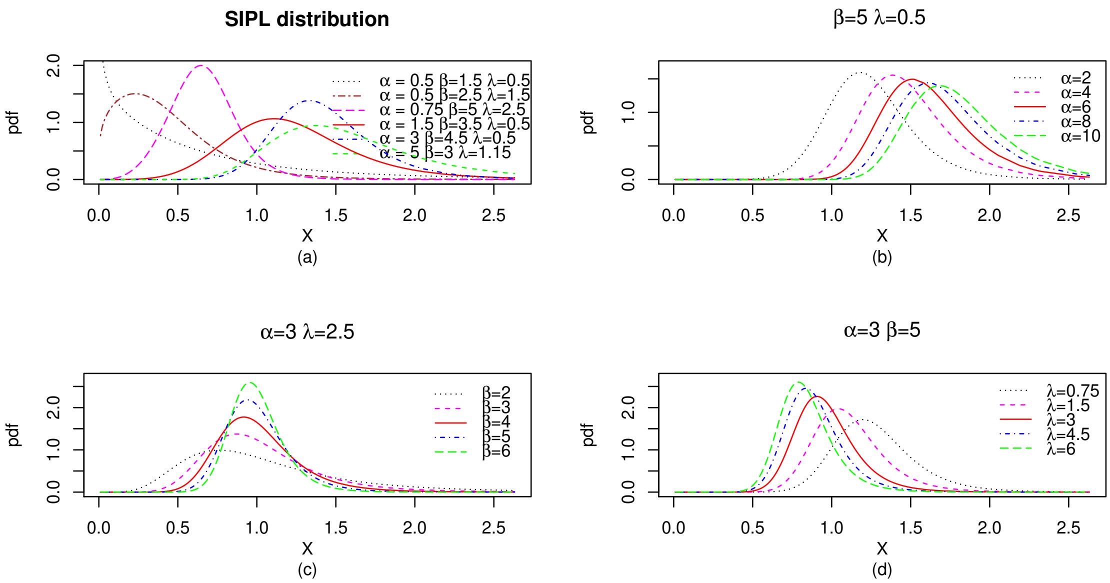

FIGURE 1. Curves of the pdf of the SIPL distribution at different parameter values

From Figure 1, the curves may be decreasing or unimodal, with a broad range of skewness, peakedness, and plateness, but they are often almost symmetrical or right-skewed. In comparison to the left tail of the IPL distribution, the left tail of the SIPL distribution clearly profits from more freedom. Because of these properties, the SIPL can be used to model a wide range of lifetime phenomena.

2.2. **Quantile function.** The quantile function (qf) is one way of prescribing a probability distribution. It is a key component in the Monte Carlo methodology and is used in a variety of mathematical applications. Reference [10] goes into great detail about the statistical uses of qfs. Mathematically, the qf is defined as the inverse function of the associated cdf. As a result, the qf of the SIPL distribution is calculated as

$$Q_{SIPL}(u;\jmath) = F_{SIPL}^{-1}(u;\jmath) = \left\{ \lambda \left[\left(\frac{2}{\pi} \arcsin u \right)^{-1/\alpha} - 1 \right] \right\}^{-1/\beta}, \quad u \in (0,1). \tag{2.1}$$

The presence of the arcsine function in the definition is a characteristic of the SIPL distribution that the IPL distribution does not have. The median of the SIPL distribution follows by taking $u = 1/2$ in the above equation. Also, in theory, a sample from the SIPL distribution can be obtained by applying this qf to a uniform distribution sample. More applications in this direction can be found in [10].

2.3. **Hazard rate function.** The analytical behavior of the hrf of a lifetime distribution is critical for managing its modeling capacity. On this topic, we can consult [2]. Here, the hrf of the SIPL distribution is written as

$$h_{SIPL}(x;\jmath) = \frac{f_{SIPL}(x;\jmath)}{1 - F_{SIPL}(x;\jmath)}$$
$$= \frac{\pi}{2}\frac{\alpha\beta}{\lambda}x^{-\beta-1}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha-1}\tan\left[\frac{\pi}{4}+\frac{\pi}{4}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}\right], \quad x > 0$$

and $h_{SIPL}(x;\jmath) = 0$ for $x \leq 0$. The inclusion of the tangent function in the definition opens up certain curvature possibilities that the hrf of the IPL distribution does not. This argument is demonstrated in Figure 2, which classifies different types of curvatures based on their diversity.

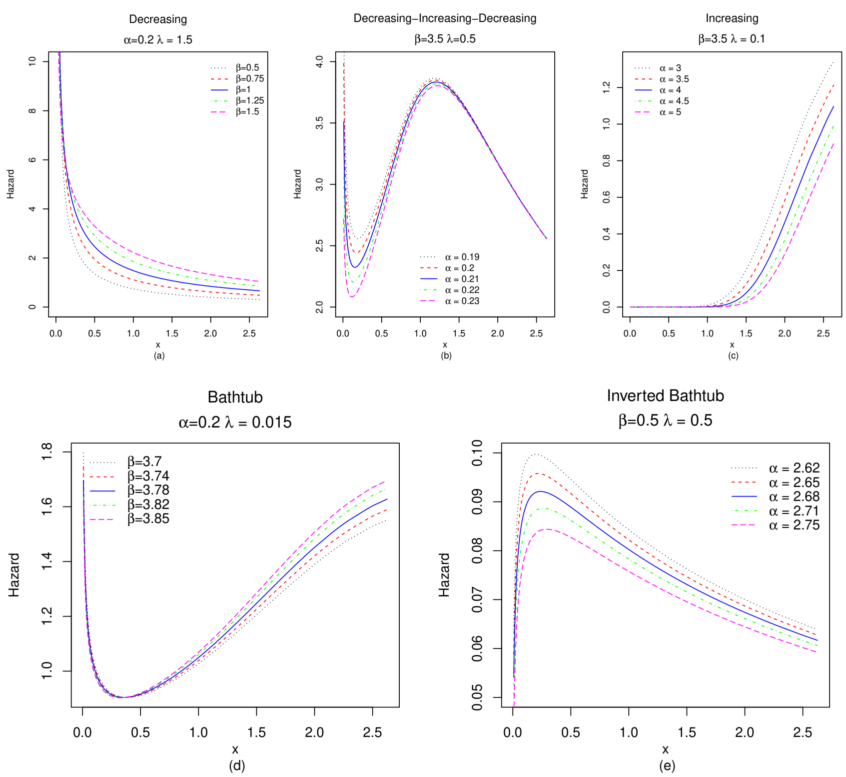

FIGURE 2. Curves of the hrf of the SIPL distribution at different parameter values

From Figure 2, we see that the hrf exhibits decreasing, increasing, bathtub, reversed bathtub, and decreasing-increasing-decreasing-shaped curves, in contrast to the hrf of the IPL distribution, which is severely limited in this regard. This is yet another reason why the proposed SIPL distribution is preferable to the IPL distribution.

3. Technical properties and specific functions

This section delves into the technical properties and specific functions of the SIPL distribution, which can be applied in a variety of theoretical and practical scenarios.

3.1. **Stochastic ordering results.** The SIPL distribution enjoys some hierarchical properties that are presented in this portion. We adopt the notion of first-order stochastic ordering as presented in [24]. First, the following result shows that the IPL distribution first-order stochastically dominates the SIPL distribution.

Proposition 3.1. *For any $x \in \mathbb{R}$, the following inequality is satisfied:*

$$G_{IPL}(x;\jmath) \leq F_{SIPL}(x;\jmath).$$

Proof. The inequality is an obvious equality for $x \leq 0$ since $G_{IPL}(x;\jmath) = F_{SIPL}(x;\jmath) = 0$. The following inequality is well-known: $\sin(y) \geq (2/\pi)y$ for $y \in [0,\pi/2]$. Then, since $(\pi/2)\left(1+x^{-\beta}/\lambda\right)^{-\alpha} \in [0,\pi/2]$, we immediately obtain:

$$F_{SIPL}(x;\jmath) = \sin\left[\frac{\pi}{2}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}\right] \geq \frac{2}{\pi}\frac{\pi}{2}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha} = \frac{\pi}{2}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}$$
$$= G_{IPL}(x;\jmath).$$

The stated result is established. □

In addition, an intrinsic first-order stochastic dominance of the SIPL distribution is exhibited in the next result.

Proposition 3.2. *For $\alpha_1 \geq \alpha_2$ and $\lambda_2 \geq \lambda_1$, by setting $\jmath_1 = (\alpha_1, \beta, \lambda_1)$ and $\jmath_2 = (\alpha_2, \beta, \lambda_2)$, we have*

$$F_{SIPL}(x;\jmath_1) \leq F_{SIPL}(x;\jmath_2).$$

Proof. It is enough to study the monotonicity of $F_{SIPL}(x;\jmath)$ according to the parameters. We have

$$\frac{\partial}{\partial\alpha}F_{SIPL}(x;\jmath) = -\frac{\pi}{2}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}\log\left(1+\frac{x^{-\beta}}{\lambda}\right)\cos\left[\frac{\pi}{2}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}\right] < 0,$$

meaning that $F_{SIPL}(x;\jmath)$ is decreasing with respect to α, and

$$\frac{\partial}{\partial\lambda}F_{SIPL}(x;\jmath) = \frac{\pi}{2}\frac{\alpha}{\lambda^2}x^{-\beta}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha-1}\cos\left[\frac{\pi}{2}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}\right] > 0,$$

meaning that $F_{SIPL}(x;\jmath)$ is increasing with respect to λ. □

It follows from Proposition 3.2 that, under some conditions on the parameters, the SIPL distribution with parameters vector $\jmath_1$ first-order stochastically dominates the SIPL distribution with parameters vector $\jmath_2$. This hierarchical distributional structure allows us to better understand the behavior of the SIPL distribution according to the parameters in terms of cdfs.

One can remark that there is no first-order stochastic dominance with respect to the parameter β since

$$\frac{\partial}{\partial\beta}F_{SIPL}(x;\jmath) = \frac{\pi}{2}\frac{\alpha}{\lambda}x^{-\beta}(\log x)\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha-1}\cos\left[\frac{\pi}{2}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}\right],$$

which can be negative or positive according to $x < 1$ and $x > 1$, respectively.

3.2. **Linear representation.** The following finding demonstrates how the pdf of the SIPL distribution can be reduced to a manageable sum.

Proposition 3.3. *The pdf of the SIPL distribution can be expanded as*

$$f_{SIPL}(x;\jmath) = \sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k)!}\left(\frac{\pi}{2}\right)^{2k+1}\frac{\alpha\beta}{\lambda}x^{-\beta-1}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-(2k+1)\alpha-1},$$

for $x > 0$.

Proof. The proof is based on the Taylor series expansion of the sine function. Precisely, for $x > 0$, we have

$$F_{SIPL}(x;\jmath) = \sin\left[\frac{\pi}{2}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}\right] = \sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}\left[\frac{\pi}{2}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}\right]^{2k+1}$$

$$= \sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-(2k+1)\alpha}.$$

Now, the differentiation of $F_{SIPL}(x;\jmath)$ with respect to x gives

$$f_{SIPL}(x;\jmath) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1} \frac{\alpha\beta}{\lambda}(2k+1)x^{-\beta-1}\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-(2k+1)\alpha-1}.$$

The stated result is obtained after the simplification of the terms $2k+1$. □

Based on Proposition 3.3, one can remark that $f_{SIPL}(x;\jmath)$ is written as an infinite mixture of pdfs of the IPL distribution as

$$f_{SIPL}(x;\jmath) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1} g_{IPL}(x;(2k+1)\alpha,\beta,\lambda).$$

Therefore, some properties of the IPL distribution can be transferred to the proposed SIPL distribution. In the next, we will use Proposition 3.3 with direct calculus, instead of this indirect approach.

3.3. **Ordinary moments.** The following result emphasizes the existence of the ordinary moments of the SIPL distribution. A series expansion for the r-th ordinary moment is provided.

Proposition 3.4. *Let $r \geq 1$ be an integer and X be a random variable with the SIPL distribution. Then, for $r < 2\beta$, the r-th ordinary moment of X exists. In the case $r < \beta$, it can be expanded as*

$$E(X^r) = \alpha\lambda^{-r/\beta}\sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!}\left(\frac{\pi}{2}\right)^{2k+1} B\left(1-\frac{r}{\beta},(2k+1)\alpha+\frac{r}{\beta}\right),$$

where E denotes the mathematical expectation and $B(a,b)$ refers to the standard beta function given as $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt = \int_0^{+\infty} t^{a-1}(1+t)^{-(a+b)}dt$ for $a,b>0$.

Proof. The convergence of the integral $\int_0^{+\infty} x^r f_{SIPL}(x;\jmath)dx$ leads to the existence of $E(X^r)$. The Riemann integrability criterion is useful in this regard. When x is in the neighborhood of 0, we have $x^r f_{SIPL}(x;\jmath) \sim (\pi/2)(\beta\alpha/\lambda)x^{r+\alpha\beta-1}$, which is integrable over an interval of the form $(0,c)$ with $c>0$ if and only if $r+\alpha\beta-1>-1$, which is always fulfilled. For the case $x\to+\infty$, we have $x^r f_{SIPL}(x;\jmath) \sim [(\pi^2/4)\beta\alpha^2/\lambda^2]x^{r-2\beta-1}$, which is integrable over an interval of the form $(c,+\infty)$ with $c>0$ if and only if $r-2\beta-1<-1$, implying that $r<2\beta$. Hence, the condition $r<2\beta$ is necessary to ensure the existence of $E(X^r)$.

Let us now suppose that $r < \beta < 2\beta$. By using Proposition 3.3, exchanging $\int$ and $\sum$ which is possible thanks to the dominated convergence theorem, and applying to the change of variables $y = x^{-\beta}/\lambda$, we obtain

$$\begin{aligned}
E(X^r) &= \int_0^{+\infty} x^r f_{SIPL}(x; \jmath) dx \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k+1} \int_0^{+\infty} \frac{\alpha\beta}{\lambda} x^{r-\beta-1} \left(1 + \frac{x^{-\beta}}{\lambda}\right)^{-(2k+1)\alpha-1} dx \\
&= \alpha\lambda^{-r/\beta} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k+1} \int_0^{+\infty} y^{-r/\beta}(1+y)^{-(2k+1)\alpha-1} dy \\
&= \alpha\lambda^{-r/\beta} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k+1} B\left(1 - \frac{r}{\beta}, (2k+1)\alpha + \frac{r}{\beta}\right).
\end{aligned}$$

The desired result is obtained. □

A computational remark is that, for K large enough, a precise approximation of $E(X^r)$ is obtained as

$$E(X^r) \approx \alpha\lambda^{-r/\beta} \sum_{k=0}^{K} \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k+1} B\left(1 - \frac{r}{\beta}, (2k+1)\alpha + \frac{r}{\beta}\right).$$

Also, the beta term can be expressed in terms of standard gamma function values as

$$B\left(1 - \frac{r}{\beta}, (2k+1)\alpha + \frac{r}{\beta}\right) = \frac{\Gamma(1 - r/\beta)\,\Gamma((2k+1)\alpha + r/\beta)}{\Gamma((2k+1)\alpha + 1)}.$$

Diverse moment measures of X can be defined from Proposition 3.4. Here, we restrict our attention on the variance basically defined by $Var = E\left[(X - E(X))^2\right]$, as well as the skewness and kurtosis defined by

$$Skewness = \frac{1}{Var^{3/2}} E\left[(X - E(X))^3\right], \quad Kurtosis = \frac{1}{Var^2} E\left[(X - E(X))^4\right].$$

Note that all the involved central moments can be expressed according to the first four ordinary moments through the standard binomial technique, that is, for any positive integer n,

$$E\left[(X - E(X))^n\right] = \sum_{\ell=0}^{n} \binom{n}{\ell} E(X^\ell)(-1)^{n-\ell}[E(X)]^{n-\ell}.$$

Table 1 shows the above measures of X for various parameter values, assuming $\beta > 4$.

TABLE 1. Moments of the SIPL distribution with different parameter values

Parameters	α	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	Var	$Skewness$	$Kurtosis$
$\beta = 4.2$ $\lambda = 0.5$	2	1.314708	1.858226	2.837872	4.731892	0.1297692	1.316197	6.917435
	4	1.61063	2.749419	5.01265	9.881754	0.1552885	1.889989	8.012451
	8	1.934942	3.943671	8.537791	19.889383	0.1996687	2.266506	8.700323
	12	2.143961	4.832425	11.549526	29.654824	0.2358558	2.405777	8.952307
	16	2.302855	5.570055	14.273422	39.262763	0.2669121	2.477989	9.08263
$\beta = 5.2$ $\lambda = 0.5$	2	1.240433	1.612441	2.200899	3.166785	0.07376731	0.7864958	5.528673
	4	1.463213	2.222023	3.514406	5.81867	0.08103127	1.2687394	6.371227
	8	1.697764	2.979161	5.424428	10.304183	0.09675889	1.591295	6.911904
	12	1.844702	3.512681	6.932349	14.2575	0.10975597	1.7112853	7.111688
	16	1.954507	3.940874	8.230482	17.903679	0.12077627	1.7736011	7.215334
$\beta = 5.5$ $\lambda = 2.5$	1.5	0.8492963	0.7572007	0.7088004	0.6980446	0.03589658	0.4855285	4.898175
	2.5	0.963462	0.9642966	1.0048163	1.0939367	0.03603753	0.8511548	5.571002
	4.5	1.0952929	1.2389178	1.4515416	1.7688539	0.03925119	1.2175016	6.186335
	6.5	1.1804516	1.4361366	1.8064142	2.3593089	0.04267061	1.3876633	6.466088
	8.5	1.2446644	1.5949947	2.1112016	2.8991188	0.04580525	1.4844368	6.624545
$\beta = 4.5$ $\lambda = 1.2$	3	1.188641	1.491579	1.986627	2.83304	0.07871263	1.445774	6.994705
	7	1.473811	2.273421	3.694251	6.385049	0.10130228	1.952929	7.900895
	11	1.64118	2.813117	5.069738	9.700662	0.11964513	2.115202	8.18612
	14	1.736128	3.145606	5.988022	12.094625	0.1314673	2.178702	8.297451
	18	1.839797	3.530313	7.113159	15.198752	0.14545817	2.231502	8.389938

Table 1 shows that the values of the considered measures vary from small to big. For symmetrical cases, the skewness is almost equal to 0, whereas for right-skewed cases, it is positive. The kurtosis also displays a wide variety of values. As a consequence, on these moment measures, we say that the SIPL distribution is a versatile model in general.

3.4. **Inverse moments.** The subsequent result is the same as Proposition 3.4, but with the inverse moments.

Proposition 3.5. *Let $s \geq 1$ be an integer and X be a random variable with the SIPL distribution. Then, for $s < \alpha\beta$, the s-th inverse moment of X exists. In the case*

$s < \alpha\beta$, it can be expanded as

$$E(X^{-s}) = \alpha\lambda^{s/\beta}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k)!}\left(\frac{\pi}{2}\right)^{2k+1}B\left(1+\frac{s}{\beta},(2k+1)\alpha-\frac{s}{\beta}\right).$$

Proof. We proceed as for the proof of Proposition 3.4. When x is in the neighborhood of 0, we have $x^{-s}f_{SIPL}(x;\jmath) \sim (\pi/2)(\beta\alpha/\lambda)x^{-s+\alpha\beta-1}$, which is integrable over an interval of the form $(0,c)$ with $c>0$ if and only if $-s+\alpha\beta-1>-1$, which is valid if and only if $s<\alpha\beta$. For the case $x\to+\infty$, we have $x^{-s}f_{SIPL}(x;\jmath) \sim [(\pi^2/4)\beta\alpha^2/\lambda^2]x^{-s-2\beta-1}$, which is integrable over an interval of the form $(c,+\infty)$ with $c>0$ if and only if $-s-2\beta-1<-1$, which is always fulfilled. Under the condition $s<\alpha\beta$, by substituting r by $-s$ in the developments of the proof of Proposition 3.4 concerning $E(X^r)$, the stated result comes directly. □

As a direct application, the first inverse moment of X, also called harmonic mean, is obtained as

$$E(X^{-1}) = \alpha\lambda^{1/\beta}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k)!}\left(\frac{\pi}{2}\right)^{2k+1}B\left(1+\frac{1}{\beta},(2k+1)\alpha-\frac{1}{\beta}\right).$$

This sum expression can be the basis for more on its numerical evaluation. As with the ordinary moments, we can define the corresponding variance, skewness and kurtosis, etc.

3.5. **Incomplete moments and income curves.** The following result discusses the incomplete moments of the SIPL distribution, with a proposition of analytical expansion.

Proposition 3.6. *Let $r\geq 1$ be an integer and X be a random variable with the SIPL distribution. Then, the r-th incomplete moment of X at $t\geq 0$ always exists and can be expanded as*

$$E(X^rI(X\leq t)) = \alpha\lambda^{-r/\beta}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k)!}\left(\frac{\pi}{2}\right)^{2k+1}B^*_{1/(1+\lambda t^\beta)}\left(1-\frac{r}{\beta},(2k+1)\alpha+\frac{r}{\beta}\right),$$

where $B^*_x(a,b)$ refers to the 'upper incomplete' beta function given as $B^*_x(a,b) = \int_x^1 t^{a-1}(1-t)^{b-1}dt$ for $a,b>0$.

Proof. The justification of the existence of $E(X^rI(X\leq t))$ is immediate: For all the values of the parameters, we have $0<E(X^rI(X\leq t))\leq t^r<+\infty$. The expansion of $E(X^r I(X \leq t))$ can be proved in a similar way to the one of $E(X^r)$. Indeed, by using Proposition 3.3, exchanging $\int$ and $\sum$ which is possible thanks to the dominated convergence theorem, and applying to the change of variables $y = x^{-\beta}/\lambda$ followed by the change of variables $z = y/(1+y)$, we get

$$
\begin{aligned}
E(X^r I(X \leq t)) &= \int_0^t x^r f_{SIPL}(x; \jmath) dx \\
&= \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k+1} \int_0^t \frac{\alpha\beta}{\lambda} x^{r-\beta-1} \left(1 + \frac{x^{-\beta}}{\lambda}\right)^{-(2k+1)\alpha-1} dx \\
&= \alpha\lambda^{-r/\beta} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k+1} \int_{t^{-\beta}/\lambda}^{+\infty} y^{-r/\beta}(1+y)^{-(2k+1)\alpha-1} dy \\
&= \alpha\lambda^{-r/\beta} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k+1} \int_{1/(1+\lambda t^{\beta})}^{1} z^{-r/\beta}(1-z)^{(2k+1)\alpha+r/\beta-1} dz \\
&= \alpha\lambda^{-r/\beta} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k+1} B^*_{1/(1+\lambda t^{\beta})} \left(1 - \frac{r}{\beta}, (2k+1)\alpha + \frac{r}{\beta}\right).
\end{aligned}
$$

The desired result is proved. □

The possible applications of the incomplete moments of the SIPL distribution are many, including the definition of various survival functions, actuarial measures, and income curves. See, for instance, [9] and [3]. In particular, the first incomplete moment allows us to define the Bonferroni and Lorenz curves, expressed as

$$
B(p) = \frac{1}{pE(X)} E(XI(X \leq q)) \quad L(p) = \frac{1}{E(X)} E(XI(X \leq q)), \quad p \in (0,1),
$$

respectively, where $q = Q_{SIPL}(p; \jmath)$ as given in (2.1). They can be expanded and approximated through the use of Proposition 3.6 with $r = 1$. These curves are essential probabilistic objects to visualize income inequality. A variety of their plots are seen in Figure 3 to study their variations.

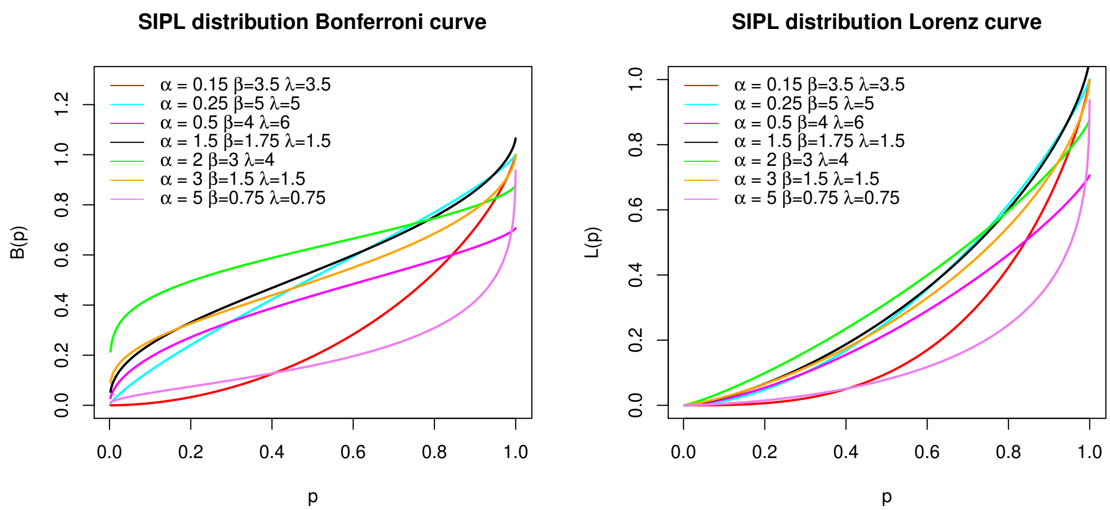

Figure 3. Bonferroni and Lorenz curves of the SIPL distribution at different parameter values

From Figure 3, the curves are found to have diverse convex properties; the Bonferroni curves have tilde shapes and J shapes, while the Lorenz curves have J shapes. This diversity of shapes reflects the adaptability of the SIPL distribution as applied in the context of income inequalities.

4. Maximum likelihood estimation

For data fitting purposes, the maximum likelihood approach can be adapted to the SIPL model. First, we consider n values $x_1, x_2, \ldots, x_n$ supposed to be the observations of n independent random variables with the SIPL distribution. Then, the associated likelihood function is

$$L(\jmath) = \prod_{i=1}^{n} f_{SIPL}(x_i; \jmath) =$$

$$= \left(\frac{\pi}{2}\right)^n \left(\frac{\alpha\beta}{\lambda}\right)^n e^{-(\beta+1)\sum\limits_{i=1}^{n}\log x_i} e^{-(\alpha+1)\sum\limits_{i=1}^{n}\log\left(1+x_i^{-\beta}/\lambda\right)} \prod_{i=1}^{n} \cos\left[\frac{\pi}{2}\left(1+\frac{x_i^{-\beta}}{\lambda}\right)^{-\alpha}\right].$$

We can derive the log-likelihood function as

$$\log L(\jmath) = n\log\left(\frac{\pi}{2}\right) + n\log\left(\frac{\alpha\beta}{\lambda}\right) - (\beta+1)\sum_{i=1}^{n}\log x_i - (\alpha+1)\sum_{i=1}^{n}\log\left(1+\frac{x_i^{-\beta}}{\lambda}\right) + \sum_{i=1}^{n}\log\left\{\cos\left[\frac{\pi}{2}\left(1+\frac{x_i^{-\beta}}{\lambda}\right)^{-\alpha}\right]\right\}.$$

Then, following the maximum likelihood technique, an accurate estimate of the parameter vector $\jmath$ is obtained as

$$\hat{\jmath} = \operatorname{argmax}_{\jmath} \log L(\jmath)$$

Thus, by denoting $\hat{\jmath} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$, the estimates $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ are called the maximum likelihood estimates (MLEs) of α, β and λ, respectively. Theoretically, these estimates are the solutions of $\nabla_{\jmath} \log L(\jmath) \mid_{\jmath=\hat{\jmath}}= 0$, where $\nabla_{\jmath}$ denotes the gradient operator with respect to $\jmath$. The components of $\nabla_{\jmath} \log L(\jmath)$ can be expanded as

$$\frac{\partial}{\partial\alpha}\log L(\jmath) = \frac{n}{\alpha} - \sum_{i=1}^{n}\log\left(1+\frac{x_i^{-\beta}}{\lambda}\right) - \sum_{i=1}^{n}\frac{\pi}{2}\left(1+\frac{x_i^{-\beta}}{\lambda}\right)^{-\alpha}\log\left(1+\frac{x_i^{-\beta}}{\lambda}\right) \tan\left[\frac{\pi}{2}\left(1+\frac{x_i^{-\beta}}{\lambda}\right)^{-\alpha}\right]$$

$$\frac{\partial}{\partial\beta}\log L(\jmath) = \frac{n}{\beta} - \sum_{i=1}^{n}\log x_i - (\alpha+1)\sum_{i=1}^{n}\frac{x_i^{-\beta}}{\lambda+x_i^{-\beta}}\log x_i + \frac{\pi}{2}\frac{\alpha}{\lambda}\sum_{i=1}^{n}x_i^{-\beta}\log x_i\left(1+\frac{x_i^{-\beta}}{\lambda}\right)^{-\alpha-1}\tan\left[\frac{\pi}{2}\left(1+\frac{x_i^{-\beta}}{\lambda}\right)^{-\alpha}\right]$$

and

$$\frac{\partial}{\partial\lambda}\log L(\jmath) = -\frac{n}{\lambda} + \frac{(\alpha+1)}{\lambda}\sum_{i=1}^{n}\frac{x_i^{-\beta}}{\lambda+x_i^{-\beta}} + \frac{\pi}{2}\frac{\alpha}{\lambda^2}\sum_{i=1}^{n}x_i^{-\beta}\left(1+\frac{x_i^{-\beta}}{\lambda}\right)^{-\alpha-1}\tan\left[\frac{\pi}{2}\left(1+\frac{x_i^{-\beta}}{\lambda}\right)^{-\alpha}\right].$$

The expressions of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ are elusive, but numerical evaluations are all that is needed in practice. Basic methods in statistical software, such as the R software (see [21]), can be used to conveniently derive these numerical values. Furthermore, MLE theory guarantees that the random version of $\hat{\jmath}$ is asymptotically distributed as a three-dimensional normal distribution, with mean vector $\jmath$ and variance-covariance matrix $\Xi = \{-\nabla^2_{\jmath} \log L(\jmath) \mid_{\jmath=\hat{\jmath}}\}^{-1}$.

The standard error (SE) of $\hat{\alpha}$ is obtained by taking the square root of the first diagonal variable of Ξ, and the SEs of the two other parameters can be obtained in a similar way. Also, various statistical measures and confidence intervals are based on the asymptotic normal distribution. One can mention that, by applying the plug-in approach, $f_{SIPL}(x;\hat{\jmath})$ is also a functional estimate of the unknown pdf $f_{SIPL}(x;\jmath)$. As described in the next section, this approximate pdf plays a critical role in fitting the normalized histogram of the data. Naturally, the plug-in technique can be applied to estimate other functions of interest.

We are now checking the accuracy of the MLEs. The qf of the SIPL distribution is used to produce the values $x_1, x_2, \ldots, x_n$. For each sample size of $n = 50, 100, 200, 300, 500$, we run 1000 Monte Carlo simulations on the following sets of parameters: Set I $= (0.5, 5, 0.05)$ and Set II $= (0.75, 4, 0.05)$. The standard mean MLE (MMLE), bias (Bias), and mean squared error (MSE) are computed in each case. The findings are presented in Table 2.

Table 2. MMLEs, biases and MSEs based on the simulations with different sample sizes

n	$\hat{\alpha}$			$\hat{\beta}$			$\hat{\lambda}$		
	MMLE	Bias	MSE	MMLE	Bias	MSE	MMLE	Bias	MSE
					Set I				
50	0.68660	0.18660	1.47316	5.37859	0.37859	2.74746	0.17659	0.12659	0.65856
100	0.53467	0.03467	0.04598	5.30920	0.30920	1.61558	0.07670	0.02670	0.00976
200	0.51542	0.01542	0.01729	5.14680	0.14680	0.78557	0.06203	0.01203	0.00252
300	0.50919	0.00919	0.01127	5.10602	0.10602	0.52277	0.05803	0.00803	0.00154
400	0.50989	0.00989	0.00888	5.07917	0.07917	0.42576	0.05668	0.00668	0.00112
500	0.50679	0.00679	0.00664	5.05466	0.05466	0.30456	0.05546	0.00546	0.00084
					Set II				
50	1.27265	0.52265	10.90739	4.22104	0.22104	1.31187	0.25624	0.20624	1.81118
100	0.83992	0.08992	0.16163	4.14842	0.14842	0.75146	0.08481	0.03481	0.01423
200	0.78879	0.03879	0.05678	4.08506	0.08506	0.36497	0.06477	0.01477	0.00374
300	0.78139	0.03139	0.03158	4.04532	0.04531	0.23523	0.06038	0.01038	0.00172
400	0.76553	0.01553	0.02039	4.04353	0.04353	0.17338	0.05607	0.00607	0.00101
500	0.76993	0.01993	0.01684	4.01312	0.01312	0.14068	0.05679	0.00679	0.00084

From Table 2, we see that, as the sample sizes increase, the biases and MSEs decrease, and the resulting mean estimates of the parameters become closer to the true value.

This demonstrates that the SIPL model parameters are well estimated by the MLEs and motivates their use for data fitting purposes.

5. The SIPL modeling strategy

The basic notations, goodness-of-fit checks, and model adequacy metrics based on the SIPL model are discussed in this portion. Let $x_1, \ldots, x_n$ be n values that we order in an ascending manner to obtain the ordered values denoted by $x_{(1)}, \ldots, x_{(n)}$. We work with the goodness-of-fit measures such as Cramér-von Mises, Anderson-Darling and Kolmogorov-Smirnov (K-S) statistics. In the setting of the SIPL distribution, they can be listed as

$$\mathrm{W}^* = \frac{1}{12n} + \sum_{i=1}^{n} \left(F_{SIPL}(x_{(i)}; \hat{\jmath}) - \frac{2i-1}{n} \right)^2,$$

$$\mathrm{A}^* = -n - \sum_{i=1}^{n} \frac{2i-1}{n} \left[\log(F_{SIPL}(x_{(i)}; \hat{\jmath})) + \log(1 - F_{SIPL}(x_{(i)}; \hat{\jmath})) \right]$$

and

$$D_n = \max_{i=1,\ldots,n} \left(\frac{i}{n} - F_{SIPL}(x_{(i)}; \hat{\jmath}), F_{SIPL}(x_{(i)}; \hat{\jmath}) - \frac{i-1}{n} \right),$$

respectively. The p-Value of the K-S test is also taken into account. For given data, these adequacy metrics are commonly used to determine which model/distribution has the best fit. The model with the lowest W* or A* value and the highest p-Value is chosen as the best. For a more in-depth look at the W* and A* statistics, see [8]. In addition, the Akaike information criterion (AIC), correct Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) are considered. In the setting of the SIPL distribution, they are defined as AIC $= -2\log L(\hat{\jmath}) + 2k$, BIC $= -2\log L(\hat{\jmath}) + k\log(n)$, CAIC $= -2\log L(\hat{\jmath}) + 2kn/(n-k-1)$ and HQIC $= -2\log L(\hat{\jmath}) + 2k\log[\log(n)]$, with $k = 3$, being the number of parameters of the SIPL model. Of course, these expressions can be adapted to any distribution effortlessly. According to prevailing knowledge, the model with the lowest AIC, CAIC, BIC, or HQIC value is chosen as the best fit for the data. In the rest of the study, we want to see how the SIPL model compares to the Lomax-based models listed in Table 3.

TABLE 3. Competitive models of the SIPL model

Model	Label	Cdf	Reference
Inverse power Lomax	IPL	$\left(1+\frac{x^{-\beta}}{\lambda}\right)^{-\alpha}$	[11]
Topp-Leone Lomax	TLGL	$\left(1-(1+\alpha x)^{-2\beta}\right)^{\lambda}$	[20]
Power Lomax	PL	$1-\left(1+\lambda x^{\beta}\right)^{-\alpha}$	[22]
Exponentiated Lomax	EL	$\left(1-\left(\frac{\beta}{x+\beta}\right)^{\alpha}\right)^{\lambda}$	[13]
Lomax	Lomax	$1-\left(\frac{\beta}{x+\beta}\right)^{\alpha}$	[15]

The applicability of the SIPL model for two real data sets is now discussed.

Data set 1: First, we consider the data reported in [16] which represent the failure components of aircraft windshields times. The data are: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663. A description of these data is provided in Table 4.

TABLE 4. Descriptive statistics of data set 1

Mean	Median	Variance	Skewness	Kurtosis	Minimum	Maximum
2.55745	2.3545	1.25177	0.09949	-0.65232	0.04	4.663

We clearly notice from Table 4 that the data are symmetrical, presenting a platykurtic nature and low dispersion. The negative value of the kurtosis is known to produce lighter tails in a distribution. Table 5 lists the MLEs, as well as the SEs in parentheses, of the SIPL model parameters as well as those of other competitors.

Table 5. MLEs with SEs for data set 1

Model	α	β	λ
SIPL	**0.55626094**	**3.83518484**	**0.00465587**
	(0.071863295)	(0.269670478)	(0.001851412)
IPL	0.387587025	5.065255714	0.002823232
	(0.055471506)	(0.387916376)	(0.001354061)
PL	2.510918	2.501948	24.858636
	(1.0039915)	(0.2813778)	(8.8454850)
EL	24.107930	30.212370	3.661293
	(13.9109419)	(18.6585652)	(0.6506768)
TLGL	3.721336	9.745047	24.585348
	(0.7759183)	(5.5473841)	(16.2933590)
Lomax	-	8.650051	21.150309
		(3.207235)	(8.180986)

The W*, A*, D_n, K-S p-Value, AIC, CAIC, BIC, and HQIC of the SIPL model, as well as those of the competitors, are mentioned in Table 6.

Table 6. Statistical metrics of the considered models applied to data set 1

Model	W*	A*	D_n	p-Value	AIC	CAIC	BIC	HQIC
SIPL	**0.0584**	**0.5987**	**0.0727**	**0.7662**	**267.8582**	**268.1582**	**275.1507**	**270.7897**
IPL	0.0607	0.6259	0.0912	0.4876	270.3969	270.6969	277.6894	273.3284
PL	0.1031	0.9686	0.1061	0.3016	275.1259	275.4259	282.4184	278.0574
EL	0.2393	1.8777	0.1236	0.1536	288.6155	288.9155	295.9079	291.547
TLGL	0.2465	1.9232	0.1204	0.1751	289.4639	289.7639	296.7563	292.3954
Lomax	0.1933	1.5824	0.3077	2.49×10^{-7}	337.4818	337.6299	342.3434	339.4361

From Table 6, we see that all the metrics are favorable to the SIPL model; it is the best according to the considered statistical criteria. This result can be confirmed visually. In this regard, Figure 4 shows the related fitted normalized histogram and probability-probability (PP) plot.

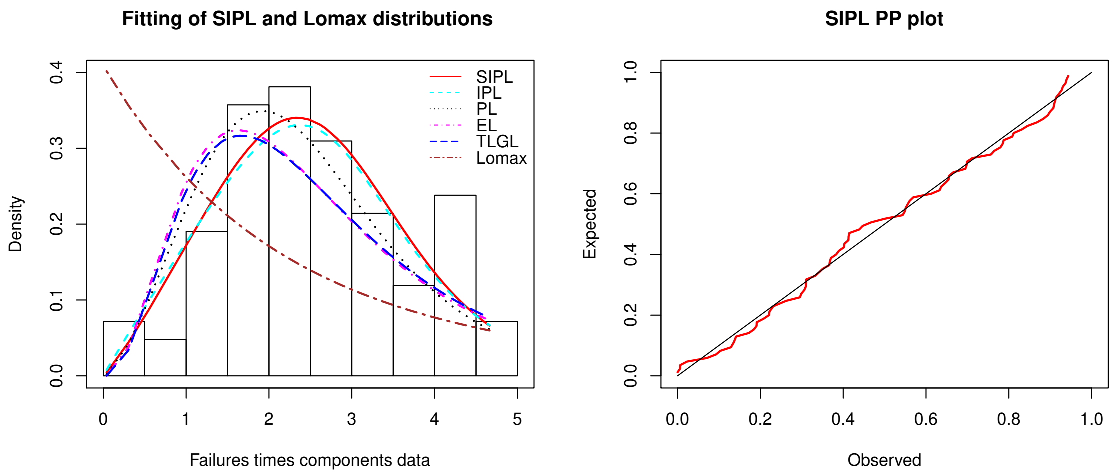

FIGURE 4. Fitted pdfs and PP plot for data set 1

From Figure 4, it is worth noting that the SIPL model has lighter tails than those of the competitors, allowing it to fit the data in a more suitable manner. The red line of the PP plot is in total adequation with the black line, confirming the fit power of the SIPL model.

Data set 2: For the second data set, we consider the uncensored data of [19], which consist of 100 observations on breaking stress of carbon fibers (in Gba). The data are: 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 2.82, 2.05, 3.65. A summary of descriptive statistics to data set 2 is reported in Table 7.

TABLE 7. Descriptive statistics of data set 2

Mean	Median	Variance	Skewness	Kurtosis	Minimum	Maximum
2.6214	2.7	1.02796	0.36815	0.10494	0.39	5.56

From Table 7, we observe that the data are approximately symmetric and platykurtic with a low variance.

The MLEs and SEs of the parameters of the considered models are reported in Table 8.

TABLE 8. MLEs with SEs for data set 2

Model	α	β	λ
SIPL	**0.603317311**	**4.677762333**	**0.002457786**
	(0.0602495373)	(0.2010761839)	(0.0004566074)
IPL	0.692945377	4.768960326	0.007351522
	(0.33184656)	(1.22920175)	(0.01435339)
PL	1.624010	3.169221	29.455632
	(0.5246620)	(0.3380815)	(8.5643898)
TLGL	25.408341	22.975388	8.504096
	(15.707237)	(15.610496)	(1.789886)
EL	8.964875	8.283858	14.222593
	(1.934670)	(3.997527)	(7.879147)
Lomax	-	9.946361	25.833924
		(3.517630)	(9.683001)

The W*, A*, D_n, K-S p-Value, AIC, CAIC, BIC, and HQIC of the SIPL model, as well as those of the competitors, are mentioned in Table 9.

TABLE 9. Statistical metrics of the considered models applied to data set 2

Model	W*	A*	D_n	p-Value	AIC	CAIC	BIC	HQIC
SIPL	**0.0631**	**0.3575**	**0.0785**	**0.5695**	**288.6301**	**288.8801**	**296.4456**	**291.7932**
IPL	0.1626	0.8279	0.0919	0.3667	293.519	293.769	301.3345	296.6821
PL	0.1750	0.8914	0.1257	0.0848	296.914	297.164	304.7295	300.077
EL	0.2549	1.3462	0.1103	0.1751	300.7922	301.0422	308.6077	303.9553
TLGL	0.2706	1.4368	0.1131	0.1552	302.1661	302.4161	309.9816	305.3292
Lomax	0.1676	0.8605	0.3139	5.52e-09	405.116	405.2397	410.3263	407.2247

Clearly, Table 9 shows that the goodness-of-fit test of the SIPL model has the largest p-value and the related model adequacy values are the smallest. As a result, the data seem to be best fitted by the SIPL model.

Figure 5 shows the related fitted normalized histogram and PP plot.

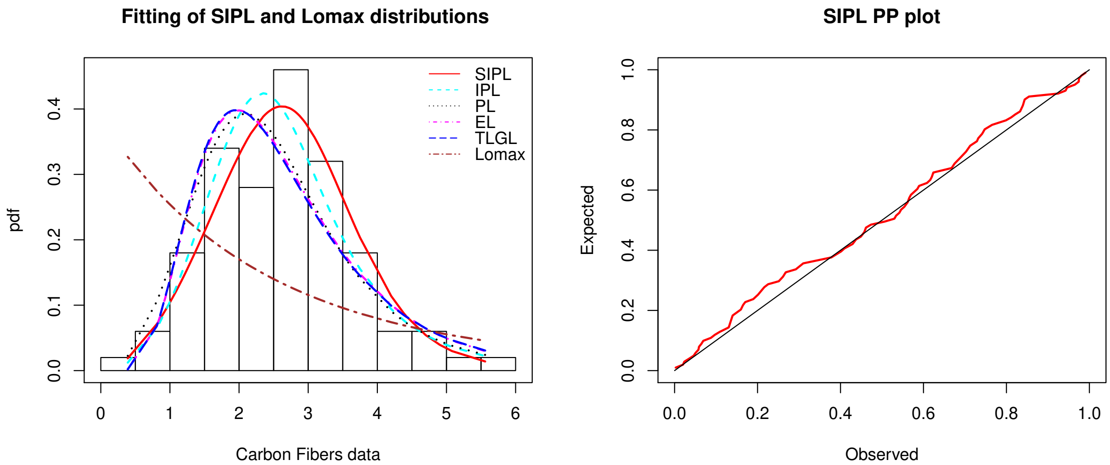

FIGURE 5. Fitted pdfs and PP plot for data set 2

From Figure 5, it is clear that the SIPL model is superior in fitting than the other Lomax-based models. In particular, the symmetry of the data is well captured, with correct adjustment of the tails. The fine adjustment of the lines of the PP plot just confirms the applicability of the SIPL model in the fitting of these data.

6. Final discussion

The key contribution of this paper is the proposal of a new extension of the inverse power Lomax distribution, named the *sine inverse power Lomax (SIPL)* distribution. It is inspired by a well-known trigonometric class of distributions: the sine formed class of distributions. The central idea is to use the oscillating properties of the sine function to improve the modeling capabilities of the inverse power Lomax distribution while maintaining the same number of parameters and a certain functional simplicity. Some statistical properties, such as first-order stochastic ordering, quantile function, diverse moments, and income curves, were developed and discussed. Several graphical works illustrated the usefulness of the SIPL model under consideration. As an application, we compared the SIPL model to the key existing Lomax-based models using two realistic data sets. When the data are symmetrical or skewed to the right, the SIPL model performs better when compared to its competitors. It also has the ability to handle moderately left-skewed data. In every case, the results are reasonably satisfactory, showing that the SIPL model can be used to evaluate a wide panel of data sets effectively.

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References

[1] I.B. Abdul-Moniem,Recurrence relations for moments of lower generalized order statistics from exponentiated Lomax distribution and its characterization, *Journal of Mathematical and Computational Science*, **2(4)** (2012), 999-1011.

[2] M.V. Aarset, How to identify bathtub hazard rate, *IEEE Transactions on Reliability*, **36** (1987), 106-108.

[3] P. Artzner, Delbaen, F., Eber, J.-M. and Heath, D. Coherent measures of risk, *Mathematical Finance*, **9(3)** (1999), 203-228.

[4] S. Al-Marzouki, F. Jamal, C. Chesneau, and M. Elgarhy, Type II Topp-Leone power Lomax distribution with applications, *Mathematics*, **8** (2020), 4.

[5] M. Ahsanullah, Record values of the Lomax distribution, *Statistica Neerlandica*, **45(1)** (1991), 21-29.

[6] M.C. Bryson, Heavy-tailed distributions: properties and tests, *Technometrics*, **16(1)** (1974), 61-68.

[7] M. Chahkandi, and M. Ganjali, On some lifetime distributions with decreasing failure rate, *Computational Statistics and Data Analysis*, **53(12)** (2009), 4433-4440.

[8] G. Chen, and N. Balakrishnan, A general purpose approximate goodness-of-fit test, *Journal of Quality Technology*, **27(2)** (1995), 154-161.

[9] G.M. Cordeiro, R.B. Silva, and A.D.C. Nascimento, *Recent Advances in Lifetime and Reliability Models*, Bentham Sciences Publishers, Sharjah, UAE, (2020).

[10] W. Gilchrist, *Statistical Modelling with Quantile Functions*, CRC Press, Abingdon, (2000).

[11] A.S. Hassan, and M. Abd-Allah, On the inverse power Lomax distribution, *Annals of Data Science*, **6(2)** (2019), 259-278.

[12] D. Kumar, U. Singh, and S.K. Singh, A new distribution using sine function: its application to bladder cancer patients data, *Journal of Statistics Applications and Probability*, **4** (2015) 417-427.

[13] A.J. Lemonte, and G.M. Cordeiro, An extended Lomax distribution, *Statistics*, **47(4)** (2013), 800-816.

[14] G.S. Lingappaiah, Bayes prediction in exponential life-testing when sample size is a random variable, *IEEE transactions on Reliability*, **35(1)** (1986), 106-110.

[15] K.S. Lomax, Business failures. Another example of the analysis of failure data, *Journal of the American Statistical Association*, **49** (1954), 847-852.

[16] D.N.P. Murthy, M. Xie, and R. Jiang, *Weibull Models*. John Wiley & Sons, New York, (2004)

[17] V. B V Nagarjuna,R. Vishnu Vardhan, and C. Chesneau, Kumaraswamy generalized power Lomax distribution and its applications, *Stats*, **4(1)** (2021), 28-45.

[18] V. B V Nagarjuna, R. Vishnu Vardhan, and C. Chesneau, On the accuracy of the sine power Lomax model for data fitting, *Modelling*, **2(1)**,(2021), 78-104.

[19] M.D. Nichols, and W.J.Padgett, A bootstrap control chart for Weibull percentiles, *Quality and Reliability Engineering International*, **22** (2006), 141-151.

[20] P.E. Oguntunde, M.A. Khaleel, H.I. Okagbue and Q.A. Odetunmibi, The Topp-Leone Lomax (TLLo) distribution with applications to airborne communication transceiver dataset, *Wireless Personal Communications*, **109(1)** (2019), 349-360.

[21] R Development Core Team *R: A language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria, (2005). ISBN 3-900051-07-0, URL: http://www.R-project.org.

[22] E.H.A. Rady, W.A. Hassanein and T.A. Elhaddad, The power Lomax distribution with an application to bladder cancer data, *SpringerPlus*, **5(1)** (2016), 1-22.

[23] M. Rajab, M. Aleem, T. Nawaz and M. Daniyal, On five parameter beta Lomax distribution, *Journal of Statistics*, **20** (2013), 102-118.

[24] M. Shaked, and J.G. Shanthikumar, *Stochastic Orders*, Wiley, New York,(2007).

[25] L/ Souza, *New trigonometric classes of probabilistic distributions*, Thesis, Universidade Federal Rural de Pernambuco, (2015).

[26] L. Souza, W.R.O. Junior, C.C.R. de Brito, C. Chesneau, T.A.E. Ferreira and L. Soares, On the Sin-G class of distributions: theory, model and application, *Journal of Mathematical Modeling*, **7(3)** (2019a), 357-379.

[27] L. Souza, W.R.O. Junior, C.C.R. de Brito, C. Chesneau, T.A.E. Ferreira and L. Soares, General properties for the Cos-G class of distributions with applications, *Eurasian Bulletin of Mathematics*, **2(2)** (2019b), 63-79.

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