Jordan Journal of Mathematics and Statistics (JJMS), 15(3B), 2022, pp 741 - 771
DOI: https://doi.org/10.47013/15.3.21

MODELING, ANALYSIS AND OPTIMAL CONTROL TO CO-DYNAMICS OF HIV/AIDS-TB DISEASES IN HOMOGENEOUS POPULATION

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Abstract. In this paper, an optimal control mathematical model of HIV/AIDS and TB co-infection with vaccination and relapse is developed and analysed by dividing the total human population under consideration into five compartments, namely, susceptible (S), TB-infected (T), HIV-infected (H), vaccinated (V) and AIDS-infected (A). We analysed the steady states behaviour of the dynamical system representing the co-infection transmission dynamics of HIV/AIDS and TB epidemic. The mathematical model possesses four equilibrium points such as disease free, HIV/AIDS infection free, TB infection free and vaccination free. The stability of aforesaid cases is also investigated. A threshold parameter reproduction number $\mathcal{R}_0$ is computed and if $\mathcal{R}_0 < 1$ the disease dies out and it becomes endemic if $\mathcal{R}_0 > 1$. It is also found that the co-infection period also influences the transmission patterns of diseases. Some important theorems and results are proved. Optimal control solutions are provided to predict the efficacy of vaccination and control strategies. The sensitivity analysis has also been facilitated to carry out the effects of certain key parameters on the diseases co-dynamics. It is found that administration of appropriate vaccine at proper time could be more effective in controlling the co-infection. The relapse factor is also considered in the model where the vaccination fails.

1. Introduction

Tuberculosis is caused by mycobacterium infectious disease that remains a problem of worldwide. TB is such a type of disease which increases due to the environmental

2010 *Mathematics Subject Classification.* 92BXX.

Key words and phrases. HIV/AIDS-TB co-infection, relapse, vaccination.

Received: March 21, 2021 Accepted: July 11, 2021 .

factors, e.g. open drainage of sewage in residential areas, open water storage tanks, discharge of household wastes in residential area, etc. Transmission of tuberculosis is occurred by air borne infectious droplets [7, 26]. Once infected with Mycobacterium TB, a person stays infected for many years, probably for the life time. It typically effects the lungs and other sensitive parts of human being. In 1993 the WHO declared Tuberculosis as a global emergency [3, 39, 40]. TB is curable and preventable. A model for tuberculosis has been an essential tool in controlling as well as assessing the spread of vital disease [30, 8].

Recently, the human immune deficiency virus (HIV) infection, which can lead to acquired immunodeficiency syndrome (AIDS), has become a life threatening diseases in both the developed and developing nations [5, 15, 33, 24]. It is a severe disease which breaks down the body's immune system and leaves the victim more vulnerable to infections and neurological disorder. If any person is infected with human immune deficiency virus, he or she will find the hurdle to fight with infectious diseases. The HIV destroys white blood cells known as $CD4$ cells. It causes mortality of millions of people and expenditure of enormous amount of money in health care and disease control. T-helper cells are called as $CD4$ cells [6, 37, 40]. Yet there is no vaccine for AIDS [9, 36]. However, antiretroviral (ART) treatment improves health, pro-longs life and substantially reduces the risk of HIV transmission. In both high income and low income countries, the life expectancy of patients infected with HIV who have access to ART is now measured in decades, and might approach that of uninfected populations in patients who receive a high HIV treatment [36]. The AIDS epidemic is now spreading rapidly in Asia, where new infection is increasing faster than anywhere else in the world. Globally, India has the highest estimated number of HIV infected people in any single country, next only to South Africa. India in epidemic is marked by heterogeneity. It is not a single epidemic but make of distinct epidemics of within the same state and continuous to be driven strongly by heterosexual transmission. Mathematical models have been used extensively in research into the epidemiology of HIV/AIDS, to help improve over understanding of major contributing factors in a given epidemic.

From the initial model, distinct rectification has been added into modelling frameworks, and specific issues have been addressed by researchers [31, 22, 38, 40]. In particular, developed a model for spread of HIV in a heterosexual population taking into account the group contact tracing and carried out equilibrium analysis. The impact of condom use amongst a homogeneously mixing male homosexual population on the sexual transmission of HIV/ AIDS [38, 2] proposed a nonlinear model for an epidemic with contact tracing and applied it to the Cuban HIV/AIDS epidemic to obtain the size of HIV epidemic. In [2, 20, 34, 13] the authors presented a theoretical frame work for transmission of HIV/AIDS epidemic in India. In [10, 40] the authors presented a homogeneous mixing population model for HIV transmission with incorporates an anti HIV preventive vaccine. In [10, 21, 38] the authors proposed a simple deterministic model to study the transmission of HIV/AIDS in a population with variable size structure. Developing a mathematical model for the individual that has the structure of two classes of sexually active male customers and commercial sex workers.

Co-infection of TB and HIV is when someone has both infectious diseases. TB increases the rate at which HIV infection develops into AIDS and HIV disease setup the activation of TB [17, 23]. It is noted that HIV infectious disease and infection with TB bacteria are two distinct infections as shown in [27]. Therefore, it is essential to study the transmission of HIV-TB in the population. Some studies have been made by taking into the account HIV-TB [19, 38] has proposed within host models for the dynamics between HIV and activated CD4T cells specific to other pathogens. In [18] a model is developed which reflects the transmission of the effect that the HIV epidemic may have on TB. They found the effect that the HIV epidemic may have on TB. They found the effect that HIV will have on the general population to be dependent on the contact structure between the general population and the HIV risk groups, as well as a possible shift in the dynamics associated with TB transmission. Co-treatment with HIV-related tuberculosis improves survival, especially in patients with CD4 less than fifty ell/mm3 [17]. Thirty-four million people living with HIV, the TB is contaminated globally by one-third of the [38, 37]. About 450,000 people were estimated with both HIV and TB died, in addition to the 1.1 million deaths from TB

and 800 000 deaths from HIV alone in the year 2014. Thus, more people died of TB in 2014 than from HIV-related infections [40]. It has built a sex-structured model in his work to capture the impact of complacency on the dynamics of HIV/AIDS, but did not include how the mix would be affected by TB [35]. But, a lot of work has been done in the mathematical modeling of co-infection of different viruses [28, 41, 29, 4]. A mathematical model has been developed for co-infection of TB-HIV [27, 17, 13, 26]. Their study did not provide treatment with anti-HIV medication. The model of co-infection with TB-HIV/AIDS and effective control treatment was developed [17]. Motivated and inspired by the work done by the people in this direction in this paper we introduced a novel vaccination class in which we have taken two parameters namely relapse rate and vaccination rate. The relapse rate in transmitting from the HIV class and reception in vaccination class. Further, the vaccination rate is transmitting from TB and reception also in vaccination class. Moreover, in the vaccination class the natural death rate is going out as usual like in other classes. The paper is arranged as follows: In Section 2, mathematical model formulation is given. Some basic properties of the proposed model are discussed in section 3. In section 4, analysis is done. In section 5, the optimal control analysis with multiple time-dependent controls is performed. The Numerical simulation is provided in section 6. Finally, conclusion is drawn in section 7.

2. Mathematical Model Description

In this model, we consider the population of size $N(t)$ with a constant recruitment rate of Q_0. The whole population is divided into five subclasses: susceptible $S(t), T(t)$ infected with tuberculosis, HIV infected with $H(t), V(t)$ vaccination and AIDS infected with $A(t)$. The susceptible population become tuberculosis infected at a rate β_1 before media alert, which becomes tuberculosis infected. The expression $\beta(t) = \left(\beta_1 - \beta_1' \frac{T}{m+T}\right)$ demonstrates the less quantity of the transmission dynamics of tuberculosis infection after media alarm. This tests the spread of TB infection from the infected person to the susceptible person. If m is equal to zero, transmission rate is constant. Therefore, the rate of transmission is not only related to the spreading potential of the disease, there are closely related to the disease awareness

of every susceptible host population. The media alert for TB infected population at the rate m exhibits the effect on communication transmission of media coverage. The reduced value of the transmission dynamic rate of HIV infection after media alert is taken as $\beta(H) = \left(\beta_2 - \beta_2' \frac{H}{n+H}\right)$. The parameter rate n reflects the impact of media coverage to the contact transmission of HIV infection and relapsed at a rate θ. At a constant rate β_3, the population in TB class is infected by HIV infection. Therefore Tuberculosis is curable disease, as such some people of Tuberculosis class are recovered at rate λ, vaccinated at a rate ϕ and enter into the susceptible class. In the vaccination class $V(t)$ we enter the vaccination, relapse and as usual the natural death rate is going out. With the infection rate δ, the HIV infection are converted in full blown AIDS and it is assumed that no anti-HIV drugs is available within population and hence some member of HIV class are bounded to develop full blown AIDS. Absolutely, once AIDS is developed in the population, no media awareness can help him to be controlled. Finally, the population dies at constant rate d. Let α be the disease-induced death rate (see Figure 1). The following system of non-

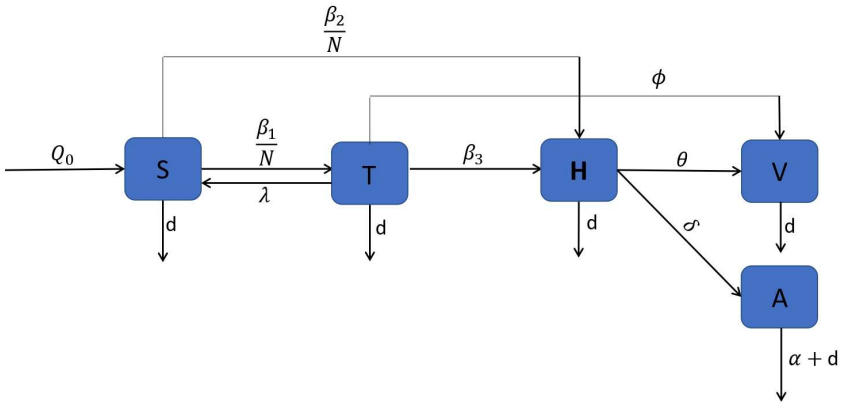

FIGURE 1. Transition diagram of the model

TABLE 1. Possible feasible ranges of malaria model (2.1)

State variables	Description
S	Susceptible host population
T	TB-Infected host population
H	HIV-Infected host population
V	Vaccinated host population
A	AIDS-Infected host population

TABLE 2. Parametric values of the HIV/AIDS-TB co-infection model (2.1)

Parameter source	Description	Parametric values	Sources
Q_0	Recruitment rate constant for host population	$(150-200)month^{-1}$	Estimated
β_1	TB infected rate	$(0.1-0.15)$	Estimated
β_1'	Chance of infection in the TB population	$(0.02-0.09)$	Estimated
β_2	Rate of infection for HIV	$(0.5-0.8)$	Estimated
β_2'	Chance of infection in the TB population	$(0.002-0.4)$	Estimated
m	Media alert rate in TB class	$(5-30)$	[21]
n	Media alert rate in HIV class	(4-10)	[15]
β_3	The population in TB class in infected by HIV infection	$(0.5-1.2)$	Estimated
λ	Recovered TB populations	0.5	[21]
δ	HIV infected people are bound to develop full blow AIDS	$(0.001-0.5)$	[22]
α	Disease induced death rate	$(0.5-2)$	[18]
d	Host natural death rate	$(0.01-0.05)$	Estimeted
ϕ	Vaccination rate for hosts	0.5	Estimated
θ	Relapse rate for hosts	0.08	Estimated

linear differential equations which govern the dynamical system.

$$\begin{aligned}
\frac{dS}{dt} =& Q_0 - \left(\beta_1 - \beta_1' \frac{T}{m+T}\right)\frac{ST}{N} - \left(\beta_2 - \beta_2' \frac{H}{n+H}\right)\frac{SH}{N} - dS + \lambda T \\
\frac{dT}{dt} =& \left(\beta_1 - \beta_1' \frac{T}{m+T}\right)\frac{ST}{N} - \beta_3 \frac{TH}{N} - (\lambda + d)T - \phi T \\
\frac{dH}{dt} =& \left(\beta_2 - \beta_2' \frac{H}{n+H}\right)\frac{SH}{N} + \beta_3 \frac{TH}{N} - (\delta + d)H - \theta H \\
\frac{dV}{dt} =& \phi T + \theta H - dV \\
\frac{dA}{dt} =& \delta H - (\alpha + d)A
\end{aligned} \tag{2.1}$$

Initial conditions, $S(0) = S_0 > 0, T(0) = T_0 \geq 0, H(0) = H_0 \geq 0, V(0) = V_0 \geq 0$ and $A(0) = A_0 \geq 0$. Since, $N(t) = S(t) + T(t) + H(t) + V(t) + A(t)$, the model (2.1) can be modified as:

$$\begin{aligned}
\frac{dN}{dt} =& Q_0 - dN - \alpha A \\
\frac{dT}{dt} =& \left(\beta_1 - \beta_1' \frac{T}{m+T}\right)\frac{(N - T - H - V - A)T}{N} - \beta_3 TH - (\lambda + d)T - \phi T \\
\frac{dH}{dt} =& \left(\beta_2 - \beta_2' \frac{H}{n+H}\right)\frac{(N - T - H - V - A)H}{N} + \beta_3 TH - (\delta + d)H - \theta H \\
\frac{dV}{dt} =& \phi T + \theta H - dV \\
\frac{dA}{dt} =& \delta H - (\alpha + d)A
\end{aligned} \tag{2.2}$$

3. Basic Mathematical Properties of the Model

In this section, we discuss some important basic properties of model which are estimated in finding the existence and uniqueness of a solution.

Uniqueness of Solution

Theorem 3.1. Suppose D denotes the domain, and

$$|t - t_0| \leq a, \|x - x_0\| \leq b, \tag{3.1}$$

where $x = (x_1, x_2, x_3, ..., x_n), x_0 = (x_{10}, x_{20}, x_{30}, ..., x_{n0})$.

Also, suppose that $g(t, x)$ satisfies the Lipchitz condition:

$$\|f(t, x_1) - f(t, x_2)\| \leq k\|x_1 - x_2\|. \tag{3.2}$$

Then, whenever the points (t, x_1) and (t, x_2) belong to the domain D, where k is used to represent a positive constant, there exists a constant $\delta > 0$ such that there exists a unique (exactly one) continuous vector solution $x(t)$ of model (2.1) in the interval $|t - t_0| \leq \delta$. It is important to note that condition (3.1) is satisfied by requirement that $\left\{ \dfrac{\partial f_i}{\partial x_j} : i, j = 1, 2, 3, ..., n \right\}$ is continuous and bounded in the domain D.

Lemma 3.1. *If $g(t, x)$ has continuous partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ on a bounded closed convex domain $\mathbb{R}$ (i.e, convex set of numbers), where $\mathbb{R}$ is used to denote real number, then it satisfies Lipchitz conditions in $\mathbb{R}$.*

Our interest is in the domain $1 \leq \epsilon \leq \mathbb{R}$. So, we look for a bounded solution $0 < \mathbb{R} < \infty$. We now prove the following existence theorem.

Existence of Solution

Theorem 3.2. Let D denote the domain defined in equation (3.1) such that (3.2) hold. Then, there exists a solution of model system of equations which is bounded in the domain D.

Proof. Let

$$
\begin{aligned}
f_1 =& Q_0 - dN - \alpha A, \\
f_2 =& \left(\beta_1 - \beta_1' \frac{T}{m+T} \right) \frac{(N - T - H - V - A)T}{N} - \beta_3 TH - (\lambda + d)T - \phi T, \\
f_3 =& \left(\beta_2 - \beta_2' \frac{H}{n+H} \right) \frac{(N - T - H - V - A)H}{N} + \beta_3 TH - (\delta + d)H - \theta H, \\
f_4 =& \phi T + \theta H - dV, \\
f_5 =& \delta H - (\alpha + d)A.
\end{aligned}
$$

We show that $\left\{ \dfrac{\partial f_i}{\partial x_j} : i, j = 1, 2, 3, 4, 5 \right\}$ are contnuious and bounded. We explored the following partial derivatives for all the model equations.

$$
\begin{aligned}
&\left| \frac{\partial f_1}{\partial N} \right| = |-d| < \infty, \left| \frac{\partial f_1}{\partial T} \right| = \left| \frac{\partial f_1}{\partial H} \right| = \left| \frac{\partial f_1}{\partial V} \right| = |0| < \infty, \left| \frac{\partial f_1}{\partial A} \right| = |-\alpha| < \infty, \\
&\left| \frac{\partial f_2}{\partial N} \right| = \left| \frac{T}{N^2} \left[\beta_1 (N - S) - \beta_1' \frac{(N - S)T}{(m + T)} \right] \right| < \infty, \\
&\left| \frac{\partial f_2}{\partial T} \right| = \left| \left[\beta_1 - \xi_1 - \beta_3 H - \beta_1 \frac{(T + N - S)}{N} - \beta_1' \frac{(4A + 2S - T)T}{N(m + T)} - \beta_1' \frac{(2N - S)T^2}{N(m + T)^2} \right] \right| < \infty,
\end{aligned}
$$

$$\left|\frac{\partial f_2}{\partial H}\right| = \left|-\frac{T}{N}\left[\beta_1 + \beta_3 N - \frac{\beta_1' T}{m+T}\right]\right| < \infty, \left|\frac{\partial f_2}{\partial V}\right| = \left|\frac{\beta_1' T^2}{N(m+T)} - \frac{\beta_1 T}{N}\right| < \infty,$$
$$\left|\frac{\partial f_2}{\partial A}\right| = \left|\frac{\beta_1' T^2}{N(m+T)} - \frac{\beta_1 T}{N}\right| < \infty,$$
$$\left|\frac{\partial f_3}{\partial N}\right| = \left|\left[\frac{\beta_2(T+H+A)HT^2}{N} + \frac{\beta_2'(T+H+A)TH^2}{N^2(n+H)}\right]\right| < \infty,$$
$$\left|\frac{\partial f_3}{\partial T}\right| = \left|\frac{H}{N}\left[\beta_3 N - \beta_2 - \xi_1 - \beta_2'\frac{H}{(n+H)}\right]\right| < \infty,$$
$$\left|\frac{\partial f_3}{\partial H}\right| = \left|\left[\beta_2 - \xi_2 - \beta_3 T - \beta_2\frac{(H+N-S)}{N} - \beta_2'\frac{(4T+2S)H}{N(n+H)} - \beta_2'\frac{(2N-S)H^2}{N(n+H)^2}\right]T\right| < \infty,$$
$$\left|\frac{\partial f_3}{\partial V}\right| = \left|\frac{H}{N}\left[\beta_2'\frac{H}{(n+H)} - \beta_2'\right]\right| < \infty, \left|\frac{\partial f_3}{\partial A}\right| = \left|\frac{H}{N}\left[\beta_2'\frac{H}{(n+H)} - \beta_2'\right]\right| < \infty,$$
$$\left|\frac{\partial f_4}{\partial T}\right| = |\phi| < \infty, \left|\frac{\partial f_4}{\partial H}\right| = |\theta| < \infty, \left|\frac{\partial f_4}{\partial V}\right| = |-d| < \infty, \left|\frac{\partial f_4}{\partial N}\right| = \left|\frac{\partial f_4}{\partial A}\right| = \left|\frac{\partial f_5}{\partial N}\right| =$$
$$\left|\frac{\partial f_5}{\partial T}\right| = \left|\frac{\partial f_5}{\partial V}\right| = |0| < \infty, \left|\frac{\partial f_5}{\partial H}\right| = |\delta| < \infty, \left|\frac{\partial f_5}{\partial A}\right| = |-(\alpha+d)| < \infty.$$

□

3.1. **Invariant and Attractive Region.** The dynamical transmission of the model (2.1) will be analyzed in the following biologically feasible region $\Omega \subset \mathbb{R}^5_+$ where $\Omega = \left\{(S,T,H,V,A) \in \mathbb{R}^5_+ : S>0, T\geq 0, H\geq 0, V\geq 0, A\geq 0 : S+T+H+V+A \leq \frac{Q_0}{d}\right\}$.

Theorem 3.3. The feasible region $\Omega \subset \mathbb{R}^5_+$ is positively invariant for the model (2.1) with respect to initial conditions in $\mathbb{R}^5_+$.

Proof. Since, $N = S+T+H+V+A$, we have

$$\frac{dN}{dt} \leq Q_0 - dN - \alpha A. \tag{3.3}$$

The solution of the equation (3.3) is

$$N(t) = \frac{Q_0}{d} + \left(N(0) - \frac{Q_0}{d}\right)e^{-d(t)}$$

Now, taking the limit, we get

$$\lim_{t\to\infty} N(t) \leq \frac{Q_0}{d}.$$

Therefore, for positive value of $t, N(t)$ converges to infinity. So, the solution of the model (2.1) with initial conditions remains in Ω. Thus, the feasible region Ω is positively invariant and attracts all solutions in $\mathbb{R}^5_+$. □

3.2. **Positivity and Boundedness.**

Theorem 3.4. The solution of the model (2.1) is positively bounded for all $(S(0), T(0), H(0), V(0), A(0)) \in \mathbb{R}^5_+$, and also defined for the positive value of time t.

Proof. In order to explore the positive solution, it is required to verify that on every hyperplane bounding the positive orthant, the vector field point $\mathbb{R}^5_+$. From the model (2.1) we have

$$\begin{aligned}
\frac{dS}{dt}(at\ S=0) =& Q_0 + \lambda T \geq 0\\
\frac{dT}{dt}(at\ T=0) =& 0 \geq 0\\
\frac{dH}{dt}(at\ H=0) =& 0 \geq 0\\
\frac{dV}{dt}(at\ V=0) =& \phi T + \theta H \geq 0\\
\frac{dA}{dt}(at\ A=0) =& \delta H \geq 0.
\end{aligned}$$

Therefore, the above target set has been achieved and hence the solution will stay in $\mathbb{R}^5_+$. Thus $\Omega = (S(0), T(0), H(0), V(0), A(0)) \in \mathbb{R}^5_+ : S(0), T(0), H(0), V(0), A(0) \geq 0$ is bounded in feasible region. Hence, all terms of the sum are positive and thus the solution of the model (2.1) is bounded. □

4. The Analysis

The model's analysis was achieved by measuring the points of equilibrium.

4.1. **Equilibrium Analysis.** The five possible points of equilibrium are obtained as follows:

(i) DFE point $E_0(\overline{N}, 0, 0, 0, 0)$ exists for all parameter values as

$$\overline{N} = \frac{Q_0}{d}, \overline{T} = 0, \overline{H} = 0, \overline{V} = 0, \overline{A} = 0.$$

(ii) IFE point on HIV/AIDS $E_1(N^*, T^*, 0, V^*, A^*)$ is given by

$$N^* = \frac{Q_0}{d}, A_1 T^{*2} + B_1 T^* + C_1 = 0, H^* = 0, V^* = \frac{\phi T^*}{d}, A^*$$

where $A_1 = -(\beta_1 - \beta_1')N^*, B_1 = -N^* m\beta_1 + (\beta_1 - \beta_1') - \xi_1, C_1 = m(\beta_1 - \xi_1)$ in which $\xi_1 = (\lambda + d + \phi)$. Further, since $C_1 > 0$, if $(\beta_1 - \xi_1) > 0$, the model has a unique HIV/AIDS free equilibrium point.

(iii) IFE point on TB $E_2(\widehat{N}, 0, \widehat{H}, \widehat{V}, \widehat{A})$ is

$$\widehat{N} = \frac{1}{d}\bigg(Q_0 - \frac{\alpha\delta\widehat{H}}{\alpha + d}\bigg), \widehat{T} = 0, A_2\widehat{H}^2 + B_2\widehat{H} + C_2 = 0, \widehat{V} = \frac{\theta\widehat{H}}{d}, \widehat{A} = \frac{\delta\widehat{H}}{\alpha + d}$$

where

$$\begin{aligned}
A_2 =& -(\beta_2 - \beta_2')\bigg(\frac{\delta}{\alpha + d}\bigg(\frac{\alpha + d}{d} + 1 - \frac{\alpha\delta\xi_2}{d(\alpha + d)}\bigg)\bigg),\\
B_2 =& -\frac{n\delta\beta_2}{(\alpha + d)}\bigg(\frac{\alpha + d}{d}\bigg) - n\beta_2 + (\beta_2 - \beta_2')\widehat{N} - \frac{n\alpha\delta\xi_2}{d(\alpha + d)} - \xi_2\widehat{N},\\
C_2 =& n\widehat{N}(\beta_2 - \xi_2)
\end{aligned}$$

in which $\xi_2 = (\delta + d + \theta)$. Clearly, $C_2 > 0$, if $\beta_2 > \xi_2$, then the model has single TB infection free equilibrium point.

(iv) The vaccination free equilibrium point $E_3(\acute{N}, \acute{T}, \acute{H}, 0, \acute{A})$ is given by

$$\begin{aligned}
\acute{N} =& \frac{1}{d}\bigg(Q_0 - \frac{\alpha\delta\acute{H}}{\alpha + d}\bigg), V' = 0, \acute{A} = \frac{\delta\acute{H}}{\alpha + d},\\
\acute{T} =& \frac{\dfrac{Q_0}{d}\big[\eta_1 - \xi_1\big] - \bigg[\eta_1\bigg\{\dfrac{\mu}{\alpha + d}\bigg\} + \beta_3 + \alpha\delta\bigg\{\dfrac{\eta_1 - \xi_2}{d(\alpha + d)}\bigg\}\bigg]\acute{H}}{\eta_1},
\end{aligned}$$

$$\acute{H} = \frac{\dfrac{Q_0}{d}\bigg[\big\{\eta_2 - (\beta_2 - \xi_2)\big\} + \bigg(\dfrac{\beta_3 - \eta_2}{\eta_1}\bigg)\big\{\eta_1 - (\beta_2 - \xi_2)\big\}\bigg]}{\bigg[\dfrac{\beta_3\mu}{\alpha + d} + \beta_3\bigg(\dfrac{\beta_3 - \eta_2}{\eta_1}\bigg) + \dfrac{\alpha\delta}{d(\alpha + d)}\big[\eta_1 - \xi_1 + \big\{\eta_2 - (\beta_2 - \xi_2)\big\}\big]\bigg]}$$

where, $\mu = \alpha + d + \lambda, \eta_1 = (\beta_1 - \beta_1' P_T)$ and $\eta_2 = (\beta_2 - \beta_2' P_H)$.

(v) The endemic co-infection equilibrium point $E_4(\widetilde{N}, \widetilde{T}, \widetilde{H}, \widetilde{V}, \widetilde{A})$ is given by

$$\begin{aligned}
\widetilde{N} =& \frac{1}{d}\bigg(Q_0 - \frac{\alpha\delta\widetilde{H}}{\alpha + d}\bigg), \widetilde{V} = \frac{\phi\widetilde{T}}{d} + \frac{\theta\widetilde{H}}{d}, \widetilde{A} = \frac{\delta\widetilde{H}}{\alpha + d},\\
\widetilde{T} =& \frac{\dfrac{Q_0}{d}\big[\eta_1 - \xi_1\big] - \bigg[\eta_1\bigg\{\dfrac{\mu}{\alpha + d}\bigg\} + \beta_3 + \alpha\delta\bigg\{\dfrac{\eta_1 - \xi_2}{d(\alpha + d)}\bigg\}\bigg]\widetilde{H}}{\eta_1},
\end{aligned}$$

$$\widetilde{H} = \frac{\frac{Q_0}{d}\left[\{\eta_2 - (\beta_2 - \xi_2)\} + \left(\frac{\beta_3 - \eta_2}{\eta_1}\right)\{\eta_1 - (\beta_2 - \xi_2)\}\right]}{\left[\frac{\beta_3\mu}{\alpha + d} + \beta_3\left(\frac{\beta_3 - \eta_2}{\eta_1}\right) + \frac{\alpha\delta}{d(\alpha + d)}\left[\eta_1 - \xi_1 + \{\eta_2 - (\beta_2 - \xi_2)\}\right]\right]}.$$

4.2. **Linear Stability Analysis.** We linearize model (2.2) by replacing the equilibrium points with a minor disturbance by replacing $N(t) = N + n(t), T(t) = T + \tau(t), H(t) = H + h(t), V(t) = V + v(t), A(t) = A + a(t)$ where $n(t), \tau(t), h(t), v(t)$ and $a(t)$, are the small disruption that is induced by points of equilibrium.

4.2.1. *Stability of Diseases Free Equilibrium Point E_0.* The model (2.2) for Jacobian matrix about equilibrium point E_0 is given by

$$J\left(\frac{Q_0}{d}, 0, 0, 0, 0\right) = \begin{bmatrix} -d & 0 & 0 & 0 & -\alpha \\ 0 & \frac{Q_0}{d}\left[\beta_1 - \xi_1\right] & 0 & 0 & 0 \\ 0 & 0 & \frac{Q_0}{d}\left[\beta_2 - \xi_2\right] & 0 & 0 \\ 0 & \varphi & \theta & -d & 0 \\ 0 & 0 & \delta & 0 & -(\alpha + d) \end{bmatrix}$$

The characterstic equation of J is $|J - \lambda' I| = 0$, i.e.

$$(d + \lambda')^2(-(\alpha + d + \lambda'))\left[\frac{Q_0}{d}(\beta_1 - \xi_1) - \lambda'\right]\left[\frac{Q_0}{d}(\beta_2 - \xi_2) - \lambda'\right] = 0 \tag{4.1}$$

Now,

$(i)\lambda' = -d \implies \lambda' < 0,$

$(ii)\ \lambda' = -d \implies \lambda' < 0,$

$(iii)\ \lambda' = \frac{Q_0}{d}(\beta_1 - \xi_1) \implies \lambda' < 0$ if $\beta_1 < \xi_1 \implies \frac{\beta_1}{\xi_1} < 1$. Here $\frac{\beta_1}{\xi_1} = \mathcal{R}_T$ is reproduction number for the Tuberculosis epidemic,

$(iv)\ \lambda' = \frac{Q_0}{d}(\beta_2 - \xi_2) \implies \lambda' < 0$ if $\beta_2 < \xi_2 \implies \frac{\beta_2}{\xi_2} < 1$. Again here $\frac{\beta_2}{\xi_2} = \mathcal{R}_H$ is reproduction number for the HIV infection,

$(v)\ \lambda' = -(\alpha + d) \implies \lambda' < 0.$

It is observed that the DFE is locally asymptotically stable if $\mathcal{R}_T, \mathcal{R}_H$ is less than one, the disease will die out and become no longer epidemic; and it becomes unstable if $\mathcal{R}_T, \mathcal{R}_H$ is greater than one in such case the disease becomes epidemic.

4.2.2. *Stability Analysis of HIV/AIDS Free point* $E_1(N^*, T^*, 0, V^*, A^*)$. The model (2.2) for Jacobian matrix about equilibrium point E_1 is given by

$$J(N^*, T^*, 0, V^*, A^*) = \begin{bmatrix} -d & 0 & 0 & 0 & -\alpha \\ D_1 & F_1 & G_1 & H_1 & I_1 \\ 0 & 0 & K_1 & 0 & 0 \\ 0 & \phi & \theta & -d & 0 \\ 0 & 0 & \delta & 0 & -(\alpha+d) \end{bmatrix}$$

where

$$\begin{aligned} D_1 =& \frac{d^2}{Q_0^2}\left[\beta_1(T^*+V^*) - T\beta_1'\frac{(T^*+V^*)}{(m+T^*)}\right]T^*, \\ F_1 =& \left[\frac{Q_0}{d}\left\{\beta_1 - \xi_1 - \frac{2\beta_1'T^*}{(m+T^*)} + \frac{\beta_1'T^{*2}}{(m+T^*)^2}\right\} + \left\{\frac{\beta_1'T^*u}{(m+T^*)^2} + \frac{\beta_1'v}{(m+T^*)} - 2\beta_1\right\}T^* \right. \\ & \left. -\beta_1 V^*\right], \\ G_1 =& -\left[\beta_1 - \beta_1'\frac{T^*}{m+T^*} - \beta_3\frac{Q_0}{d}\right]T^*, \\ H_1 =& \frac{Q_0}{d}\left[-\beta_1 - \beta_1'\frac{T^*}{m+T^*}\right]T^*, \\ I_1 =& -\frac{d}{Q_0}\left[-\beta_1 - \beta_1'\frac{T^*}{m+T^*}\right]T^*, \end{aligned}$$

$$K_1 = \left[\frac{Q_0}{d}(\beta_3 - \xi_2) - \left(\beta_2 + \frac{Q_0}{d}\beta_3\right)T^* - \beta_3 V^*\right],$$

in which $u = V^* - T^*$ and $v = 3T^* - 2V^*$.

The characterstic equation of J is given as

$$|J - \lambda' I| = 0$$

$$\lambda'^5 + a_1\lambda'^4 + a_2\lambda'^3 + a_3\lambda'^2 + a_4\lambda' + a_5 = 0 \tag{4.2}$$

where

$$
\begin{aligned}
a_1 =&[3d - F_1 - K_1 + \alpha],\\
a_2 =&[d(2d - 2F_1 - 3K_1) + F_1K_1 - \phi H_1 + (d - 2F_1 - K_1)(\alpha + d)],\\
a_3 =&[-d(2dF_1 + 2\phi H_1 - K_1F_1 - 2\phi K_1 - dK_1) - (dF_1 + 2\phi H_1 - 2dK_1 + K_1F_1)\\
&\times(\alpha + d)],\\
a_4 =&(\alpha + d)(2dK_1F_1 - d^2(F_1 + K_1) + \phi H_1(K_1 - d)) + K_1(d^2F_1 + \alpha\phi H_1).
\end{aligned}
$$

Hence, E_1 is unstable due to positive values F_1 and K_1 and this implies that $a_1 > 0$. This fails the Routh-Hurwit criterion for stability. By this criterion, E_1 is locally asymptotically stable if $a_i > 0 (i = 1, 2, 3, 4, 5), a_1a_2 - a_3 > 0, a_1^2a_4 - a_3^2 > 0$ and $a_1a_2a_3^2 - a_1a_2^2a_4 - a_1a_3a_4 + a_1a_2a_5 - a_3^2 + a_2a_3a_4 - a_1a_3a_5 > 0$, otherwise unstable.

4.2.3. *Stability Analysis of TB Free Equilibrium point E_2.* The model (2.2) for Jacobian matrix about equilibrium point E_2 is given by

$$
J(\widehat{N}, 0, \widehat{H}, \widehat{V}, \widehat{A}) = \begin{bmatrix}
-d & 0 & 0 & 0 & -\alpha \\
0 & D_2 & 0 & 0 & 0 \\
F_2 & G_2 & H_2 & V_2 & K_2 \\
0 & \phi & \theta & -d & 0 \\
0 & 0 & \delta & 0 & -(\alpha + d)
\end{bmatrix}
$$

where

$$
\begin{aligned}
D_2 =&\widehat{N}(\beta_1 - \xi_1) - \beta_1(\widehat{V} + \widehat{A}) - (\beta_1 + \beta_3\widehat{N})\widehat{H},\\
F_2 =&\frac{1}{\widehat{N}^2}\left[\beta_2(\widehat{N}^2 + \widehat{V} + \widehat{A}) - \beta_2\frac{\widehat{H}(\widehat{H} + \widehat{V} + \widehat{A})}{(n + \widehat{H})}\right]\widehat{H},\\
G_2 =&\left[\left(\widehat{N}\beta_3 - \beta_3 + \beta_2'\frac{\widehat{H}}{n + \widehat{H}}\right)\right]\widehat{H},\\
H_2 =&\widehat{N}\left[\left(\beta_2'\frac{\widehat{H}^2}{(n + \widehat{H})^2} + \beta_2' - 2\beta_2'\frac{\widehat{H}}{n + \widehat{H}}\right) - \xi_2\right] - \left[\beta_2'\frac{\widehat{H}w}{(n + \widehat{H})^2} + 3\beta_2\widehat{H} - \beta_2'\frac{x}{(n + \widehat{H})}\right]\\
&\times\widehat{H} - \beta_2(\widehat{V} + \widehat{A}),\\
V_2 =&-\frac{1}{\widehat{N}}\left[\beta_2 + \beta_2'\frac{\widehat{H}}{n + \widehat{H}}\right]\widehat{H},\\
K_2 =&-\frac{1}{\widehat{N}}\left[\beta_2 + \beta_2'\frac{\widehat{H}}{n + \widehat{H}}\right]\widehat{H},
\end{aligned}
$$

in which $w = \widehat{H} + \widehat{V} + \widehat{A}$ and $x = 3\widehat{H} + 2\widehat{V} + 2\widehat{A}$.

The characterstic equation of J is given as

$$|J - \lambda' I| = 0$$

$$\lambda'^5 + b_1\lambda'^4 + b_2\lambda'^3 + b_3\lambda'^2 + b_4\lambda' + b_5 = 0 \tag{4.3}$$

where

$$b_1 = \big[(\alpha + 3d) - (D_2 + H_2)\big],$$

$$b_2 = \big[(d - H_2 - V_2)(\alpha + d) - (3D_2 + 2H_2 + V_2)d + (H_2 - \alpha)D_2 - \theta V_2\big],$$

$$b_3 = \big[(dH_2 - \theta V_2 - dD_2 + D_2(H_2 + V_2)) + (D_2H_2 - dH_2 - \theta V_2 + D_2(H_2 + V_2))d(\delta D_2 \times K_2 + \theta D_2V_2)\big],$$

$$b_4 = \big[dD_4H_2 - 2d^2H_2 - 2d\theta V_2 - \theta D_2V_2\big](\alpha + d) - dD_2(dH_2 + \theta V_2),$$

$$b_5 = \big[2d(\alpha + d)D_2(dH_2 + \theta V_2)\big].$$

Hence, E_2 will be asymptotically stable if $\beta_1 < \xi_1, \beta_2 < \xi_2$ and is unstable due to two +ve eigenvalues. According to Ruth-Hurwitz's E_2 will be asymptotically stable if $b_i(i = 1, 2, 3, 4, 5), b_1b_2 - b_3 > 0, b_1^2b_4 - b_3^2 > 0$ and $b_1b_2b_3^2 - b_1b_2^2b_4 - b_1b_3b_4 + b_1b_2b_5 - b_3^3 + b_2b_3b_4 - b_1b_3b_5 > 0$ and unstable otherwise.

4.2.4. *Stability Analysis of Vaccine Free Equilibrium point* $E_3(\acute{N}, \acute{T}, \acute{H}, 0, \acute{A})$. The model (2.2) for Jacobian matrix about equilibrium point E_3 is given by

$$J(\acute{N}, \acute{T}, \acute{H}, 0, \acute{A}) = \begin{bmatrix} -d & 0 & 0 & 0 & -\alpha \\ D_3 & F_3 & G_3 & V_3 & H_3 \\ K_3 & L & M & V_4 & N \\ 0 & \phi & \theta & -d & 0 \\ 0 & 0 & \delta & 0 & -(\alpha + d) \end{bmatrix}$$

where

$$D_3 = \frac{1}{N'^2}\left[\left(\beta_1\frac{(\acute{T} + \acute{H} + \acute{A})\acute{T}}{(m + \acute{T})} - \beta_1'\frac{(\acute{T} + \acute{H} + \acute{A})\acute{T}}{(m + \acute{T})}\right)\right]\acute{T},$$

$$F_3 = \acute{N}\left[\beta_1 - \xi_1 - \beta_3\acute{H} - 2\beta_1'\frac{T'^2}{(m+\acute{T})^2}\right] - \left[2\beta_1 - 3\beta_1'\frac{\acute{T}}{(m+\acute{T})}S + \beta_1'\frac{(\acute{T}+\acute{H}+\acute{A})\acute{T}}{(m+\acute{T})^2}\right]$$
$$\times\acute{T} + \left[\beta_1 + 2\beta_1'\frac{\acute{T}}{m+\acute{T}}\right]\acute{H} + \left[\beta_1 + 2\beta_1'\frac{\acute{T}}{m+\acute{T}}\right]\acute{A},$$
$$G_3 = -\frac{1}{\acute{N}}\left[\beta_1 + \beta_3\acute{N} - \beta_1'\frac{\acute{T}}{m+\acute{T}}\right]\acute{T},$$
$$V_3 = -\frac{1}{\acute{N}}\left[\beta_1 - \beta_1'\frac{\acute{T}}{m+\acute{T}}\right]\acute{T},$$
$$H_3 = -\frac{1}{\acute{N}}\left[\beta_1 - \beta_1'\frac{\acute{T}}{m+\acute{T}}\right]\acute{T},$$
$$K_3 = \frac{1}{N'^2}\left[\left(\beta_2(\acute{T}+\acute{H}+\acute{A}) - \acute{H}\frac{\beta_1'\acute{T} + \beta_2'(\acute{H}+\acute{A})}{(n+\acute{H})^2}\right)\right]\acute{H},$$
$$L = \left[\beta_3 - \frac{\beta_3}{N} - \beta_2'\frac{\acute{H}}{\acute{N}(n+\acute{H})}\right]\acute{H},$$

$$M = \frac{1}{\acute{N}}\left[-2\beta_2\acute{H} - \beta_2\acute{A} + \acute{H}\frac{(2\beta_1'\acute{T} + \beta_2'(3\acute{H}+2\acute{A}))}{(n+\acute{H})} - H'^2\frac{(\beta_1'\acute{T} + \acute{A} + \beta_2'(\acute{H}+\acute{A}))}{(n+\acute{H})^2}\right]$$
$$+\left[\beta_2\xi_2 + \beta_2'\frac{2\acute{H}}{(n+\acute{H})} + \beta_2'\frac{H'^2}{(n+\acute{H})^2}\right],$$
$$V_4 = \frac{1}{\acute{N}}\left[\beta_2'\frac{\acute{H}}{(n+\acute{H})} - \beta_2\right]\acute{H},$$
$$N = \frac{1}{\acute{N}}\left[\beta_2'\frac{\acute{H}}{(n+\acute{H})} - \beta_2\right]\acute{H}.$$

The characterstic equation of J is given as

$$|J - \lambda' I| = 0$$

$$\lambda'^5 + c_1\lambda'^4 + c_2\lambda'^3 + c_3\lambda'^2 + c_4\lambda' + c_5 = 0 \tag{4.4}$$

where

$$c_1 = [2d - F_3 - M + (\alpha + d)],$$
$$c_2 = [dF_3 - MF_3 + G_3L + V_3\phi + dM + V_4\theta - d(F_3 + M - d) + (d - F_3 - M + d)$$
$$\times(\alpha + d) - \delta N],$$

$$
\begin{aligned}
c_3 =&[\delta(hl+\alpha V_3-2dN)+(-MdF_3-F_3V_3\theta+G_3Ld+D_3V_4\phi+V_3L\theta-V_3M\phi)\\
&+d(-MF_3+dF_3+G_3L+V_3\phi+dM+V_4\theta)+(-MF_3+dF_3+G_3L+V_3\phi\\
&+dM+V_4\theta)+d(F_3+M-d)(\alpha+d)+\delta NF_3],\\
c_4 =&[\delta(\phi(H_3V_4-NV_3)-d(H_3L-\alpha K_3-dN))-d(F_3(d-M)+G_3L+\phi V_3+dM\\
&+\theta V_4)-(\alpha+d)(dG_3L+D_3V_4+\theta LV_3-dF_3M-\theta F_3V_4-\phi MV_3-d(\phi(F_3+V_3)\\
&+M(d-F_3))+G_3L-\theta V_4)+d\delta NF_3],\\
c_5 =&[d(\alpha+d)(F_3dM-F_3V_4\theta-G_3V_4\phi-V_3L\theta+V_3M\phi)+\delta\phi(dH_3V_3-dH_3V_4+D_3\\
&\times V_4-\alpha V_3K_3)+d(\alpha D_3L\delta-\alpha F_3K_3\theta-L\delta H_3d-F_3NL\delta)+\delta(aD_3L-\alpha F_3K_3\\
&-LH_3d-dF_3H)].
\end{aligned}
$$

Clearly, $c_i>0(i=1,2,3,4,5)$.

When $E_3<0$ then $\beta_1<\xi_1, M<0, \beta_2<\xi_2$. From Ruth-Hurwitz that the equation (4.4) has negative real parts iff $\beta_1<\xi_1, \beta_2<\xi_2, c_1c_2-c_3>0, c_1^2c_2-c_3>0$ and $c_1c_2c_3^2-c_1c_2^2c_4-c_1c_3c_4+c_1c_2c_5-c_3^3+c_2c_3c_4-c_1c_3c_5>0$. Under these conditions E_3 will be asymptotically stable otherwise unstable.

4.2.5. *Stability Analysis of Endemic Equilibrium point* $E_4(\widetilde{N},\widetilde{T},\widetilde{H},\widetilde{V},\widetilde{A})$. The model (2.2) for Jacobian matrix about equilibrium point E_4 is given by

$$
J(\widetilde{N},\widetilde{T},\widetilde{H},\widetilde{V},\widetilde{A})=\begin{bmatrix}
-d & 0 & 0 & 0 & -\alpha\\
D_4 & F_4 & G_4 & V_5 & H_4\\
K_4 & L & M & V_6 & N\\
0 & \phi & \theta & -d & 0\\
0 & 0 & \delta & 0 & -(\alpha+d)
\end{bmatrix}
$$

where

$$
\begin{aligned}
D_4 =&\frac{1}{\widetilde{N}^2}\left[\beta_1(\widetilde{N}-\widetilde{S})-\beta_1'\frac{(\widetilde{N}-\widetilde{S})\widetilde{T}}{(m+\widetilde{T})}\right]\widetilde{T},\\
F_4 =&\widetilde{N}\left[\beta_1-\xi_1-\beta_3\widetilde{H}-\beta_1\frac{(2\widetilde{N}-\widetilde{S})}{\widetilde{N}}-\beta_1'\frac{(4\widetilde{A}+2\widetilde{S}-\widetilde{T})\widetilde{T}}{\widetilde{N}(m+\widetilde{T})}-\beta_1'\frac{(2\widetilde{N}-\widetilde{S})\widetilde{T}^2}{(m+\widetilde{T})^2}\right],
\end{aligned}
$$

$$G_4 = -\frac{1}{\widetilde{N}}\left[\beta_1 + \beta_3\widetilde{N} - \beta_1'\frac{\widetilde{T}}{m+\widetilde{T}}\right]\widetilde{T},$$

$$V_5 = -\frac{1}{\widetilde{N}}\left[\beta_1 - \beta_1'\frac{\widetilde{T}}{m+\widetilde{T}}\right]\widetilde{T},$$

$$H_4 = -\frac{1}{\widetilde{N}}\left[\beta_1 - \beta_1'\frac{\widetilde{T}}{m+\widetilde{T}}\right]\widetilde{T},$$

$$K_4 = \frac{1}{\widetilde{N}^2}\left[\left(\beta_2(\widetilde{T}+\widetilde{H}+\widetilde{A}) - \widetilde{H}\frac{\beta_1'\widetilde{T}+\beta_2'(\widetilde{H}+\widetilde{A})}{(n+\widetilde{H})^2}\right)\right]\widetilde{H},$$

$$L = \left[\beta_3 - \frac{\beta_3}{\widetilde{N}} - \beta_2'\frac{\widetilde{H}}{\widetilde{N}(n+\widetilde{H})}\right]\widetilde{H},$$

$$M = \frac{1}{\widetilde{N}}\left[-2\beta_2\widetilde{H} - \beta_2\widetilde{A} + \widetilde{H}\frac{(2\beta_1'\widetilde{T}+\beta_2'(3\widetilde{H}+2\widetilde{A}))}{(n+\widetilde{H})} - \widetilde{H}^2\frac{(\beta_1'\widetilde{T}+\widetilde{A}+\beta_2'(\widetilde{H}+\widetilde{A}))}{(n+\widetilde{H})^2}\right]$$
$$+\left[\beta_2\xi_2 + \beta_2'\frac{2\widetilde{H}}{(n+\widetilde{H})} + \beta_2'\frac{\widetilde{H}^2}{(n+\widetilde{H})^2}\right],$$

$$V_6 = \frac{1}{\widetilde{N}}\left[\beta_2'\frac{\widetilde{H}}{(n+\widetilde{H})} - \beta_2\right]\widetilde{H},$$

$$N = \frac{1}{\widetilde{N}}\left[\beta_2'\frac{\widetilde{H}}{(n+\widetilde{H})} - \beta_2\right]\widetilde{H}.$$

The characterstic equation of J is given as

$$|J - \lambda' I| = 0$$

$$\lambda'^5 + d_1\lambda'^4 + d_2\lambda'^3 + d_3\lambda'^2 + d_4\lambda' + d_5 = 0 \tag{4.5}$$

where

$$d_1 = [2d - F_3 - M + (\alpha + d)],$$

$$d_2 = [d^2 - MF_3 + G_3L + V_3\phi + V_4\theta + (2d - F_3 - M + d)(\alpha) - \delta N],$$

$$d_3 = [\delta(hl + \alpha V_3 - 2dN) + (-MdF_3 - F_3V_3\theta + G_3Ld + D_3V_4\phi + V_3L\theta - V_3M\phi)$$
$$+d(-MF_3 + dF_3 + G_3L + V_3\phi + dM + V_4\theta) + (-MF_3 + dF_3 + G_3L + V_3\phi$$
$$+dM + V_4\theta) + d(F_3 + M - d)(\alpha + d) + \delta NF_3],$$

$$
\begin{aligned}
d_4 =&[\delta(\phi(H_3V_4 - NV_3) - d(H_3L - \alpha K_3 - dN)) - d(F_3(d - M) + G_3L + \phi V_3 \\
&+dM + \theta V_4) - (\alpha + d)(dG_3L + D_3V_4 + \theta LV_3 - dF_3M - \theta F_3V_4 - \phi MV_3 \\
&-d(\phi(F_3 + V_3) + M(d - F_3)) + G_3L - \theta V_4) + d\delta NF_3], \\
d_5 =&[d(\alpha + d)(F_3dM - F_3V_4\theta - G_3V_4\phi - V_3L\theta + V_3M\phi) + \delta\phi(dH_3V_3 - dH_3V_4 \\
&+D_3V_4 - \alpha V_3K_3) + d(\alpha D_3L\delta - \alpha F_3K_3\theta - L\delta H_3d - F_3NL\delta) + \delta(aD_3L - \alpha \\
&\times F_3K_3 - LH_3d - dF_3H)].
\end{aligned}
$$

Clearly, $d_i > 0 (i = 1, 2, 3, 4, 5)$.

When $E_4 < 0$ then $\beta_1 < \xi_1, M < 0, \beta_2 < \xi_2$. From Ruth-Hurwitz that the equation (4.5) has negative real parts iff $\beta_1 < \xi_1, \beta_2 < \xi_2, d_1d_2 - d_3 > 0, d_1^2d_2 - d_3 > 0$ and $d_1d_2d_3^2 - d_1d_2^2d_4 - d_1d_3d_4 + d_1d_2d_5 - d_3^3 + d_2d_3d_4 - d_1d_3d_5 > 0$. Under these conditions E_4 will be asymptotically stable, otherwise unstable.

5. Optimal Control Model

The HIV/AIDS-TB co-infection model (2.2) is modified to include the optimal control functions as follows:

$$
\begin{aligned}
\frac{dN}{dt} =& Q_0 - dN - \alpha A \\
\frac{dT}{dt} =& \left(\beta_1 - \beta_1' \frac{T}{m + T}\right) \frac{(N - T - H - V - A)T}{N} - \beta_3 TH - (\xi_1 + l_1)T + l_3V \\
\frac{dH}{dt} =& \left(\beta_2 - \beta_2' \frac{H}{n + H}\right) \frac{(N - T - H - V - A)H}{N} + \beta_3 TH - (\xi_2 + l_2)H \\
\frac{dV}{dt} =& \phi T + \theta H - (d + l_3)V \\
\frac{dA}{dt} =& (\delta + l_2)H - (\alpha + d)A - l_1T,
\end{aligned} \tag{5.1}
$$

where, l_1 represents to aware the infected people in TB class, l_2 represents to aware the infected people in HIV class and l_3 represents to vaccinated the co-infected people. Pontryagin's Maximum Principle is employed to analyze the optimal control model given by the non-autonomous model (5.1) with the aim of minimizing infectious hosts, while keeping the corresponding costs of control implementation as low as possible. Now, with the help of optimal control theory we will take the behaviour of our model (5.1). Thus, the objective functional for the optimal control model (5.1) is given

below:

(5.2)
$$K(l_i,\Omega)=\int_0^{T_f}\big(D_1N^2+D_2T^2+D_3H^2+D_4V^2+D_5A^2+w_1l_1^2+w_2l_2^2+w_3l_3^2\big)dt,$$

where T_f is the final time of control implementation, Ω is the set of all compartmental variables, D_1, D_2, D_3, D_4, D_5 are the positive weight constants for the variables N, T, H, V, A, respectively. The weight constants for the optimal controls are $l_i,\ i=1,2,3$. The terms $w_1l_1^2, w_2l_2^2$ and $w_3l_3^2$ are the cost functions associated with infected people in TB, HIV and V, respectively. In accordance with the literature on optimal control problems, a quadratic cost on controls has been used. The interest is to find a control $l^*=l_i^*,\ i=1,2,3$ satisfying from $T=0$ to T_f such that

(5.3)
$$H(l_i(t))=min\left\{\frac{K(l_i,\Omega)}{l_i}\in F\right\},\ i=1,2,3,$$

where F is the smooth function for the interval $[0,1]$. Following existence results due to Rischel and Fleming, the optimal control l^* exist.

5.1. **The Control Characterization.** The Pontryagin's maximum principle gives the necessary conditions for the existence of the optimal control triplet (l_1^*, l_2^*, l_3^*) of the optimal control model (5.1). This principle converts the state model (5.1) and equation (5.2) with equation (5.3) into a problem of minimizing pointwise a M, with respect to $l_1(t), l_2(t)$ and $l_3(t)$. Therefore, the Lagrangian function M related to objective function is given by

(5.4)
$$\begin{aligned}M(\Omega,D_i)&=D_1N^2+D_2T^2+D_3H^2+D_4V^2+D_5A^2+w_1l_1^2+w_2l_2^2+w_3l_3^2\\&+\lambda_1(Q_0-dN-\alpha A)\\&+\lambda_2\left(\left(\beta_1-\beta_1'\frac{T}{m+T}\right)\frac{(N-T-H-V-A)T}{N}-\beta_3TH-(\xi_1+l_1)T+l_3V\right)\\&+\lambda_3\left(\left(\beta_2-\beta_2'\frac{H}{n+H}\right)\frac{(N-T-H-V-A)H}{N}+\beta_3TH-(\xi_2+l_2)H\right)\\&+\lambda_4\big(\phi T+\theta H-(d+l_3)V\big)+\lambda_5\big((\delta+l_2)H-(\alpha+d)A-l_1T\big).\end{aligned}$$

The co-adjoint equation variables $\lambda_i=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)$ for the system is calculated by taking the partial derivative of M with respect to each state variable N, T, H, V, A.

The necessary conditions for the existence of these adjoint state variables are provided in the following theorem.

Theorem 5.1. Let the optimal control be (l_1^*, l_2^*, l_3^*) that minimizes objective functional (5.2) over F subject to the model (5.1). Then, there exist adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 satisfying the adjoint model

$$\begin{aligned}
\dot{\lambda_1} =&\frac{\partial M}{\partial N} = -2D_1N + \lambda_1 d - (\beta_1\lambda_2T^2 + \beta_2\lambda_3H^2)\frac{1}{N^2} - (\beta_1\lambda_2T + \beta_2\lambda_3H)\frac{1}{N^2} - (\beta_1\lambda_2 \\
&+\beta_2\lambda_3)\frac{TH}{N^2} + \frac{N-S}{N^2}\left(\frac{\lambda_2\beta_1'T^2}{m+T} + \frac{\lambda_2\beta_2'H^2}{n+H}\right), \\
\dot{\lambda_2} =&\frac{\partial M}{\partial T} = -2D_2T + (\lambda_2 - \lambda_3)\beta_3H - \lambda_2(\beta_1 - \xi_1) + (\lambda_2 - \lambda_5)l_1 - \lambda_3\beta_3H - \lambda_4\phi + \lambda_2 \\
&\times\frac{\beta_1}{N}[T + N - S] + \frac{\lambda_2\beta_1'T}{N(m+T)^2}[3mT + 2m(S+T) - 2T^2 + T(S+T)] + \frac{\lambda_3H}{N} \\
&\times\left[\beta_2 - \frac{\beta_2'}{N(m+H)}\right], \\
\dot{\lambda_3} =&\frac{\partial M}{\partial H} = -2D_3H(\lambda_2 - \lambda_3)\beta_3T - \lambda_3(\beta_2 - \xi_2) + (\lambda_3 - \lambda_5)l_2 - \lambda_4\theta - \lambda_5\delta + \frac{\lambda_3\beta_2}{N}[H + \\
&N - S] + \frac{\lambda_3\beta_2'H}{N(n+H)^2}[2n(N + T - VH - 2A) - 3nH + HN] + \frac{\lambda_2T}{N}\left[\beta_1 - \frac{\beta_1'T}{(n+T)}\right], \\
\dot{\lambda_4} =&\frac{\partial M}{\partial V} = -2D_4V(\lambda_2\beta_1T + \lambda_3\beta_2H)\frac{1}{N} - \left(\frac{\lambda_2\beta_1'T^2}{m+T} + \frac{\lambda_3\beta_2'H^2}{n+H}\right)\frac{1}{N} + (\lambda_4 - \lambda_2)l_3 + \lambda_4 d, \\
\dot{\lambda_5} =&\frac{\partial M}{\partial A} = -2D_5A + (\lambda_1 + \lambda_5)\alpha + (\lambda_2\beta_1T + \lambda_3\beta_2H)\frac{1}{N} - \left(\frac{\lambda_2\beta_1'T^2}{m+T} + \frac{\lambda_3\beta_2'H^2}{n+H}\right)\frac{1}{N} \\
&+\lambda_4(\alpha + d),
\end{aligned} \tag{5.5}$$

with final-time conditions

$$\lambda_i(T_f) = 0, \ for\ i = 1, 2, 3, 4, 5 \tag{5.6}$$

subject to

$$\begin{aligned}
l_1^* =& max\left(c_1, min\left(d_1, \frac{T(\lambda_2 - \lambda_5)l_1}{2w_1}\right)\right), \\
l_2^* =& max\left(c_2, min\left(d_2, \frac{H(\lambda_3 - \lambda_5)l_2}{2w_2}\right)\right), \\
l_3^* =& max\left(c_3, min\left(d_3, \frac{V(\lambda_4 - \lambda_2)l_3}{2w_3}\right)\right).
\end{aligned} \tag{5.7}$$

Proof. The adjoint model (5.5) is derived from the Hamiltonian M in (5.4) as follows:

$$\dot{\lambda_1} = \frac{\partial M}{\partial N}, \dot{\lambda_2} = \frac{\partial M}{\partial T}, \dot{\lambda_3} = \frac{\partial M}{\partial H}, \dot{\lambda_4} = \frac{\partial M}{\partial V}, \dot{\lambda_5} = \frac{\partial M}{\partial A},$$

with the final-time conditions

$$\lambda_1(T_f) = \lambda_2(T_f) = \lambda_3(T_f) = \lambda_4(T_f) = \lambda_5(T_f) = 0. \tag{5.8}$$

Further, in order to find the optimal control solution to the objective function (5.7), we set

$$\frac{\partial M_i}{\partial l_i} = 0, where, i = 1, 2, 3.$$

Then, optimal solution set is

$$l_i^* = \begin{cases} 0 \ if \ \widetilde{l_1} \leq 0 \\ l_i^* \ if \ 0 \leq \widetilde{l_i} \leq 1 \\ 1 \ if \ \widetilde{l_1} \geq 1 \end{cases}$$

for $i = 1, 2, 3$ and where

$$\begin{aligned} \widetilde{l_1} &= max\left(c_1, min\left(d_1, \frac{T(\lambda_2 - \lambda_5)l_1}{2w_1}\right)\right), \\ \widetilde{l_2} &= max\left(c_2, min\left(d_2, \frac{H(\lambda_3 - \lambda_5)l_2}{2w_2}\right)\right), \\ \widetilde{l_3} &= max\left(c_3, min\left(d_3, \frac{V(\lambda_4 - \lambda_2)l_3}{2w_3}\right)\right). \end{aligned}$$

Hence proved. □

6. Numerical Illustrations

In this section, the fourth order Runga-Kutta method is used to solve the system of ODEs. We provide a numerical simulation for the model. In MATLAB software the computer program is encoded. We used different default parameters for the computing purposes. To select the values of the parameters β_1' and β_2', we realize that it should be less than β_1 and β_2. The default parameter values pick for numerical simulation purpose are shown in Table 2.

Figure 2 shows the variation of TB versus time for $\lambda = 0.1, 0.5$. It has been observed that as we increase the recovery rate of TB population λ, the TB infected population goes on decreasing. Figure 3 shows the variation of TB versus time for $\phi = 0.0009, 0.1$.

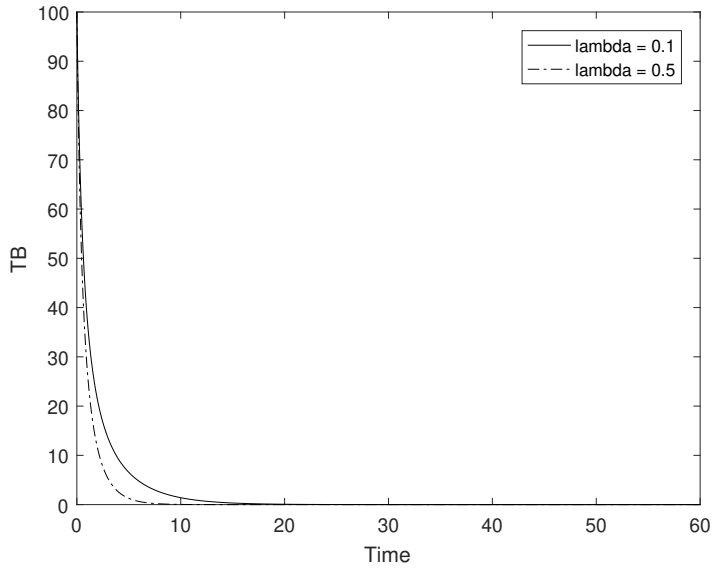

FIGURE 2. Variation of TB population with time for different values of $\lambda = 0.1, 0.5$.

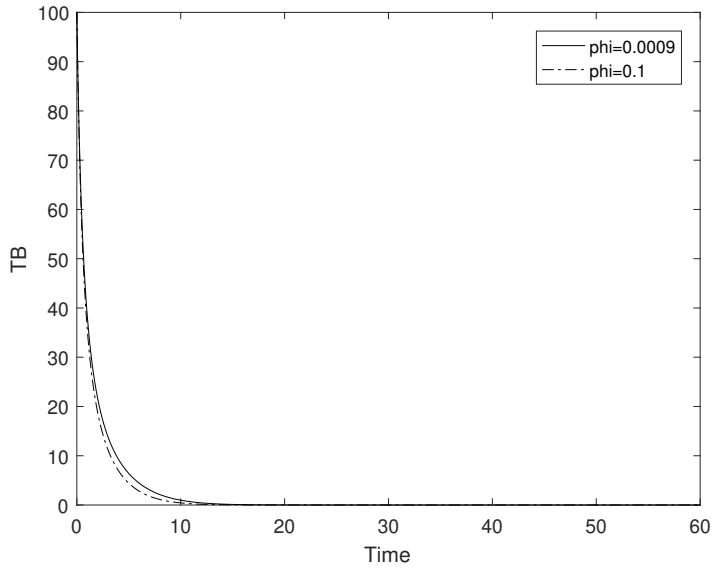

FIGURE 3. Variation of TB population with time for different values of $\phi = 0.0009, 0.1$.

It has been observed that as we increase the vaccination rate for host ϕ, the TB infected population goes on decreasing. Figure 4 shows the variation of HIV versus time for $n = 100, 150$ with time and then reaches its equilibrium position. It has been observed that as we increase the media rate for HIV infected population n,

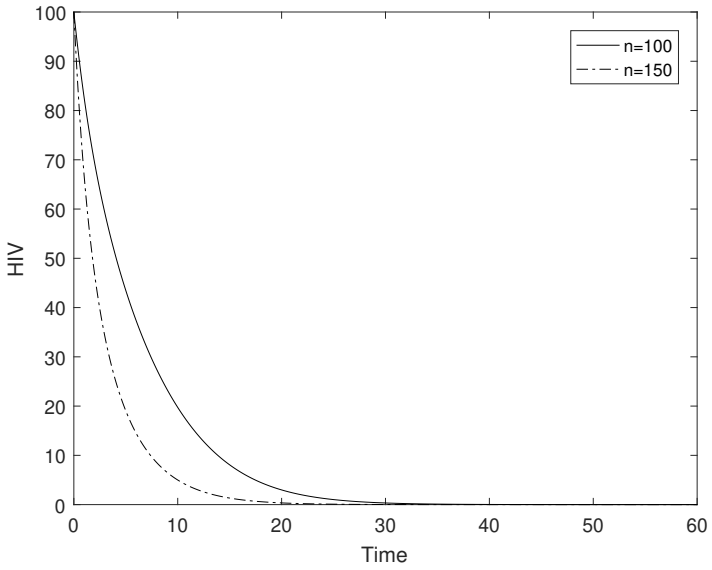

FIGURE 4. Variation of HIV population with time for different values of $n = 100, 150$.

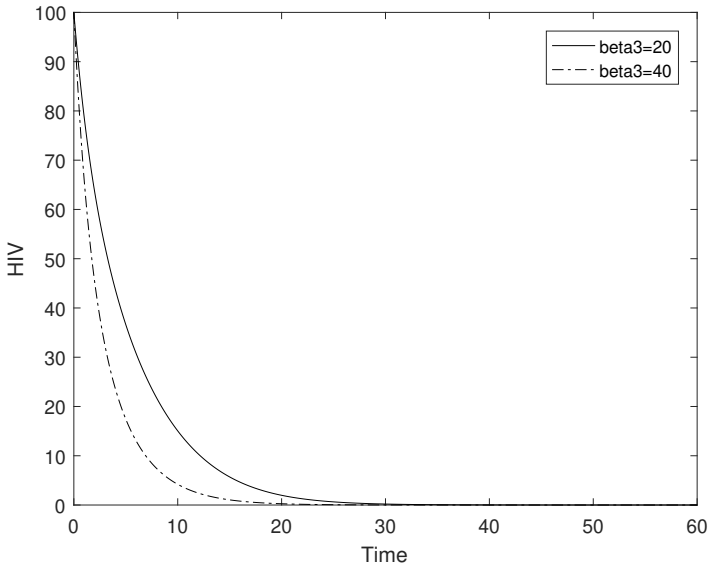

FIGURE 5. Variation of HIV population with time for different values of $\beta_3 = 20, 40$.

the HIV infected population goes on decreasing. Figure 5 shows the variation of HIV versus time for $\beta_3 = 20, 40$. It has been observed that as we increase the HIV infected rate β_3, the HIV infected population goes on decreasing. Figure 6 shows the variation of AIDS versus time for $\alpha = 0.005, 0.009$. It has been observed that as we

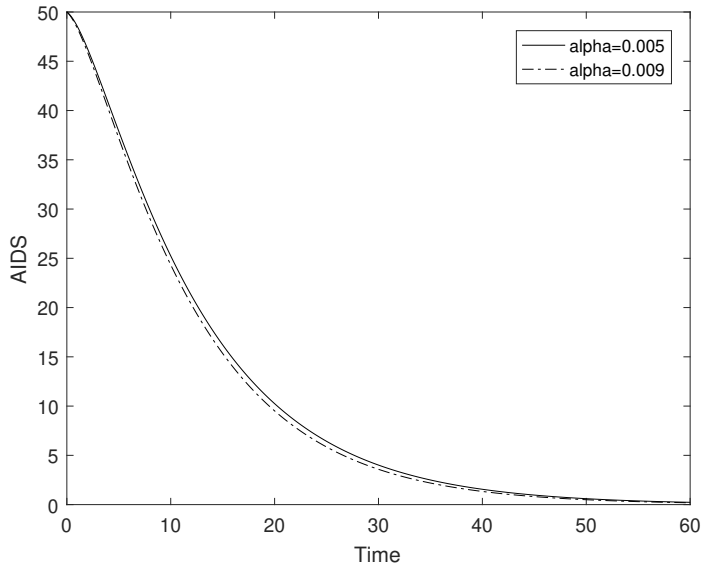

FIGURE 6. Variation of AIDS population with time for different values of $\alpha = 0.005, 0.009$.

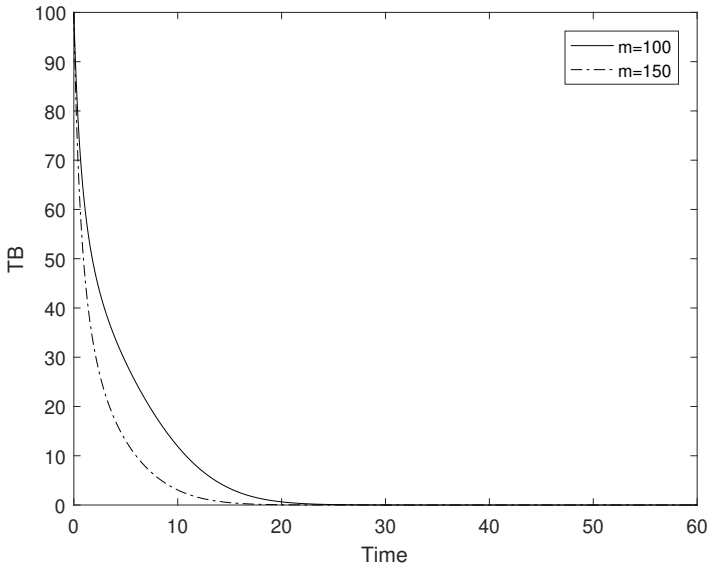

FIGURE 7. Variation of TB population with time for different values of $m = 100, 150$.

increase the induced death rate for AIDS infected population α, the AIDS infected population goes on decreasing. Figure 7 shows the variation of TB versus time for $m = 100, 150$. It has been observed that as we increase the media rate for TB infected population m, the TB infected population goes on decreasing. Figure 8 shows the

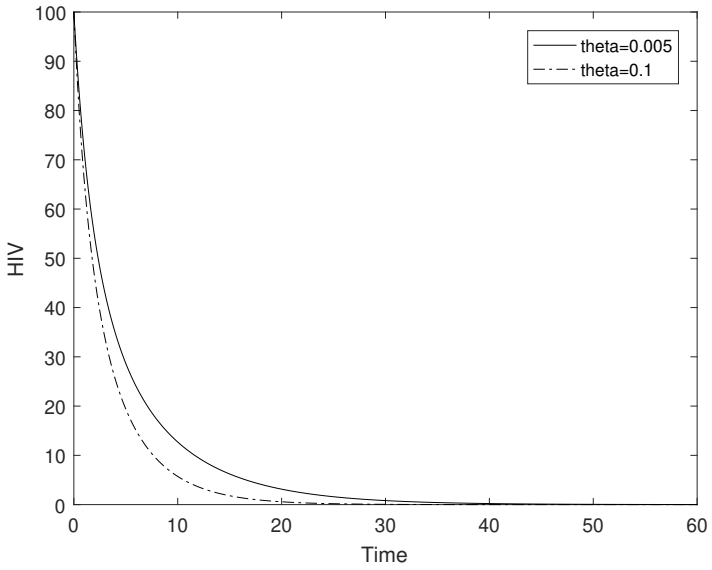

FIGURE 8. Variation of HIV population with time for different values of $\theta = 0.005, 0.1$.

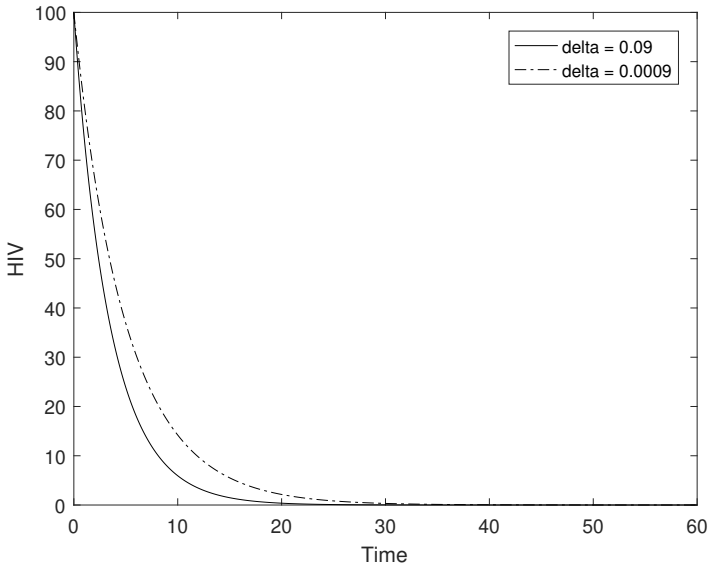

FIGURE 9. Variation of HIV population with time for different values of $\delta = 0.09, 0.0009$.

variation of HIV versus time for $\theta = 0.005, 0.1$. It has been observed that as we increase the relapse rate θ for hosts population, the HIV infected population goes on decreasing. It is clear that the HIV infected population decreases meaning thereby that it increases the full-blown AIDS population. Figure 9 shows the variation of

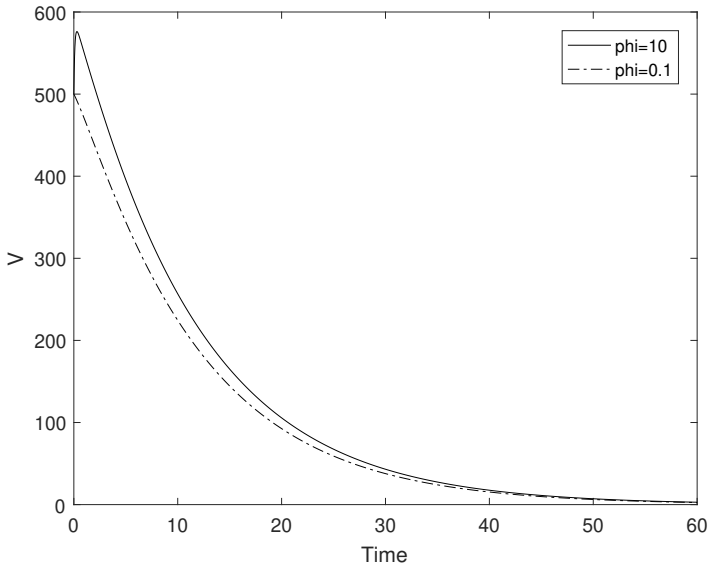

FIGURE 10. Variation of Vaccinated population with time for different values of $\phi = 10, 0.1$.

HIV versus time for different values of $\delta = 0.09, 0.0009$. It has been observed that as we increase δ (transmission rate from HIV population to full blown AIDS), the curve of HIV infected population decreases exponentially with time. Finally, Figure 10 demonstrates the variation of vaccinated population (V) versus time for different values of $\phi = 10, 0.1$. It has been observed that as we increase the vaccination rate ϕ for hosts population, the vaccinated population decreases. So, it is clear that as we administrate the vaccination to the infected people, the positive results are seen. Overall, it is concluded that if TB infection is treated properly then HIV infection can be kept under control and the effective media awareness can lower the transmission dynamics of epidemics, not completely but to a greater extent.

CONCLUSION

To assess the role of knowledge in the prevention and control of co-infection with TB & HIV/AIDS, a non-linear compartment model was developed. The stability of E_0, E_1, E_2, E_3 and E_4 is presented. The DFE E_0 is stable if $\beta_1 < \xi_1$ and $\beta_2 < \xi_2$. The point E_1 is local asymptotically stable if it satisfies the condition given in stability criteria of E_1. The point E_2 shows the Tuberculosis free population and is unstable due to the presence of positive eigen values. The vaccination free point E_4 is locally

asymptotically stable if it satisfies the condition given in stability criteria of E_4. The endemic point E_5 is locally asymptotically stable if it satisfies the condition given in stability criteria of E_5. Similarly, the same case will occur in the case of HIV epidemic. It is observed that if β_1, and β_2, are equal to zero, there is no change in the basic reproduction numbers as they are independent of media effects. But it has been noticed that values of the parameters m, n, β_1, and β_2, change the equilibrium points, local stability of HIV/AIDS free point and TB free point holds good when $\mathcal{R}_H, \mathcal{R}_T$ are greater than unity, otherwise these points become diseases free points. The two cases for media alert for TB and HIV epidemics are $\beta(T) = \left(\beta_1 - \beta_1' \frac{T}{m+T}\right)$ and $\beta(H) = \left(\beta_2 - \beta_2' \frac{H}{n+H}\right)$. This implies that when Tuberculosis begins to spread, the media works for the awareness immediately and advises the masses to take protective measures to fight against the disease.

Again when $\frac{\partial \beta(H)}{\partial \beta_2'} = \frac{H}{n+H} < 0$, it is clear that for the bigger values of the rate β_1', the transmission rate is smaller. The investigation done may be helpful to health organizations, social workers and practitioners in hospital in understanding the complex nature of dynamics of the fatal diseases namely; HIV/AIDS and TB; the final outcome may be further used for the prevention and control of these diseases. The impact of covid-19 on this model which will be studied as the future reference of this course.

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