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# SOME PROPERTIES OF BALANCING NUMBERS

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Abstract. In this paper we discuss some aspects and properties of Balancing Numbers and some other related numbers. We prove, among other things, that a balancing number cannot be a power of a prime integer. We give some identities concerning these numbers and its related numbers. We use linear algebra techniques to write a balancing number and its related numbers in the Binet form.

## 1. Introduction

In ([1]), Behera and Panda gave the definition of *a balancing number* as follows:

**Definition 1.1.** A natural number $n$ is a balancing number if there is a natural number $r$ such that the ordered pair $(n; r)$ is a solution for the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1.1}$$

The natural number $r$ is called the balancer for the balancing number $n$. Let

$$T_n = 1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2} \tag{1.2}$$

be the $n$-th triangular number and let $S_n = n^2$ be the $n$-th square number. Notice that

$$T_{n-1} + T_n = n^2 = S_n$$

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By adding $1+2+\cdots+(n-1)+n$ to both sides of (1.1) we have

$$S_n = n^2 = T_n + T_{n-1} = T_{n+r} \tag{1.3}$$

So (1.1) can be rephrased as to find two natural numbers $n$ and $r$ such that the sum of two consecutive triangular numbers is a triangular number which is at the same time a square number. So we can say that the positive integer $n$ is a balancing number with balancer $r$ if and only if the $(n+r)$-th triangular number is the $n$-th square number. For example $T_6+T_5 = T_8$ (here $n = 6, r = 2$) and $T_{35}+T_{34} = T_{49}$ (here $n = 35, r = 14$).

Now equation (1.1) is indeed of the form $T_{n-1} = nr + T_r$. So it can be written as a quadratic equation (in $r$):

$$r^2 + (2n+1)r + (n-n^2) = 0 \tag{1.4}$$

Solving this equation for $r$ we have

$$r = \frac{(-2n-1) + \sqrt{8n^2+1}}{2} \tag{1.5}$$

Clearly the numerator is an even integer, and hence in order that $n$ becomes a balancing number and $r$ a balancer for $n$, $r$ must be a root of equation (1.4) and $8n^2+1$ is an odd perfect square.

On the other hand equation (1.1) can also be regarded as a quadratic equation (in $n$)

$$n^2 - (2r+1)n - (r^2+r) = 0 \tag{1.6}$$

Hence

$$n = \frac{(2r+1) + \sqrt{8r^2+8r+1}}{2} \tag{1.7}$$

In this case we can say that in order $n$ to be a balancing number with balancer $r$, $n$ must be a root of equation (1.6) and $8r^2+8r+1$ must be an odd perfect square.

Let us denote the $n$-th balancing number by $B_n$ and its corresponding $n$-th balancer by $r_n$. Therefore Equations ((1.5) and (1.7)) above become

$$r_n = \frac{(-2B_n-1) + \sqrt{8B_n^2+1}}{2} \tag{1.8}$$

and

$$B_n = \frac{(2r_n+1)+\sqrt{8r_n^2+8r_n+1}}{2} \tag{1.9}$$

respectively.

## 2. Some Further Results on Balancing Numbers

There is a clear relation between the $n$-th balancing number $B_n$ and its $n$-th balancer $r_n$ as we can see in the following theorem.

**Theorem 2.1.** $\lim_{n\to\infty} \frac{r_n}{B_n} = \sqrt{2}-1$

*Proof.* Using formulae ((1.8) and (1.9)) we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{r_n}{B_n} &= \lim_{n\to\infty} \frac{\frac{(-2B_n-1)+\sqrt{8B_n^2+1}}{2}}{B_n} \\
&= \lim_{n\to\infty} -1 - \frac{1}{2B_n} + \frac{\sqrt{8+\frac{1}{B_n^2}}}{2} \\
&= -1 + \frac{\sqrt{8}}{2} \text{ (since } \lim_{n\to\infty} \frac{1}{B_n} = 0) \\
&= \sqrt{2}-1 \approx 0.4142.
\end{aligned}$$

Clearly $\lim_{n\to\infty} \frac{B_n}{r_n} = \sqrt{2}+1$.

Table 1 below gives the first eight balancing numbers $B_n$'s with their corresponding balancers $r_n$'s and the ratios (upto the first four decimals) $\frac{r_n}{B_n}$. We see from Table 1 that we achieved the approximate ratio of the $n$-th balancer by the $n$-th balancing number $(\sqrt{2}-1)$ immediately at the seventh place.

Behera and Panda in ([1]) showed that the balancing numbers satisfy the second order linear recurrence relation by the identity

$$B_{n+1} = 6B_n - B_{n-1} \quad \text{for} \quad n \in \mathbb{N} \tag{2.1}$$

Identity (2.1) shows that $B_n$ and $B_{n+1}$ have the same parity i.e., both are even or both are odd.

TABLE 1.

| $n$th | $B_n$ | $r_n$ | $\frac{r_n}{B_n}$ |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 6 | 2 | 0.3333 |
| 3 | 35 | 14 | 0.4000 |
| 4 | 204 | 84 | 0.4117 |
| 5 | 1189 | 492 | 0.4138 |
| 6 | 6930 | 2870 | 0.4141 |
| 7 | 40391 | 16730 | 0.4142 |
| 8 | 235416 | 97512 | 0.4142 |

Now if $B_n$ is the $n$-th balancing number, then, as we mentioned above, $8B_n^2+1$ is an odd perfect square, say $m^2$, for some odd integer $m$. So we have

$$8B_n^2 + 1 = m^2. \tag{2.2}$$

Let us change notation for a while and write this equation as $8x^2 + 1 = y^2$, where $x, y$ are integers. This is in fact a Pell's equation (See [12] Page 553) of the form

$$y^2 - 8x^2 = 1. \tag{2.3}$$

Clearly the ordered pair $(1,3) = (x_1, y_1)$ is a fundamental solution of (2.3). The $n$-th solution $(x_n, y_n)$ of (2.3) can be found by the following equations (See [12] Theorem 13.12):

$$y_n + x_n\sqrt{8} = (3+\sqrt{8})^n \qquad \text{and } y_n - x_n\sqrt{8} = (3-\sqrt{8})^n \tag{2.4}$$

In fact $x_n$ is the $n$-th balancing number $B_n$. Let us denote $3+\sqrt{8}$ by $\gamma$ and $3-\sqrt{8}$ by $\delta$. Then $\gamma+\delta = 6, \gamma\delta = 1$. Let us denote $y_n$ by $C_n$. Clearly from the two equations in (2.4), we have

$$B_n = \frac{\gamma^n - \delta^n}{2\sqrt{8}} \tag{2.5}$$

and

$$C_n = \frac{\gamma^n + \delta^n}{2} \tag{2.6}$$

$C_n$ is called the $n-$th Lucas-balancing number (See [7]). Formulae ((2.5) and (2.6)) are the Binet form of $B_n$ and $C_n$ respectively (See [2]).

Now

$$(2.7) \qquad \sqrt{8B_n^2+1} \;=\; \sqrt{8(\frac{\gamma^n-\delta^n}{4\sqrt{2}})^2+1} \;=\; \sqrt{(\frac{\gamma^n+\delta^n}{2})^2} \;=\; \frac{\gamma^n+\delta^n}{2}.$$

This shows that $C_n = \sqrt{8B_n^2+1}$.

**Theorem 2.2.** *For each integer $n \geq 1$, we have*

*(1) $C_{n+1} = 6C_n - C_{n-1}$. (See [13] proved for $n \geq 2$)*

*(2) $C_{n+1} = 3C_n + 8B_n$.*

*(3) $2C_n^2 - C_{2n} = 1$.*

*(4) $\frac{1}{8}(C_{n-1}C_{n+1} - C_n^2) = 1$.*

*Proof.* (1) Since $B_n$ is a balancing number, clearly $C_n = \sqrt{8B_n^2+1}$ is an integer.

$$\begin{aligned} 6C_n - C_{n-1} &= 6(\frac{\gamma^n+\delta^n}{2}) - (\frac{\gamma^{n-1}+\delta^{n-1}}{2}) \\ &= \frac{6\gamma^n + 6\delta^n - \gamma^{n-1} - \delta^{n-1}}{2} \\ &= \frac{\gamma^{n-1}(6\gamma-1) + \delta^{n-1}(6\delta-1)}{2} \\ &= \frac{\gamma^{n-1}(17+6\sqrt{8}) + \delta^{n-1}(17-6\sqrt{8})}{2} \\ &= \frac{\gamma^{n-1}(\gamma^2) + \delta^{n-1}(\delta^2)}{2} \\ &= \frac{\gamma^{n+1}+\delta^{n+1}}{2} \;=\; C_{n+1}. \end{aligned}$$

For (2), consider $C_{n+1}$.

$$\begin{aligned} C_{n+1} &= \frac{\gamma^{n+1}+\delta^{n+1}}{2} \;=\; \frac{\gamma\gamma^n+\delta\delta^n}{2} \\ &= \frac{(3+\sqrt{8})\gamma^n + (3-\sqrt{8})\delta^n}{2} \\ &= \frac{3(\gamma^n+\delta^n)}{2} + \frac{\sqrt{8}(\gamma^n-\delta^n)}{2} \\ &= \frac{3(\gamma^n+\delta^n)}{2} + \frac{8(\gamma^n-\delta^n)}{4\sqrt{2}} \\ &= 3C_n + 8B_n. \end{aligned}$$

For (3), we have

$$\begin{aligned} 2C_n^2 - C_{2n} &= 2(\frac{\gamma^n + \delta^n}{2})^2 - \frac{\gamma^{2n} + \delta^{2n}}{2} \\ &= \frac{\gamma^{2n} + \delta^{2n} + 2\gamma^n\delta^n - \gamma^{2n} - \delta^{2n}}{2} \\ &= \gamma^n\delta^n = 1 \text{ (since } \gamma\delta = 1). \end{aligned}$$

For (4), we have

$$\begin{aligned} C_{n-1}C_{n+1} - C_n^2 &= \frac{(\gamma^{n-1} + \delta^{n-1})(\gamma^{n+1} + \delta^{n+1})}{4} - \frac{(\gamma^n + \delta^n)^2}{4} \\ &= \frac{\gamma^{2n} + \gamma^{n-1}\delta^{n-1}\delta^2 + \delta^{n-1}\gamma^{n-1}\gamma^2 + \delta^{2n}}{4} \\ &= \frac{\delta^2 + \gamma^2 - 2}{4} = 8. \end{aligned}$$

**Theorem 2.3.** *For each integer $n \geq 1$, we have*
*(1)* $\gamma^n = C_n + \sqrt{8}B_n$ *and*
*(2)* $\delta^n = C_n - \sqrt{8}B_n$.

*Proof.* Straightforward from the definition of $B_n$ and $C_n$.

Let $\Gamma_n = \gamma^n$ and $\Delta_n = \delta^n$. Then $\Gamma_n$ and $\Delta_n$ can be regarded as elements in the quadratic number ring $\mathbb{Z}[\sqrt{2}]$. Although these two elements are not integers, they do satisfy Identity (2.1) as the following theorem shows.

**Theorem 2.4.** *For each integer $n \geq 0$, we have*
*(1)* $\Gamma_{n+2} = 6\Gamma_{n+1} - \Gamma_n$.
*(2)* $\Delta_{n+2} = 6\Delta_{n+1} - \Delta_n$.

*Proof.* Direct calculations by mathematical induction on $n$.

**Theorem 2.5.** *For each integer $n \geq 1$, we have*
*(1)* $\Gamma_n = \gamma B_n - B_{n-1}$.
*(2)* $\Delta_n = \delta B_n - B_{n-1}$.

*Proof.* By mathematical induction on $n$.
For $n = 1$, it is clear that $\gamma = 3 + \sqrt{8} = 1.(3 + \sqrt{8}) - 0 = \gamma B_1 - B_0$, since $B_1 = 1$ and

$B_0 = 0$. Similar calculations show that for $n = 2$, we have $\gamma^2 = \gamma B_2 - B_1 = 6\gamma - 1$. Now suppose that the statement is true for $n = k$. Then $\Gamma_k = \gamma B_k - B_{k-1}$. For $n = k+1$ we have

$$\begin{aligned}\Gamma_{k+1} &= \gamma\Gamma_k = \gamma(\gamma B_k - B_{k-1}) = \gamma^2 B_k - \gamma B_{k-1} \\ &= (6\gamma - 1)B_k - \gamma B_{k-1} \text{ (since } \gamma^2 = 6\gamma - 1) \\ &= 6\gamma B_k - B_k - \gamma B_{k-1} = \gamma(6B_k - B_{k-1}) - B_k \\ &= \gamma B_{k+1} - B_k \text{ by Identity (2.1).}\end{aligned}$$

Hence we have (1). Similarly we can prove (2).

**Theorem 2.6.** *For each integer $n \geq 1$, we have*
*(1)* $2B_nC_n = B_{2n}$
*(2)* $C_{2n} = 1 + 16B_n^2$

*Proof.* Using the Binet (Formulae (2.5) and (2.6)) forms of $B_n$ and $C_n$ it is easy to establish (1) and (2).

Now using the Binomial Theorem we can prove the following theorem

**Theorem 2.7.** *For each integer $n \geq 1$, we have*
*(1)* $B_n = \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} 3^{n-2k-1} 2^{3k}$, *and*
*(2)* $C_n = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} 3^{n-2k} 2^{3k}$

where $[x]$ denotes the greatest integer function.
*Proof.* (1) By the Binet (Formula (2.5)) form we have $B_n = \frac{\gamma^n - \delta^n}{2\sqrt{8}} = \frac{(3+\sqrt{8})^n - (3-\sqrt{8})^n}{2\sqrt{8}}$. Now by the Binomial Theorem we have

$$(3 + \sqrt{8})^n = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} 2^{3k/2}$$

and

$$(3 - \sqrt{8})^n = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (-1)^k 2^{3k/2}$$

Now if $k = 2l$ an even integer then the $k-$th coefficient of $(3+\sqrt{8})^n - (3-\sqrt{8})^n$ equals zero. But if $k = 2l+1$ an odd integer, then the $k-$th coefficient of $(3+\sqrt{8})^n - (3-\sqrt{8})^n$

equals $2\sqrt{8}\binom{n}{2l+1}3^{n-2l-1}2^{3k}$. This implies that $B_n = \sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}3^{n-2k-1}2^{3k}$.
(2) Similar to (1).

**Theorem 2.8.** *A balancing number cannot be a power of a prime integer.*

*Proof.* Suppose that $p$ is a prime integer and $p^n$ is a balancing number for some positive integer $n$. Then, by equation (1.5), $8(p^n)^2+1$ must be an odd perfect square, say $m^2$ with $m = 2k+1$ is an odd integer for some positive integer $k$. So we have $8p^{2n}+1 = m^2$, and hence $8p^{2n} = m^2-1 = (2k+1)^2-1 = 4k^2+4k$. Canceling 4 from both sides we get

$$2p^{2n} = k(k+1) \tag{2.8}$$

Clearly $k, k+1$ are relatively prime. We consider two cases:
Case (1) The prime integer $p = 2$. In this case (2.8) becomes $2^{2n+1} = k(k+1)$. This implies, being $k, k+1$ are relatively prime, that either $k = 2p^{2n+1}$ or $k+1 = 2p^{2n+1}$. If $k = 2p^{2n+1}$, then $k+1 = 1$, which is a contradiction since $k$ is a positive integer. If $k+1 = 2p^{2n+1}$, then $k = 1$ and hence we have $2^{2n+1} = 2$ which means $n = 0$, another contradiction since $n$ is assumed to be a positive integer.
Case (2) The prime integer $p$ is odd. Equation (2.8) implies that $2|k(k+1)$. Since $k, k+1$ are relatively prime integers, we have $2|k$ or $2|k+1$. If $2|k$, then $k = 2^a b$ for some positive integers $a$ and $b$. But this implies $2p^{2n} = 2^a b(2^a b+1)$. Cancelling 2 from both sides, we get $p^{2n} = 2^{a-1}b(2^a b+1)$. Since the left hand side is not divisible by 2, $a$ must equal to 1, and the equation becomes $p^{2n} = b(2b+1)$. Again, since $b, b+1$ are relatively prime, we have either $b = p^{2n}$ or $2b+1 = p^{2n}$. If $b = p^{2n}$, then $2b+1 = 1$ and hence $b = 0$, a contradiction. If $2b+1 = p^{2n}$, then $b = 1$ and $p^{2n} = 3$. But this implies $p = 3$ and $2n = 1$, another contradiction since $n$ is an integer.
So the only balancing number that is a power of a prime integer is $B_1 = 1 = p^0$.

**Lemma 2.1.** $\lim_{n\to\infty}\frac{B_{n+1}}{B_n} = \gamma$.

*Proof.* Let $\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = \mu$. Then

$$\begin{aligned} \mu = \lim_{n\to\infty} \frac{B_{n+1}}{B_n} &= \lim_{n\to\infty} \frac{6B_n - B_{n-1}}{B_n}, \text{ (by (2.1))} \\ &= 6 - \lim_{n\to\infty} \frac{B_{n-1}}{B_n} \\ &= 6 - \frac{1}{\mu}. \end{aligned}$$

This implies $\mu^2 - 6\mu + 1 = 0$. Hence $\mu = \gamma$.

**Lemma 2.2.** $\sum_{n=1}^{\infty} \frac{1}{B_n}$ *is a convergent series.*

*Proof.* Since the series $\sum_{n=1}^{\infty} \frac{1}{B_n}$ is a series of positive terms of real numbers, we apply the ratio test for convergence. Consider

$$\lim_{n\to\infty} \frac{1/B_{n+1}}{1/B_n} = \lim_{n\to\infty} \frac{B_n}{B_{n+1}} = \frac{1}{\gamma} \text{ (by (2.1))}.$$

But $\frac{1}{\gamma} = 3 - \sqrt{8} < 1$, hence the series converges.

**Theorem 2.9.** $\sum_{n=1}^{\infty} \frac{B_n}{6^n} = 6$.

*Proof.* Let $S = \sum_{n=1}^{\infty} \frac{B_n}{6^n}$. Then

$$\begin{aligned} S = \sum_{n=1}^{\infty} \frac{B_n}{6^n} &= \frac{1}{6} + \sum_{n=2}^{\infty} \frac{B_n}{6^n} \\ &= \frac{1}{6} + \sum_{n=1}^{\infty} \frac{B_{n+1}}{6^{n+1}} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{6} + \sum_{n=1}^{\infty} \frac{6B_n - B_{n-1}}{6^{n+1}} \text{ (by (2.1))} \\ &= \frac{1}{6} + \sum_{n=1}^{\infty} \frac{B_n}{6^n} - \sum_{n=1}^{\infty} \frac{B_{n-1}}{6^{n+1}} \\ &= \frac{1}{6} + S - \frac{1}{36} \sum_{n=1}^{\infty} \frac{B_{n-1}}{6^{n-1}} \\ &= \frac{1}{6} + S - \frac{1}{36} \sum_{n=0}^{\infty} \frac{B_n}{6^n} \\ &= \frac{1}{6} + S - \frac{1}{36}(0 + \sum_{n=1}^{\infty} \frac{B_n}{6^n}) \\ &= \frac{1}{6} + S - \frac{1}{36}(0 + S). \end{aligned}$$

Hence $\frac{S}{36} = \frac{1}{6}$. Therefore $S = 6$.

**Lemma 2.3.** *Let $n$ be positive integer. If $8n + 1$ is a perfect square, then $n$ is a triangular number.*

*Proof.* Suppose $8n + 1 = m^2$, a perfect square integer. Then clearly $m$ is an odd integer. Now since $m$ is odd, we have

$$n = \frac{m^2 - 1}{8} = \frac{(m-1)(m+1)}{8} = \frac{(\frac{m-1}{2})(\frac{m+1}{2})}{2} = \frac{(\frac{m-1}{2})(\frac{m-1}{2} + 1)}{2} = T_{\frac{m-1}{2}}.$$

Hence $n$ is a triangular number.

Remark 1. (1) As a quick application of this lemma and since $C_n = \sqrt{8B_n^2 + 1}$ and since $8B_n^2 + 1$ is a perfect square, we have $B_n^2 = T_{\frac{C_n - 1}{2}}$ a triangle integer.
(2) In ([6]), Luo proved that the only triangular numbers whose squares are also triangular numbers are 1 and 6. Hence by the above lemma the only balancing numbers which are also triangular numbers are 1 and 6.

**Theorem 2.10.** *For each integer $n \geq 1$, we have*
*(1) $(B_{n+1} - 2B_n)^2 - 1$ is a triangular number.*
*(2) $(B_{n+1} - 4B_n)^2 - 1$ is a triangular number.*

*Proof.* (1) Let $A = (B_{n+1} - 2B_n)^2 - 1$. Then

$$\begin{aligned} A &= (3B_n + \sqrt{8B_n^2 + 1} - 2B_n)^2 - 1 \text{ (by ([7])} \\ &= (B_n + \sqrt{8B_n^2 + 1})^2 - 1 \\ &= 9B_n^2 + 2B_n\sqrt{8B_n^2 + 1}. \end{aligned}$$

Now

$$\begin{aligned} 8A + 1 &= 72B_n^2 + 16B_n\sqrt{8B_n^2 + 1} + 1 \\ &= 64B_n^2 + 16B_n\sqrt{8B_n^2 + 1} + 8B_n^2 + 1 \\ &= (8B_n + \sqrt{8B_n^2 + 1})^2 \text{ is a perfect square.} \end{aligned}$$

Therefore, by Lemma (2.3), $A = (B_{n+1} - 2B_n)^2 - 1$ is a triangular number. By similar arguments we can prove (2).

## 3. Binet Form of the Balancing Numbers by Linear Algebra

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 6 \end{bmatrix}.$$

Then clearly $det(A) = 1$, and its inverse is

$$A^{-1} = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}$$

As P. K. Ray ([10] and [11]) observed, Formula (2.1) can be written in matrix form as

$$\begin{bmatrix} B_n \\ B_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} B_{n-1} \\ B_n \end{bmatrix}$$

and for each positive integer $n$, the matrix $A^n$, (See [10]), equals

$$A^n = \begin{bmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{bmatrix}$$

which can be proved by induction on the natural number $n$ and using Identity (2.1). For instance since $det(A^n) = (det(A)^n = 1$, we have the Cassini formula (See [13]) $B_n^2 - B_{n-1}B_{n+1} = 1$ for each positive integer $n$.

Now let us consider the eigenvalues and eigenvectors of the matrix $A$. The characteristic equation of $A$ is $det(\lambda I - A) = \lambda^2 - 6\lambda + 1 = 0$. This equation has two real roots, $\lambda = 3 + \sqrt{8}$ and $\lambda = 3 - \sqrt{8}$. Let us write the two roots as $\gamma = 3 + \sqrt{8}$ and $\delta = 3 - \sqrt{8}$ and observe that $\gamma\delta = 1$. The eigenvectors corresponding to $\gamma$ can be found by solving the matrix equation

$$\begin{bmatrix} \gamma & -1 \\ 1 & -\delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which implies $\gamma x - y = 0$ and hence $y = \gamma x$. Now a basis for the eigenspace corresponding to the eigenvalue $\gamma$ is

$$\left\{ \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \right\}$$

Similarly a basis for the eigenspace corresponding the eigenvalue $\delta$ is

$$\left\{ \begin{bmatrix} 1 \\ \delta \end{bmatrix} \right\}$$

Now since the matrix $A$ has two distinct eigenvalues, it is diagonalizable, and $A$ is similar to the diagonal matrix

$$D = \begin{bmatrix} \gamma & 0 \\ 0 & \delta \end{bmatrix}$$

The matrix $P$ that diagonalizes the matrix $A$ is

$$P = \begin{bmatrix} 1 & 1 \\ \gamma & \delta \end{bmatrix}$$

Clearly $P^{-1}AP = D$ and of course $A = PDP^{-1}$ and clearly for each positive integer $n$ we have $A^n = PD^nP^{-1}$. This last equation gives us another way to find a closed form for the value of the balancing number $B_n$ which is called the Binet formula for $B_n$ (See for example [9]).

**Theorem 3.1.** $B_n = \frac{\gamma^n - \delta^n}{4\sqrt{2}}$

*Proof.* We have $A^n = PD^nP^{-1}$. Hence

$$A^n = \begin{bmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \gamma^n & 0 \\ 0 & \delta^n \end{bmatrix} \begin{bmatrix} \frac{-\delta}{4\sqrt{2}} & \frac{1}{4\sqrt{2}} \\ \frac{\gamma}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} \end{bmatrix}$$

Therefore

$$A^n = \begin{bmatrix} -B_{n-1} & B_n \\ -B_n & B_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{\delta^{n-1} - \gamma^{n-1}}{4\sqrt{2}} & \frac{\gamma^n - \delta^n}{4\sqrt{2}} \\ \frac{\delta^n - \gamma^n}{4\sqrt{2}} & \frac{\delta^{n+1} - \gamma^{n+1}}{4\sqrt{2}} \end{bmatrix}$$

This implies that $B_n = \frac{\gamma^n - \delta^n}{4\sqrt{2}}$.

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