Jordan Journal of Mathematics and Statistics (JJMS), 15(2), 2022, pp 157 - 175
DOI: https://doi.org/10.47013/15.2.1

NUMERICAL SOLUTION OF FRACTIONAL-ORDER POPULATION GROWTH MODEL USING FRACTIONAL-ORDER MUNTZ–LEGENDRE COLLOCATION METHOD AND PADE–APPROXIMANTS

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Abstract. This paper presents a numerical solution for a nonlinear fractional Volterra integro-differential equation to study the behavior solution of the population growth model. The technique applied based on the fractional-order Muntz–Legendre polynomials and the Pade approximants. Finally, some numerical examples are presented to show the efficiency and validity of the proposed method.

1. Introduction

It is well known that fractional calculus is one of the important parts of the numerical analysis, statistical mechanics, and physics in recent years. For example Bagley [1], Mainardi [2] and Rossikhin et al. [3] presented a survey of the application of fractional derivatives in mechanics and continuum. In addition, Ichise[4] demonstrated the applications of fractional derivatives and fractional integrals in the areas of electrochemical processes.

One of different types of the fractional order differential equations is population growth model [5, 6, 7]. This equation is nonlinear fractional Volterra integro-differential equation of the following form:

$$\frac{d^\alpha p}{d\tilde{t}^\alpha} = ap - bp^2 - cp\int_0^{\tilde{t}} p(s)ds,$$
$$p(0) = p_0, \quad 0 < \alpha \le 1,$$

2000 *Mathematics Subject Classification.* 26A33, 34A08, 74G10.

Key words and phrases. Fractional-order Muntz–Legendre polynomials (FMLPs), Nonlinear fractional Volterra integro-differential equation, Population growth model, Caputo fractional derivative.

Received: Aug. 29, 2020 Accepted: Jun. 29, 2021 .

where $p(\tilde{t})$ is the scaled population of identical individuals, $\tilde{t}$ denotes the time, α is a constant describing the order of time-fractional derivative, $a > 0$ is the birth rate coefficient, $b > 0$ is the crowing coefficient and $c > 0$ is the toxicity coefficient. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long term. We choose $t = \frac{c\tilde{t}}{b}$ and $y = \frac{bp}{a}$, for the scale time and population, respectively, to obtain the following problem

$$k\frac{d^{\alpha}y}{dt^{\alpha}} = y - y^2 - y\int_0^t y(s)ds, \quad 0 < \alpha \leq 1, \quad 0 \leq t, s \leq T < \infty, \tag{1.1}$$

$$y(0) = \beta, \tag{1.2}$$

where $k = c/(ab)$ is a prescribed non-dimensional parameter and β is the initial population. This model is a fractional-order integro-differential equation where the term $cy\int_0^t y(s)ds$ represents the effect of toxin accumulation on the species. For $\alpha = 1$, this equation will represents a first-order integro-ordinary differential equation. Recently, several analytical and numerical methods have been proposed to solve the model (1.1). For example, the Adomian decomposition method [5], Pade-approximation method by the Adomian decomposition method [8], Sinc and rational Legendre collection method [9], fractional shifted Legendre polynomial method [10] and other methods [11, 12].

The aim of this paper is to apply the factional-order Muntz–Legendre polynomials to solve the population growth model (1.1), using collocation method.

This paper is organized as follows. Review of Caputo fractional derivative, is briefly provided in section 2. In section 3, we present fractional-order Muntz–Legendre polynomials and its properties. Numerical method for solving model (1.1) is established in section 4. Finally, we illustrate some numerical examples to show the efficiency and accuracy of the proposed method in section 5.

2. Review of Caputo fractional derivative

Definition 2.1. The fractional derivative of $y(t)$ in the Caputo sense is defined as

$$D_*^{\alpha}y(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}y'(\tau)d\tau \tag{2.1}$$

for $0 < \alpha < 1$, $t > 0$.

2.1. **Properties.** Some properties of the Caputo fractional derivative and operator D_*^α, which will be used later, are as follows

i) $D_*^\alpha C = 0$, where C is a constant.

ii)

$$D_*^\alpha t^v = \begin{cases} 0, & v \in \mathbb{N}_0, v < [\alpha], \\ \frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)} t^{v-\alpha}, & (v \in \mathbb{N}_0, \quad v \geq [\alpha]) \quad \text{or} \\ & (v \notin \mathbb{N}_0, v > [\alpha]), \end{cases} \tag{2.2}$$

where $[\alpha]$ is the smallest integer greater than or equal to α and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

iii) the Caputo fractional derivative is a linear operation:

$$D_*^\alpha \Big(\sum_{i=1}^n a_i y_i(t) \Big) = \sum_{i=1}^n a_i D_*^\alpha y_i(t)$$

For more details about the properties of the Caputo fractional derivative see [13].

3. Fractional Muntz-Legendre polynomials

Definition 3.1. The fractional Muntz-Legendre polynomials $L_i(t;\alpha)$ on the interval $[0, T]$ is given by the following formula [14]:

$$L_i(t;\alpha) = \sum_{k=0}^i C_{i,k} \big(\frac{t}{T}\big)^{k\alpha}, \qquad C_{i,k} = \frac{(-1)^{i-k}}{\alpha^i k!(i-k)!} \prod_{v=0}^{i-1} ((k+v)\alpha + 1). \tag{3.1}$$

According to the Eq. (3.1), the analytic form of $L_i(t;\alpha)$ can be written as follows:

$$L_i(t;\alpha) = \sum_{k=0}^i b_{k,i} \big(\frac{t}{T}\big)^{k\alpha}, \tag{3.2}$$

where

$$b_{k,i} = \frac{(-1)^{i-k}\Gamma(\frac{1}{\alpha}+k+i)}{k!(i-k)!\Gamma(\frac{1}{\alpha}+k)}. \tag{3.3}$$

Also, we have

$$L_i(t;\alpha) = P_i^{(0,\frac{1}{\alpha}-1)}(2\big(\frac{t}{T}\big)^\alpha - 1), \qquad \alpha > 0, \tag{3.4}$$

where $P_i^{(\alpha,\beta)}$ are the Jacobi polynomial with parameters $\alpha, \beta > -1$ [14, 15]. Using Eq.(3.4) and recurrence relation between the Jacobi polynomials [15], can be obtained the following recurrence formula:

$$L_{i+1}(t;\alpha) = a_i^\alpha L_i(t;\alpha) - b_i^\alpha L_{i-1}(t;\alpha), \qquad i = 1, 2, \ldots$$

where

$$\begin{aligned} a_i^\alpha &= \frac{(2i+\frac{1}{\alpha})[(2i+\frac{1}{\alpha}-1)(2i+\frac{1}{\alpha}+1)(2\left(\frac{t}{T}\right)^\alpha - 1) - (\frac{1}{\alpha}-1)^2]}{2(i+1)(i+\frac{1}{\alpha})(2i+\frac{1}{\alpha}-1)}, \\ b_i^\alpha &= \frac{i(i+\frac{1}{\alpha}-1)(2i+\frac{1}{\alpha}+1)}{(i+1)(i+\frac{1}{\alpha})(2i+\frac{1}{\alpha}-1)}, \\ L_0(t;\alpha) &= 1, \quad L_1(t;\alpha) = (\frac{1}{\alpha}+1)\left(\frac{t}{T}\right)^\alpha - \frac{1}{\alpha}. \end{aligned}$$

Note that $L_i(1;\alpha) = 1$. Also, for $\alpha = 1$ and $T = 1$, we obtain shift Legendre polynomials [16] as follows:

$$L_i(t;1) = P_i^{(0,0)}(2t-1), \qquad t \in [0,1].$$

For $\alpha \neq 1$, we have

$$L_i(t;\alpha) = P_i^{(0,\frac{1}{\alpha}-1)}(2t^\alpha - 1) \neq P_i^{(0,0)}(2t^\alpha - 1),$$

where $P_i^{(0,0)}(2t^\alpha - 1)$ are fractional shift Legendre polynomials [17].

3.1. **Properties.** The FMLPs are orthogonal on the interval $[0,1]$ with the orthogonality relation as follows:

$$\int_0^1 l_i(t;\alpha) l_j(t;\alpha) dt = \frac{1}{2i\alpha+1}\delta_{ij}, \tag{3.5}$$

where $l_i(\frac{t}{T};\alpha) = L_i(t;\alpha)$ and δ_{ij} is the Kronecker function.

Theorem 3.1. *The FMLPs are orthogonal on the interval $[0,T]$ with the orthogonality relation:*

$$\int_0^T L_i(t;\alpha) L_j(t;\alpha) dt = \frac{T}{2i\alpha+1}\delta_{ij}, \tag{3.6}$$

proof. In Eq. (3.5), let $t = \frac{x}{T}$, then

$$\int_0^1 l_i(t;\alpha)l_j(t;\alpha)dt = \int_0^T l_i(\frac{x}{T};\alpha)l_j(\frac{x}{T};\alpha)\frac{1}{T}dx$$
$$= \int_0^T L_i(x;\alpha)L_j(x;\alpha)\frac{1}{T}dx = \frac{1}{2i\alpha+1}\delta_{ij},$$

hence we have

$$\int_0^T L_i(x;\alpha)L_j(x;\alpha)dx = \frac{T}{2i\alpha+1}\delta_{ij}.$$

So, the proof is completed.

3.2. **Function approximation.**

Definition 3.2. An arbitrary function $y(t)$ which is integrable in $[0,T]$ can be expanded as follows:

$$y(t) = \sum_{i=0}^{\infty} a_i L_i(t;\alpha), \tag{3.7}$$

where

$$a_i = \frac{2i\alpha+1}{T}\int_0^T y(t)L_i(t;\alpha)dt, \quad i = 0,1,\dots$$

In practice, only the first $(m+1)$-terms FMLPs are considered. Then it can be written the Eq. (3.7) as follows

$$y(t) \simeq y_m(t) = \sum_{i=0}^{m} a_i L_i(t;\alpha) = A^T\phi(t;\alpha), \tag{3.8}$$

where

$$A = [a_0, a_1, \dots, a_m]^T, \tag{3.9}$$
$$\phi(t;\alpha) = [L_0(t;\alpha), L_1(t;\alpha), \dots, L_m(t;\alpha)]^T.$$

In the other hand,

$$\phi(t;\alpha) = B^T X(t;\alpha),$$

where

$$B = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0m} \\ 0 & b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{mm} \end{bmatrix}, \qquad X(t;\alpha) = \begin{bmatrix} 1 \\ \left(\frac{t}{T}\right)^{1\alpha} \\ \vdots \\ \left(\frac{t}{T}\right)^{m\alpha} \end{bmatrix}, \tag{3.10}$$

and b_{ij} are defined in (3.3). So, we can write

$$y(t) \simeq y_m(t) = A^T B^T X(t;\alpha). \tag{3.11}$$

3.3. **Convergence analysis.** Following Theorems show approximation converges of FMLPs to $y(t)$.

Theorem 3.2. *Suppose $D^{k\alpha}y(t) \in C[0,1]$ for $k = 0, 1, \dots, m$ and $2m\alpha \geq 0$. If $y_m(t)$ in Eq. (3.11) is the best approximation to $y(t)$ from*

$$M_{m,\alpha} = span\{l_0(t;\alpha), l_1(t;\alpha), \dots, l_m(t;\alpha)\},$$

then.

$$\|y(t) - y_m(t)\|_\omega \leq \frac{M_\alpha}{\Gamma(m\alpha+1)\sqrt{2m\alpha+1}},$$

where $M_\alpha \geq |D^{m\alpha}y(t)|$, $t \in [0,1]$.

Proof. By applying the generalized Taylor's expansion formula (see [18, 19]), we have

$$y(t) = \sum_{k=0}^{m-1} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} D^{k\alpha}y(0^+) + \frac{t^{m\alpha}}{\Gamma(m\alpha+1)} D^{m\alpha}y(\xi), \quad 0 < \xi < t, \quad t \in [0,1].$$

Also,

$$|y(t) - \sum_{k=0}^{m-1} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} D^{k\alpha}y(0^+)| \leq \frac{M_\alpha t^{m\alpha}}{\Gamma(m\alpha+1)}.$$

On the other hand, we have $y_m(t) = A^T\phi(t;\alpha)$ is the best approximation to $y(t)$, and $\sum_{k=0}^{m-1} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} D^{k\alpha}y(0^+) \in M_{m,\alpha}$. So, it can be written

$$\begin{aligned}\|y(t) - y_m(t)\|_\omega^2 \leq & \|y(t) - \sum_{k=0}^{m-1} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} D^{k\alpha}y(0^+)\|_\omega^2 \\ \leq & \frac{M_\alpha^2}{\Gamma(m\alpha+1)^2} \int_0^1 t^{2m\alpha} dt = \frac{M_\alpha^2}{\Gamma(m\alpha+1)^2(2m\alpha+1)}.\end{aligned}$$

Hence,

$$\|y(t) - y_m(t)\|_\omega \leq \frac{M_\alpha}{\Gamma(m\alpha+1)\sqrt{2m\alpha+1}},$$

and this completes the proof.

Theorem 3.3. *Suppose* $D^{k\alpha}y(t) \in C[0,T]$ *for* $k = 0, 1, \dots, m$ *and* $2m\alpha \geq 0$. *If* $y_m(t)$ *in Eq.* (3.11) *is the best approximation to* $y(t)$ *from*

$$M_{m,\alpha} = span\{L_0(t;\alpha), L_1(t;\alpha), \dots, L_m(t;\alpha)\},$$

then,

$$\|y(t) - y_m(t)\|_\omega \leq \frac{M_\alpha}{\Gamma(m\alpha+1)}\sqrt{\frac{T}{2m\alpha+1}},$$

where $M_\alpha \geq |D^{m\alpha}y(t)|$, $t \in [0,T]$.

Proof. It is the result of the Theorem 3.2.

4. Functions approximation and description of the method

Let $y_m(t)$ be an approximation for the solution of Eqs. (1.1) and (1.2), then it can be written as

$$y(t) \simeq y_m(t) = A^T\phi(t;\alpha) = A^T B^T X(t), \quad t \in [0,T], \tag{4.1}$$

where A, B and X are defined in relations (3.9) and (3.10).
To get a solution of the problem (1.1), we employ the collocation method with suitably choice of collocation Chebyshev–Gauss–Lobatto points defined by:

$$t_j = \frac{T}{2} - \frac{T}{2}cos(\frac{\pi j}{m}), \qquad j = 0, \dots, m. \tag{4.2}$$

By substituting this collocation points into Eq. (4.1), we have

$$y_m(t_j) = A^T B^T X(t_j). \tag{4.3}$$

By using the relation (2.2), we get the Caputo fractional derivative of $y_m(t)$ of order α in relation (4.1) as:

$$D_*^\alpha y_m(t) = A^T B^T D_*^\alpha X(t), \tag{4.4}$$

where

$$D_*^\alpha X(t) = \begin{bmatrix} 0 \\ \frac{\Gamma(\alpha+1)}{T^\alpha} \\ \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)T^{2\alpha}}t^\alpha \\ \vdots \\ \frac{\Gamma(m\alpha+1)}{\Gamma((m-1)\alpha+1)T^{m\alpha}}t^{(m-1)\alpha} \end{bmatrix}. \tag{4.5}$$

By putting the collocation points into Eq. (4.4), we get

$$D_*^\alpha y_m(t_j) = A^T B^T D_*^\alpha X(t_j), \qquad j = 0, \ldots, m. \tag{4.6}$$

For the integral part of Eq. (1.1), we can write

$$\int_0^t y_m(s)ds = A^T B^T \Big(\int_0^t X(s)ds \Big) = A^T B^T IX(t), \tag{4.7}$$

where

$$IX(t) = \begin{bmatrix} t \\ \frac{t^{\alpha+1}}{(\alpha+1)T^\alpha} \\ \frac{t^{2\alpha+1}}{(2\alpha+1)T^{2\alpha}} \\ \vdots \\ \frac{t^{m\alpha+1}}{(m\alpha+1)T^{m\alpha}} \end{bmatrix}. \tag{4.8}$$

By putting the collocation points into Eq. (4.7), we get

$$\int_0^{t_j} y_m(s)ds = A^T B^T IX(t_j), \qquad j = 0, \ldots, m. \tag{4.9}$$

Now, by substituting the approximate solution y_m and putting the collocation points into Eqs. (1.1) and (1.2) as follows

$$kD_*^\alpha y_m(t_j) - y_m(t_j) + y_m^2(t_j) + y_m(t_j) \int_0^{t_j} y_m(s)ds = 0, \quad j = 1, \ldots, m. \tag{4.10}$$

$$y_m(0) - \beta = 0 \tag{4.11}$$

Then, by substituting the Eqs. (4.3), (4.6) and (4.9) in Eq. (4.10) and (4.11), we get a nonlinear system of algebraic equations as

$$\begin{cases} kCD_*^\alpha X(t_j) - CX(t_j) + (CX(t_j))^2 + (CX(t_j))(CIX(t_j)) = 0, j = 1, \ldots, m, \\ CX(t_0) - \beta = 0. \end{cases} \tag{4.12}$$

where $C = A^T B^T$. After solving the nonlinear system (4.12) of $(m+1)$ equations for the $(m+1)$ unknown coefficients a_j by using fslove command in Matlab, we can find the following approximate solution

$$y_m(t) = CX(t).$$

5. Illustrative Examples

In this section, we present some examples to study the mathematical behavior of the solution of population growth model (1.1). The accuracy of our method is estimated by the error function $E_m^\alpha(t)$, which is given as follows:

$$E_m^\alpha(t) = |k\frac{D_*^\alpha y_m(t)}{dt^\alpha} - y_m(t) + y_m^2(t) + y_m(t)\int_0^t y_m(s)ds|, \tag{5.1}$$

where $\frac{D_*^\alpha y_m(t)}{dt^\alpha}$ is the Caputo fractional derivative of $y_m(t)$ of order α in relation (4.1).

Example 5.1. *In the population growth model* (1.1) *with condition* (1.2), *we take* $k = 0.8, \alpha = 1, T = 5$ *and* $\beta = 0.1$. *For* $m = 4, 6, 8$ *and* 12, *using the present method, we obtain the approximate solutions* $y_m(t)$ *as follows:*

$$\begin{aligned}
y_4(t) =& 0.08165192454951264 - 1.271345945354403(\frac{t}{5}) \\
&+ 4.567059600747760(\frac{t}{5})^2 - 5.845176215680037(\frac{t}{5})^3 \\
&+ 2.532289533473607(\frac{t}{5})^4, \\
y_6(t) =& 0.051035763136624 - 1.633888140122556(\frac{t}{5}) \\
&+ 14.97271853121058(\frac{t}{5})^2 - 54.03608160535557(\frac{t}{5})^3 \\
&+ 90.74008607748452(\frac{t}{5})^4 - 71.46001230339654(\frac{t}{5})^5 \\
&+ 21.40936113327181(\frac{t}{5})^6 \\
y_8(t) =& 0.0999999999999 + 0.7778253301087412(\frac{t}{5}) \\
&- 2.893403760675780(\frac{t}{5})^2 + 33.21988373820448(\frac{t}{5})^3
\end{aligned}$$

$$
\begin{aligned}
&- 129.4368169807767(\frac{t}{5})^4\\
&+ 229.8686697548550(\frac{t}{5})^5\\
&- 214.0759639892768(\frac{t}{5})^6 + 102.7333048401608(\frac{t}{5})^7\\
&- 20.16481519185317(\frac{t}{5})^8,\\
y_{12}(t) =&0.1 + 0.5640171129058788(\frac{t}{5}) + 1.169875600818050(\frac{t}{5})^2\\
&+ 2.489039531857378(\frac{t}{5})^3 - 29.61864282688904(\frac{t}{5})^4\\
&+ 185.9605147254196(\frac{t}{5})^5 - 931.4457561115956(\frac{t}{5})^6\\
&+ 2708.712629788791(\frac{t}{5})^7 - 4651.846775084796(\frac{t}{5})^8\\
&+ 4890.555176130227(\frac{t}{5})^9 - 3118.301393754186(\frac{t}{5})^{10}\\
&+ 1113.113097660241(\frac{t}{5})^{11} - 171.3218700205367(\frac{t}{5})^{12}.
\end{aligned}
$$

Fig. 1 shows a graph of the above approximate solutions. Table 1 shows the absolute error

$$\epsilon^m(t) = |y_{max}(t) - y^m_{max}(t)|,$$

where y^m_{max} is the maximum approximate solution of present method and y_{max} is the maximum exact solution of given in [7] as

$$y_{max} = 1 + k\log(\frac{k}{1+k-\beta}).$$

Numerical experiments show the absolute error is large when $m < 8$. Furthermore, we present the logarithmic graph of $E^{\alpha}_m(t), (log_{10}E^{\alpha}_m(t))$ for various values of m in Fig. 2.

Table 1 and Figs. 1 and 2 show that by increasing m, the error is smaller.

TABLE 1. Absolute error $\epsilon^m(t)$ for various values of m (Example 5.1)

m	$\epsilon^m(t)$
4	$3.1533e-01$
6	$3.4594e-01$
8	$7.1002e-04$
9	$1.4619e-04$
10	$1.0780e-04$
12	$2.6653e-05$

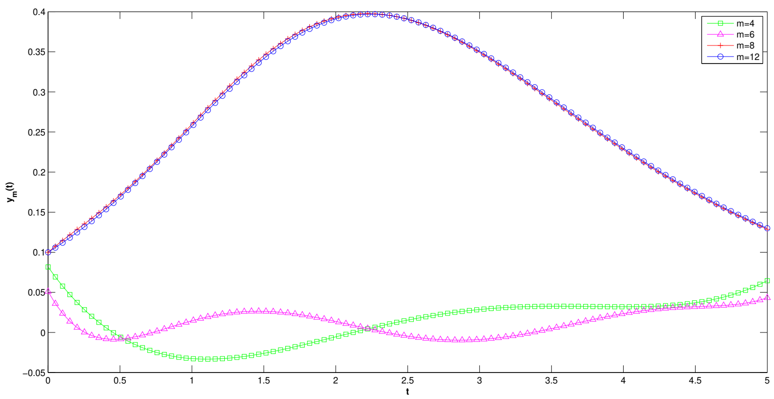

FIGURE 1. The approximate solutions of Example 5.1 for various values of m.

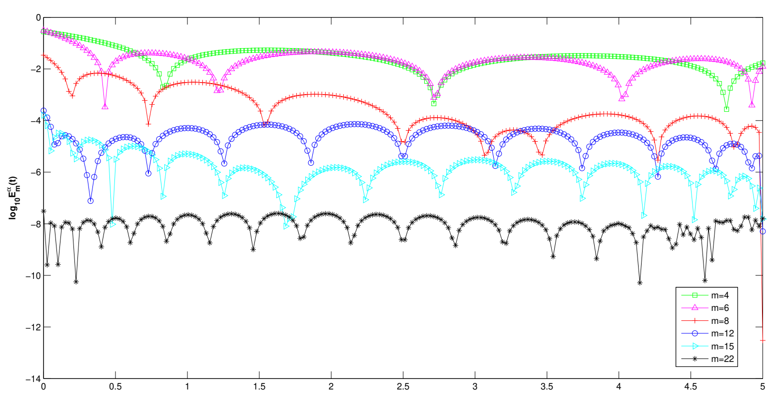

FIGURE 2. The logarithmic graph of $E_m^{\alpha}(t)$ of Example 5.1 for various values of m.

Example 5.2. *In the population growth model* (1.1) *with condition* (1.2)*, we take* $k = 1, \alpha = \frac{17}{20}, T = 5$ *and* $\beta = 0.1$*. For* $m = 4, 6, 8$ *and* 12*, using present method, we obtain the approximate solutions* $y_m(t)$ *as follows:*

$$
\begin{aligned}
y_4(t) =& 0.07120726429698099 - 1.145700387156755(\frac{t}{5})^{\frac{17}{20}} \\
&+ 3.485514593623768(\frac{t}{5})^{\frac{17}{10}} - 3.676334522253686(\frac{t}{5})^{\frac{51}{20}} \\
&+ 1.286081714613704(\frac{t}{5})^{\frac{17}{5}}, \\
y_6(t) =& 0.02828601827181646 - 0.6592826537323773(\frac{t}{5})^{\frac{17}{20}} \\
&+ 4.777469686547025(\frac{t}{5})^{\frac{17}{10}} - 14.01714932871185(\frac{t}{5})^{\frac{51}{20}} \\
&+ 19.28983999076952(\frac{t}{5})^{\frac{17}{5}} - 12.46186346208233(\frac{t}{5})^{\frac{17}{4}} \\
&+ 3.059345106749583(\frac{t}{5})^{\frac{51}{10}}, \\
y_8(t) =& 0.1 + 0.5127344573375179(\frac{t}{5})^{\frac{17}{20}} - 1.823618792794863(\frac{t}{5})^{\frac{17}{10}} \\
&+ 16.69002679734211(\frac{t}{5})^{\frac{51}{20}} - 53.13733983700161(\frac{t}{5})^{\frac{17}{5}} \\
&+ 77.48462031728099(\frac{t}{5})^{\frac{17}{4}} - 59.13470565994732e(\frac{t}{5})^{\frac{51}{10}} \\
&+ 23.17736678419399e(\frac{t}{5})^{\frac{119}{20}} - 3.702676336081774e(\frac{t}{5})^{\frac{34}{5}}, \\
y_{12}(t) =& 0.1 + 0.3736300070681236(\frac{t}{5})^{\frac{17}{20}} + 0.6686166199181787(\frac{t}{5})^{\frac{17}{10}} \\
&+ 0.3281639985722173(\frac{t}{5})^{\frac{51}{20}} - 5.735074247981428(\frac{t}{5})^{\frac{17}{5}} \\
&+ 39.26678909412409(\frac{t}{5})^{\frac{17}{4}} - 200.9243703150748(\frac{t}{5})^{\frac{51}{10}} \\
&+ 518.7613298537932(\frac{t}{5})^{\frac{119}{20}} - 751.3470212276436(\frac{t}{5})^{\frac{34}{5}} \\
&+ 652.0726945959808(\frac{t}{5})^{\frac{153}{20}} - 339.4568935003740(\frac{t}{5})^{\frac{17}{2}} \\
&+ 98.26842958790718(\frac{t}{5})^{\frac{187}{20}} - 12.20957478224640(\frac{t}{5})^{\frac{51}{5}},
\end{aligned}
$$

The results given in Figs. 3 and 4.

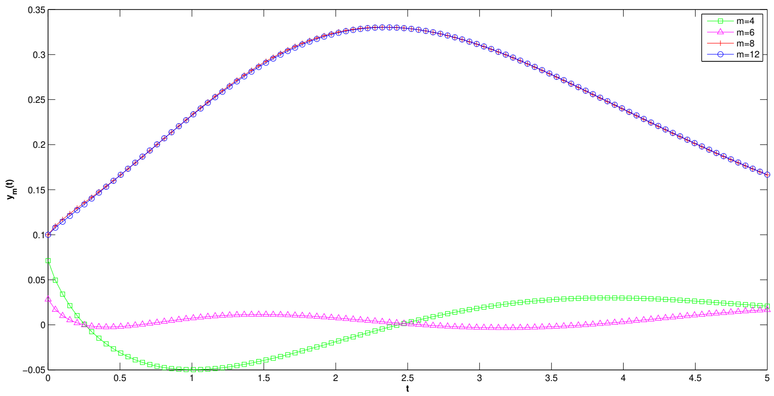

FIGURE 3. The approximate solutions of Example 5.2 for various values of m.

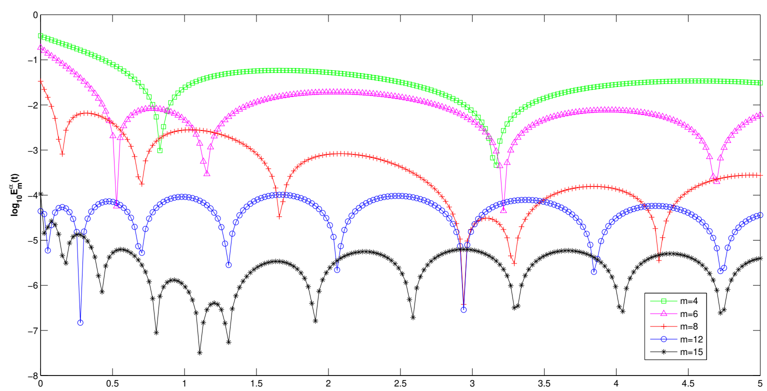

FIGURE 4. The logarithmic graph of $E_m^{\alpha}(t)$ of Example 5.2 for various values of m.

Example 5.3. *We take* $\alpha = \frac{1}{2}, T = 5$ *and* $\beta = 0.1$*, by applying proposed method, for* $k = 0.3, 0.4, 0.5$ *and* $m = 4$ *we obtain the approximate solutions* $y_4(t)$ *as follows:*

$$
\begin{aligned}
k = 0.3: \quad y_4(t) =& 0.1 + 4.313001991695015(\frac{t}{5})^{\frac{1}{2}} - 12.69683886911276(\frac{t}{5}) \\
& + 12.54222563261552(\frac{t}{5})^{\frac{3}{2}} - 4.193850305674383(\frac{t}{5})^2, \\
k = 0.4: \quad y_4(t) =& 0.1 + 3.771577461838058(\frac{t}{5})^{\frac{1}{2}} - 10.65954849582612(\frac{t}{5})
\end{aligned}
$$

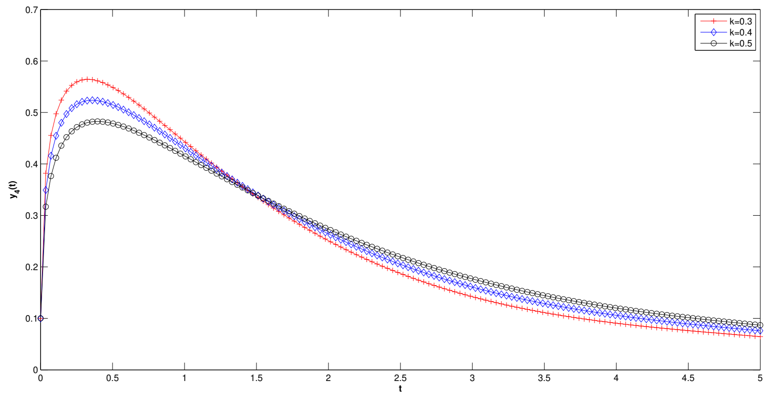

FIGURE 5. The approximate solutions of Example 5.3 for various values of k.

$$+10.15690159873609(\frac{t}{5})^{\frac{3}{2}} - 3.292960223068623(\frac{t}{5})^2,$$

$$k = 0.5: \quad y_4(t) = 0.1 + 3.243291933028762(\frac{t}{5})^{\frac{1}{2}} - 8.738930363136635(\frac{t}{5}) + 7.961921766381884(\frac{t}{5})^{\frac{3}{2}} - 2.479449160538878(\frac{t}{5})^2.$$

Fig. 5 shows the behavior of the above approximate solutions. It can be seen that y_{max} decreases as k increases. Moreover, the rapid rise along the logistic curve followed by the slow exponential decay after reaching the maximum point. This results is similar to the results in other research [7, 20, 21]. In this example, for small m, we can obtain the high accuracy. Fig. 6 shows the logarithmic graph of $E_m^\alpha(t), (log_{10}E_m^\alpha(t))$ of the above approximate solutions for $m = 4$.

It is well known that to study the rapid growth along the logistic curve that will reach a peak, then followed by the slow exponential decay [7]. The analytical solution of the population growth model (1.1) with condition (1.2) is obtained as follows [7, 21]:

$$y(t) = y(0)e^{\left(\frac{1}{k}\int_0^t [1-y(\tau)-\int_0^\tau y(x)dx]d\tau\right)}. \tag{5.2}$$

This solution shows that $y(t) > 0$ if $y(0) > 0$. Also, $y(t) \to 0$ as $t \to \infty$. Sometime, by applying the present method, for some values of α (almost near to zero),we do not see these properties. To solve this problem, we applied the Pade approximants [22] which approximate solution $y_m(t)$ leads to the a rational function to give a better

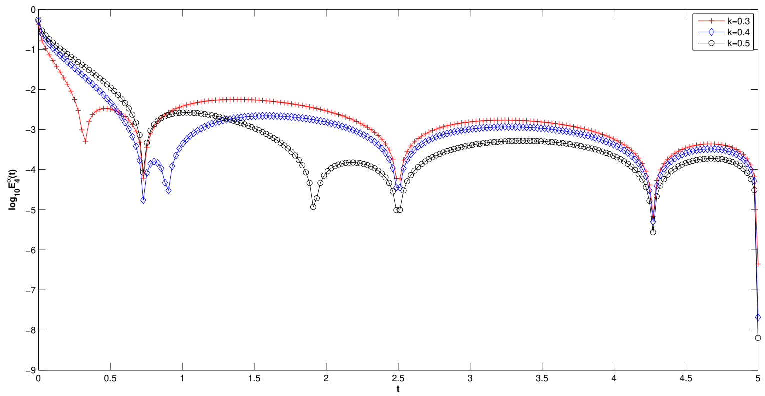

FIGURE 6. The logarithmic graph of $E_4^{\frac{1}{2}}(t)$ of Example 5.3 for various values of k.

mathematical behavior of the solution than the approximate solution $y_m(t)$ [22, 23]. Consider the following example:

Example 5.4. *We take* $\alpha = \frac{1}{5}, k = 0.8, T = 5$ *and* $\beta = 0.1$*, by applying proposed method, for* $m = 3, 4, 5$ *we obtain the approximate solutions* $y_m(t)$ *as follows:*

$$\begin{aligned}
y_3(t) =& 0.09999999999999965 - 11.58175897951440(\frac{t}{5})^{\frac{1}{5}} + 26.30859058474087(\frac{t}{5})^{\frac{2}{5}} \\
&- 14.60417753583053(\frac{t}{5})^{\frac{3}{5}}, \\
y_4(t) =& 0.1 - 22.63351582890925(\frac{t}{5})^{\frac{1}{5}} + 77.80108676581074(\frac{t}{5})^{\frac{2}{5}} \\
&- 86.77829769996865(\frac{t}{5})^{\frac{3}{5}} + 31.65776067327077(\frac{t}{5})^{\frac{4}{5}} \\
y_5(t) =& 0.1 - 27.96942253972512(\frac{t}{5})^{\frac{1}{5}} + 244.9973483564507(\frac{t}{5})^{\frac{2}{5}} \\
&- 630.4311305594935(\frac{t}{5})^{\frac{3}{5}} + 639.5413214449511(\frac{t}{5})^{\frac{4}{5}} \\
&- 226.0159898258375(\frac{t}{5})
\end{aligned}$$

Fig. 7 shows the behavior of the above approximate solutions and other values of m ($m = 10, 15, 20$) . Clearly, we do not get the correct mathematical structure of the solution of this population growth model. Therefore, we applied Pade approximants. To achieve a desirable accuracy, setting $m = 20$ and we have

$$y_{20}(t) = 0.1 + 0.12253t^{\frac{1}{5}} + 0.12680t^{\frac{2}{5}} + 0.10713t^{\frac{3}{5}} + 0.070850t^{\frac{4}{5}} - 0.093675t$$

$$+ 1.3158t^{\frac{6}{5}} - 12.155t^{\frac{7}{5}} + 75.626t^{\frac{8}{5}} - 349.42t^{\frac{9}{5}} + 1203.7t^{2}$$
$$- 3135.0t^{\frac{11}{5}} + 6193.9t^{\frac{12}{5}} + O(t^{\frac{13}{5}}).$$

Let $t^{\frac{1}{5}} = x$, then

$$\begin{aligned} y_{20}(t) =& 0.1 + 0.12253x + 0.12680x^2 + 0.10713x^3 + 0.070850x^4 - 0.093675x^5 \\ &+ 1.3158x^6 - 12.155x^7 + 75.626x^8 - 349.42x^9 + 1203.7x^{10} \\ &- 3135.0x^{11} + 6193.9x^{12} + O(x^{13}). \end{aligned}$$

Now, it can be calculated the [6/6] Pade approximants as follows

$$[6/6] = \frac{0.1 + 1.3478x + 8.4061x^2 + 28.640x^3 + 41.935x^4 - 25.553x^5 + 6.6709x^6}{1 + 12.253x + 67.780x^2 + 186.75x^3 + 90.755x^4 - 683.87x^5 + 539.81x^6}.$$

By recalling $x = t^{\frac{1}{5}}$, we get

$$[6/6] = \frac{0.1 + 1.3478t^{\frac{1}{5}} + 8.4061t^{\frac{2}{5}} + 28.640t^{\frac{3}{5}} + 41.935t^{\frac{4}{5}} - 25.553t + 6.6709t^{\frac{6}{5}}}{1 + 12.253t^{\frac{1}{5}} + 67.780t^{\frac{2}{5}} + 186.75t^{\frac{3}{5}} + 90.755t^{\frac{4}{5}} - 683.87t + 539.81t^{\frac{6}{5}}}.$$

Similarly, for $\alpha = \frac{1}{6}$, we get

$$[6/6] = \frac{0.1 + 0.78523t^{\frac{1}{6}} - 1.2827t^{\frac{2}{6}} - 38.375t^{\frac{3}{6}} - 183.90t^{\frac{4}{6}} - 9.1786t^{\frac{5}{6}} + 6.6397t}{1 - 22.139t^{\frac{1}{6}} - 366.35t^{\frac{2}{6}} - 1374.7t^{\frac{3}{6}} + 90.755t^{\frac{4}{6}} + 3518.3t^{\frac{5}{6}} - 2214.4t}.$$

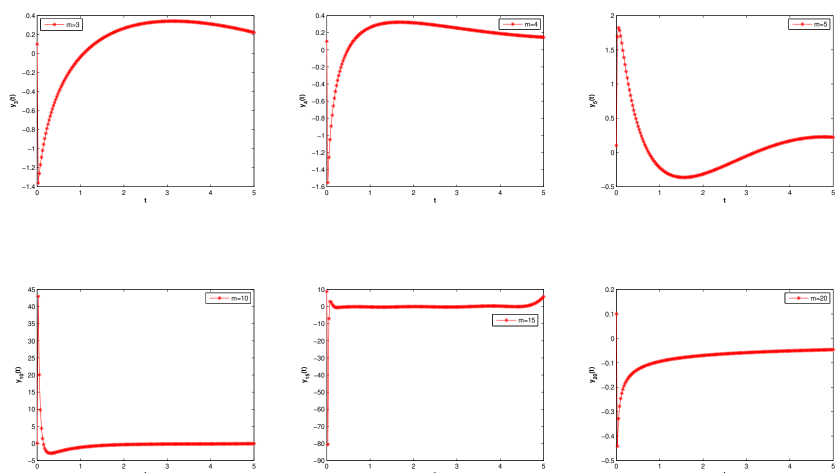

FIGURE 7. The approximate solutions $y_m(t)$ of Example 5.4 for $m = 3, 4, 5, 10, 15$ and 20.

Fig. 8 shows the behavior of the above [6/6] Pade approximants of approximate solution $y_{20}(t)$ for $\alpha = \frac{1}{5}$ and $\frac{1}{6}$. It can be seen that by decreasing the fractional derivative order, the amplitude of $y(t)$ decreases.

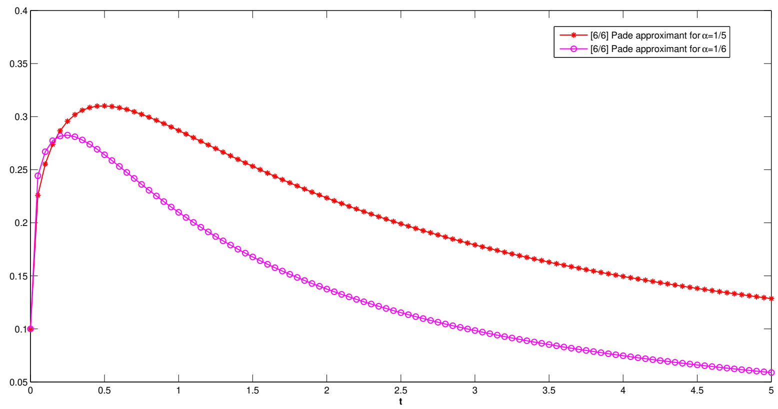

FIGURE 8. [6/6] Pade approximants of $y_{20}(t)$ for $\alpha = \frac{1}{5}$ and $\frac{1}{6}$ of Example 5.4.

6. Conclusion

In this paper, we have studied the mathematical behavior of the solution of the fractional order population growth model using fractional Muntz–Legendre polynomials. This polynomials were used as an approximation basis for present method. Sometime, for some values of α, specially near to zero, it cannot be seen the proper properties of the graph of the model (1.1). For this purpose, we applied the Pade approximants that was very successful. Numerical results state the efficiency and accuracy of the present method. In the above presented numerical results, one can see that by decreasing the fractional derivative order, the amplitude of solution $y(t)$ decreases.

Acknowledgement

I would like to thank the anonymous referee for his or her suggestions and corrections.

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