Jordan Journal of Mathematics and Statistics (JJMS), 15(2), 2022, pp 177 - 197
DOI: https://doi.org/10.47013/15.2.2

ALGORITHMS AND IDENTITIES FOR BÉZIER CURVES VIA POST-QUANTUM BLOSSOM

ALAA M. OBAD (1), KHALID KHAN (2), D.K. LOBIYAL (3) AND ASIF KHAN (4)

Abstract. In this paper, a new analogue of blossom based on post-quantum calculus is introduced. The post-quantum blossom has been adapted for developing identities and algorithms for Bernstein basis and Bézier curves. By applying the post-quantum blossom, various new identities and formulae expressing the monomials in terms of the post-quantum Bernstein basis and a post-quantum variant of Marsden's identity are investigated. For each post-quantum Bézier curves of degree m, a collection of $m!$ new, affine invariant, recursive evaluation algorithms are derived.

1. Introduction

Approximation theory basically deals with approximation of functions by simpler functions or more easily calculated functions. Broadly it is divided into theoretical and constructive approximation [5, 15].

Mursaleen et al applied post-quantum calculus in constructive approximation theory and introduced the first post-quantum analogue of Bernstein operators [19] based on post-quantum integers.

The post-quantum Bernstein operators introduced by them are generalization of well known classical Bernstein [5] operators and Phillips quantum Bernstein operators

2020 *Mathematics Subject Classification.* MSC: primary 65D17; secondary 41A10, 41A25, 41A36.
Key words and phrases. post-quantum integers; post-quantum blossom; de Casteljau algorithm; Marsden's identity; post-quantum Bernstein polynomials; post-quantum Bézier curve; quantum Bernstein polynomials.

Received: Aug. 30, 2020 Accepted: Jan. 17, 2021 .

(polynomials) [21]. For recent literatures related to constructive approximation theory, quantum calculus and post-quantum calculus, one can see [1, 2, 4, 16, 17, 18, 22, 26, 24].

Similarly (Computer Aided Geometric Design (CAGD)) is a discipline which deals with computational aspects of geometric objects. If emphasizes on the mathematical development such that it becomes compatible with computers. It has several applications in Approximation theory and Numerical analysis. Basis of Bernstein operators has been used to draw curves and surfaces. Bézier curves were independently developed by P. de-Casteljau at Citroen [6] and by P. Bézier at Renault [9]. For details on Bézier curves and surfaces' approximation, one can refer [3, 7, 10, 11, 23].

Khan and Lobiyal [12] recently constructed post-quantum analogue of Lupaş quantum Bernstein operators (rational) and investigated various properties of Lupaş post-quantum Bézier curves and surfaces. For some applications of the extra parameter 'p' of post-quantum analogue in terms of flexibility to design geometric shapes and for flexibility in approximation, one can refer [12, 20].

In CAGD, blossoming method deals with representation of curves into simpler form like representing a polynomial of degree m into monomial in m variables each of degree one. Blossoming method is used to reduce computational complexity for construction of Bézier curves and surfaces. This provides a powerful tool for deriving identities and developing change of basis algorithms for basis and Bézier curves. In [25], some algorithms and identities for quantum Bernstein basis and quantum Bézier curves using the method of quantum Blossoming are constructed.

Motivated by above mentioned work, the idea of post-quantum calculus and its importance, in next sections, we investigate and derive several results via post-quantum analogue of blossoming. The post-quantum blossom will be used for developing identities and algorithms for Bernstein bases and Bézier curves. By applying the post-quantum blossom, various new identities and formulae expressing the monomials in terms of the post-quantum Bernstein basis functions and a post quantun variant of Marsden's identity are investigated. For each post-quantum Bézier curves of degree m, a collection of $m!$ new, affine invariant, recursive evaluation algorithms are derived.

Let us recall certain notations and definitions from post-quantum calculus. The post-quantum number is defined by, for any number m

$$[m]_{p,q} = p^{m-1} + p^{m-2}q + p^{m-3}q^2 + \cdots + pq^{m-2} + q^{m-1} = \begin{cases} \frac{p^m - q^m}{p-q}, & \text{when } p \neq q \\ m\, p^{m-1}, & \text{when } p = q. \end{cases}$$

The formula for post-quantum binomial expansion is as follows:

$$(au+bv)^m_{p,q} = \sum_{r=0}^{m} p^{\frac{(m-r)(m-r-1)}{2}} q^{\frac{r(r-1)}{2}} \begin{bmatrix} m \\ r \end{bmatrix}_{p,q} a^{m-r} b^r u^{m-r} v^r,$$

$$(u+v)^m_{p,q} = (u+v)(pu+qv)(p^2u+q^2v)\cdots(p^{m-1}u+q^{m-1}v),$$

$$(1-v)^m_{p,q} = (1-v)(p-qv)(p^2-q^2v)\cdots(p^{m-1}-q^{m-1}v),$$

where post-quantum binomial coefficients are defined by

$$\begin{bmatrix} m \\ r \end{bmatrix}_{p,q} = \frac{[m]_{p,q}!}{[r]_{p,q}!\ [m-r]_{p,q}!}.$$

Details on (p, q)-calculus can be found in [12, 19].

Three main contributions of this paper are:

Blossoming: The post-quantum blossom, a new variant of the blossom is introduced which will prove new identities for post-quantum Bernstein bases and generate new approach for post-quantum Bézier curves.

Identities: Using post-quantum blossom, new identities are derived for the post-quantum Bernstein bases, and a post-quantum variant of Marsden's identity and monomials get represented using an explicit formula in terms of the post-quantum Bernstein basis functions.

Recursive Evaluation Algorithms: Using post-quantum blossom technique, for a given post-quantum Bézier curve of degree m, $m!$ new affine invariant, recursive evaluation algorithms has been constructed.

This paper has been arranged in the following way: In Section 2 and 3, we introduce the basic definitions, fundamental formulas, and explicit notation for post-quantum Bernstein bases and post-quantum Bézier curves. In Section 4, we define the post-quantum blossom and establish the existence and the uniqueness of this blossom. In Section 5, we invoke post-quantum blossoming to develop novel evaluation algorithms for post-quantum Bézier curves and in Section 6, we use the post-quantum blossom to derive new identities involving the post-quantum Bernstein basis functions, including a post-quantum version of Marsdens identity as well as formulas for representing monomials in terms of the post-quantum Bernstein basis functions.

2. Post-quantum Bernstein basis functions

The post-quantum Bernstein basis function [13, 19] is as follows

$$B_{p,q}^{r,m}(t) = \frac{1}{p^{\frac{m(m-1)}{2}}} \begin{bmatrix} m \\ r \end{bmatrix}_{p,q} p^{\frac{r(r-1)}{2}} \, t^r (1-t)_{p,q}^{m-r}, \; t \in [0,1] \tag{2.1}$$

where

$$(1-t)_{p,q}^{m-r} = \prod_{s=0}^{m-r-1} (p^s - q^s t).$$

Theorem 2.1. [13] *Each post-quantum Bernstein function of degree m is a linear combination of two post-quantum Bernstein functions of degree $m+1$.*

$$B_{p,q}^{r,m}(t) = \left(\frac{p^r \; [m+1-r]_{p,q}}{[m+1]_{p,q}} \right) B_{p,q}^{r,m+1}(t) + \left(1 - \frac{p^{r+1} \; [m-r]_{p,q}}{[m+1]_{p,q}} \right) B_{p,q}^{r+1,m+1}(t). \tag{2.2}$$

Throughout the paper onwards, we use $B_r^m(t;p,q)$ in place of $B_{p,q}^{r,m}(t)$.

3. Post-quantum Bernstein Bézier curves

The post-quantum Bézier curves [13] of degree m using the post-quantum analogues of the Bernstein basis functions are as follows:

$$\mathbf{P}(t) = \sum_{i=0}^{m} \mathbf{P_i} \; B_i^m(t;p,q) \tag{3.1}$$

where $\mathbf{P_i} \in R^3$ $(i = 0, 1, \ldots, m)$, $\mathbf{P_i}$ are control points. Joining up adjacent points $\mathbf{P_i}$, $i = 0, 1, 2, \ldots, m$ to obtain a polygon which is called the control polygon of post-quantum Bézier curves.

3.1. **de Casteljau algorithm.** Let $\tilde{\mathbf{P}}_{\mathbf{i}}^{\mathbf{0}}(t) = \hat{\mathbf{P}}_{\mathbf{i}}^{\mathbf{0}}(t) = \mathbf{P}_{\mathbf{i}}$, $i = 0, 1, \ldots, m$. Define

$$\tilde{\mathbf{P}}_{\mathbf{i}}^{\mathbf{k}}(t) = (p^{m-k} - p^{i}\ q^{m-k-i}\ t)\tilde{\mathbf{P}}_{\mathbf{i}}^{\mathbf{k-1}}(t) + p^{i}\ q^{m-k-i}\ t\ \tilde{\mathbf{P}}_{\mathbf{i+1}}^{\mathbf{k-1}}(t) \tag{3.2}$$

and

$$\hat{\mathbf{P}}_{\mathbf{i}}^{\mathbf{k}}(t) = q^{i}(p^{m-k-i} - q^{m-k-i}\ t)\hat{\mathbf{P}}_{\mathbf{i}}^{\mathbf{k-1}}(t) + p^{m-k}\ t\ \hat{\mathbf{P}}_{\mathbf{i+1}}^{\mathbf{k-1}}(t), \tag{3.3}$$

for $i = 0, 1, \ldots, m-k,\ k = 1, \ldots, m$. Then point P corresponding to the parameter t is given by $\tilde{\mathbf{P}}_{\mathbf{0}}^{\mathbf{m}}(t) = \hat{\mathbf{P}}_{\mathbf{0}}^{\mathbf{m}}(t) = \mathbf{P}(\mathbf{t})$. Now for illustration purpose, we present Figure 1 and 2 for cubic post-quantum Bézier curves using the above two de Casteljau algorithms. In both algorithms, the property of affine invariant holds at only for the final node at the top of diagram. However for p=1, the property of affine invariant holds at every intermediate nodes in first algorithm, in second algorithm this property holds only for the final node at the top of diagram.

P. Simeonova et al. [25] gave a new approach to identities and algorithms for quantum Bernstein basis and quantum Bézier curves using quantum blossom. In this paper we extend these results for post-quantum Bernstein bases and post-quantum Bézier curves using post-quantum blossoming.

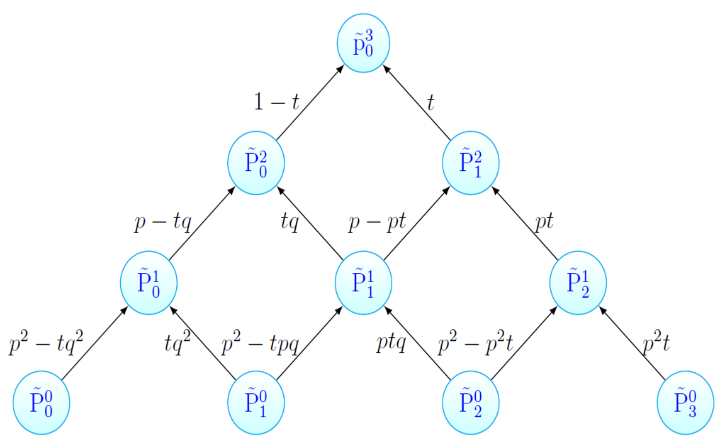

FIGURE 1. 'The first de-Casteljau evaluation algorithm for a cubic post-quantum Bézier curve on the interval $[0, 1]$.'

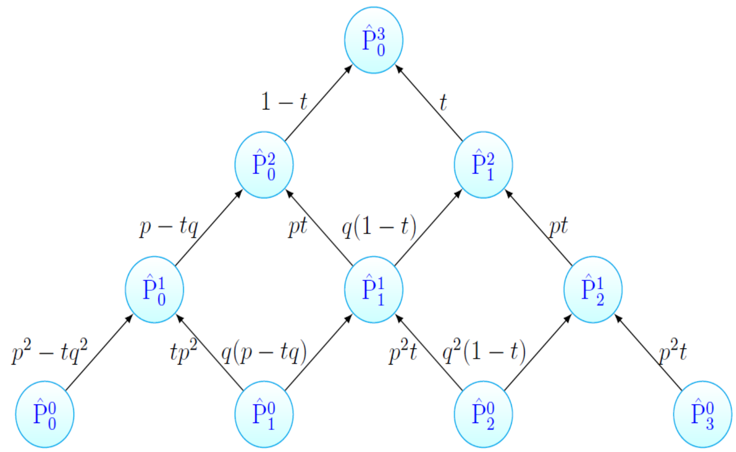

FIGURE 2. 'The second de-Casteljau evaluation algorithm for a cubic post-quantum Bézier curve on the interval $[0, 1]$.'

4. Post-quantum blossoming

Blossoming has given new approach for deriving identities and developing change of basis algorithms for standard Bernstein bases and Bézier curves [25]. In this section, post-quantum blossoming as an extension of standard quantum blossoming is achieved.

The post-quantum blossom or post-quantum polar form of a polynomial $S(t)$ of degree m is the unique symmetric multiaffine function $s(u_1, \dots, u_m; p, q)$ that reduces to $S(t)$ along the post-quantum diagonal. That is, $s(u_1, \dots, u_m; p, q)$ is the unique multivariate polynomial satisfying the following three axioms:

Post-quantum Blossoming axioms

1. **Symmetry:** $s(u_1, \dots, u_m; p, q) = s(u_{\sigma(1)}, \dots, u_{\sigma(m)}; p, q)$ for every permutation σ of the set $\{1, 2, \dots, m\}$.

2. **Multiaffine:** $s(u_1, ..., (1-\alpha)u_k + \alpha v_k, ..., u_m; p, q) =$
$(1-\alpha)\ s(u_1, ..., u_k, ..., u_m; p, q) + \alpha\ s(u_1, ..., v_k, ..., u_m; p, q),$ for ever $\alpha \in \mathbb{R}$

3. **Post-quantum Diagonal:** $s(p^{m-1}t, p^{m-2}tq, \dots, tq^{m-1}; p, q) = S(t)$.

The multiaffine property is equivalent to the fact that each variable $u_1, \dots, u_m$ appears to at most the first power that is, $s(u_1, \dots, u_m; p, q)$ is a polynomial of degree at most one in each variable. The interest in post-quantum blossoming is due to the following important properties, which will be used in section 5 to relate post quantum blossom of a polynomial to its post quantum Bézier control points.

Dual functional property

Let S(t) be a post-quantum Bézier curve of degree m over the interval $[0,1]$ with control points $\mathbf{P_0}, \ldots, \mathbf{P_m}$ and let $s(u_1, \ldots, u_m; p, q)$ be the post-quantum blossom of S(t). Then

$$\mathbf{P_k} = s(0, \ldots, 0, p^{m-1}, p^{m-2}q, \ldots, p^{m-k}q^{k-1}; p, q),\ \ k = 0, 1, \ldots, m. \tag{4.1}$$

This Dual functional property gets proved in Theorem 5.2.

Now we establish those functions existence and uniqueness which satisfy post-quantum blossoming axioms, subject to restrictions on p, q for all polynomials of degree m. But before proceeding to it let's get feel of post-quantum blossom by computing the post-quantum blossom for some simple cases.

Post-quantum Blossom of cubic polynomials

Let us consider a cubic polynomial represented by the monomial $1, t, t^2,$ and t^3. Now these monomials can be easily post-quantum blossomed for any $p \neq 0$, and $q \neq 0$, since in each case the associated function $s(u_1, u_2, u_3; p, q)$ given below can be easily verified as it is symmetric, multiaffine, and reduces to the required monomial along the post-quantum diagonal:

$$S(t) = 1 \Rightarrow s(u_1, u_2, u_3; p, q) = 1,$$

$$S(t) = t \Rightarrow s(u_1, u_2, u_3; p, q) = \frac{u_1 + u_2 + u_3}{(p^2 + pq + q^2)},$$

$$S(t) = t^2 \Rightarrow s(u_1, u_2, u_3; p, q) = \frac{u_1u_2 + u_2u_3 + u_3u_1}{pq(p^2 + pq + q^2)},$$

$$S(t) = t^3 \Rightarrow s(u_1, u_2, u_3; p, q) = \frac{u_1u_2u_3}{p^3q^3}.$$

In the right hand side of the above equation, it can be seen that functions in numerator are combinations of three variables which are written in symmetrical fashion while in case of denominator function is evaluated in symmetrical order at p^2, pq and q^2.

Using these results, any cubic polynomial $S(t) = a_3t^3 + a_2t^2 + a_1t + a_0$ for $p \neq 0, q \neq 0$ can be post-quantum blossom by setting

$$s(u_1, u_2, u_3; p, q) = a_3\frac{u_1u_2u_3}{p^3q^3} + a_2\frac{u_1u_2 + u_2u_3 + u_3u_1}{pq(p^2 + pq + q^2)} + a_1\frac{u_1 + u_2 + u_3}{(p^2 + pq + q^2)} + a_0.$$

Note: For $p = q = 1$, above blossoming reduces into classical blossoming of cubic polynomials.

Similarly, we can apply post-quantum blossom techniques for polynomials of degree m by first post-quantum blossoming the monomials t^k , for $k = 0, \ldots , m$, and then applying linearity. Indeed, let

$$\phi_{m,k}(u_1, u_2, \ldots , u_m) = \sum_{1\leq i_1<i_2<\ldots<i_k\leq m} u_{i_1} \cdots u_{i_k}$$

where the sum runs over all subsets $\{i_1, \ldots , i_k\}$ of $\{1, \ldots , m\}$, denote the k-th elementary symmetric function in the variables $u_1, \ldots , u_m$. Then we get the result as follows.

Proposition 4.1. *The post-quantum blossom of the monomial $M^m_k(t) = t^k$ (considered as a polynomial of degree m) is given by*

$$R^m_k(u_1, \ldots , u_m; p, q) = \frac{\phi_{m,k}(u_1, u_2, \ldots , u_m)}{\phi_{m,k}(p^{m-1}, p^{m-2}q, \ldots , q^{m-1})}, \tag{4.2}$$

provided that $\phi_{m,k}(p^{m-1}, p^{m-2}q, \ldots , q^{m-1}) \neq 0$.

Proof. The three blossoming axioms need to be verified now. One can see that the function $R^m_k(u_1, \ldots , u_m; p, q)$ is symmetric, due to presence of elementary symmetric function divided by a constant in the expression on the right hand side of 4.2. Also, since each variable appears to at most the first power, hence the function on the right hand side of 4.2 is multiaffine. Finally observe that since $\phi_{m,k}(u_1, u_2, \ldots , u_m)$ is a homogeneous polynomial of total degree k in the variables $u_1, \ldots , u_m$,

$$\phi_{m,k}(tu_1, \ldots , tu_m) = t^k\phi_{m,k}(u_1, \ldots , u_m).$$

Therefore along the post-quantum diagonal

$$R^m_k(p^{m-1}t, p^{m-2}qt, \ldots , q^{m-1}t; p, q) = \frac{\phi_{m,k}(p^{m-1}t, p^{m-2}qt, \ldots , q^{m-1}t)}{\phi_{m,k}(p^{m-1}, p^{m-2}q, \ldots , q^{m-1})} = t^k.$$

We can use Proposition 4.1 to establish the existence of the post-quantum blossom for arbitrary polynomials of degree m. But before we proceed, we need to determine explicit conditions for which

$\phi_{m,k}(p^{m-1}, p^{m-2}q, \ldots, q^{m-1}) \neq 0,\ k = 0, 1, \ldots, m$

Lemma 4.1.

$$\phi_{m,k}(p^{m-1}, p^{m-2}q, \ldots, q^{m-1}) = (pq)^{\frac{k(k-1)}{2}} \begin{bmatrix} m \\ k \end{bmatrix}_{p,q} \quad k = 0, 1, \ldots, m. \tag{4.3}$$

Proof. Using induction on m, we get the required result.

Corollary 4.1. $\phi_{m,k}(p^{m-1}, p^{m-2}q, \ldots, q^{m-1}) = 0$ *if and only if one of the following three conditions is satisfied:*

1. $p = 0$ *and* $m > 1,\ k > 1,\ (p = 0$ *and* $2 \leq k \leq m)$

2. $q = 0$ *and* $m > 1,\ k > 1,\ (q = 0$ *and* $2 \leq k \leq m)$

3. $p = -q$ *and* m *is even,* k *is odd.*

Proof. It can be observed that the only real root of a post-quantum binomial coefficient can be $p = -q$ because $[m]_{p,q} = \frac{p^m - q^m}{p - q}$ when $p \neq q$. Condition 1 and 2 follows from 4.3 while Condition 3 follows from the observation that $p = -q$ is a zero of the binomial coefficient $\begin{bmatrix} m \\ k \end{bmatrix}_{p,q}$ of multiplicity

$$\left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{m-k}{2} \right\rfloor = \begin{cases} 1, & \text{if m is even and k is odd} \\ 0, & \text{otherwise.} \end{cases}$$

Now the existence and uniqueness of the post-quantum blossom will be established for all polynomials of degree m and for all real values of p, q that satisfy:

$$(1)\ q \neq 0 \ \text{ and } p \neq 0 \ \text{ for all } \ m > 1 \tag{4.4}$$

and

$$(2)\ q \neq -p \ \text{ for all even } \ m > 1. \tag{4.5}$$

Conditions 4.4 and 4.5 are now the standard restrictions on the value of $p,\ q$.

From now on, whenever there is a talk of (p,q)-blossom or (p,q)-Bernstein basis functions or (p,q)-Bézier curves, the standard restrictions stated above will be applicable for the value of $p,\ q$ until it is explicitly mentioned otherwise.

Theorem 4.1. ***(Existence and Uniqueness of the post-quantum Blossom).*** *Corresponding to every polynomial S(t) of at most degree m, there exists a unique symmetric multiaffine function $s(u_1,\dots,u_m;p,q)$ that reduces to S(t) along the post-quantum diagonal. That is, there exists a unique post-quantum blossom $s(u_1,\dots,u_m;p,q)$ for every polynomial S(t) provided that $p,\ q$ satisfies the standard restrictions given by 4.4 and 4.5.*

Proof. By Proposition 4.1 and Corollary 4.1 when $p,\ q$ satisfies the constraints given by 4.4 and 4.5, then post-quantum blossom exists for the monomials $t^k, k = 0,1,\dots,m$. Since any polynomial can be written as linear combination of monomials and post-quantum blossom of the sum is actually the sum of the post-quantum blossoms, so for every given polynomial S(t), post-quantum blossom for it always exists while $p,\ q$ satisfies the restrictions given by 4.4 and 4.5. For verifying the uniqueness of the post-quantum blossom, suppose that a polynomial S(t) of degree m has two post-quantum blossoms $s(u_1,\dots,u_m;p,q)$ and $r(u_1,\dots,u_m;p,q)$. Since every symmetric multiaffine polynomial of degree m has a unique representation in terms of the $(m+1)$ symmetric polynomials of degree m, there are constants $a_0,\dots,a_m$ and $b_0,\dots,b_m$ such that

$$s(u_1,\dots,u_m;p,q) = \sum_{k=0}^{m} a_k\ R_k^m(u_1,\dots,u_m;p,q)$$

and

$$r(u_1,\dots,u_m;p,q) = \sum_{k=0}^{m} b_k\ R_k^m(u_1,u_2,\dots,u_m;p,q).$$

Evaluating on the post-quantum diagonal ($u_i = p^{m-i}tq^{i-1},\ i = 1,2,\dots,m$) yields

$$\begin{aligned} S(t) &= \sum_{k=0}^{m} a_k\ t^k \\ &= \sum_{k=0}^{m} b_k\ t^k. \end{aligned}$$

Thus $a_k = b_k,\ k = 0, 1, \ldots, m,$ so $s(u_1, \ldots, u_m; p, q) = r(u_1, \ldots, u_m; p, q)$. Hence the post-quantum blossom of S(t) is unique.

From Proposition 4.1, Lemma 4.1, Theorem 4.1, and the linearity of the post-quantum blossom we deduce the following result.

Corollary 4.2. *The post-quantum blossom of the polynomial* $S(t) = \sum_{k=0}^{m} a_k\, t^k$ *is*

$$s(u_1, \ldots, u_m; p, q) = \sum_{k=0}^{m} a_k \frac{\phi_{m,k}(u_1, u_2, \ldots, u_m)}{(pq)^{\frac{k(k-1)}{2}} \begin{bmatrix} m \\ k \end{bmatrix}_{p,q}}. \tag{4.6}$$

In this section, study has been done on the post-quantum blossom of a polynomial using the monomial representation. In the next section, investigation will be carried out that how post-quantum blossom of polynomial is related to the post-quantum Bernstein representation.

5. Post-quantum blossoming and Post-quantum de Casteljau algorithms

In the diagrams below, we use the multiplicative notation $u_1 \cdots u_m$ to represent the post-quantum blossom value $s(u_1, \ldots, u_m; p, q)$. Though an abuse of notation, this multiplicative notation is highly suggestive. For example, multiplication is commutative and the post-quantum blossom is symmetric

$$u_1 \cdots u_m = u_{\sigma(1)} \cdots u_{\sigma(m)} \longleftrightarrow s(u_1, \ldots, u_m; p, q) = s(u_{\sigma(1)}, \ldots, u_{\sigma(m)}; p, q).$$

Moreover, multiplication distributes through addition and the post-quantum blossom is multiaffine. Thus

$$u = \frac{b-u}{b-a}a + \frac{u-a}{b-a}b$$

implies both

$$u_1 \cdots u_m u = \frac{b-u}{b-a}u_1 \cdots u_m a + \frac{u-a}{b-a}u_1 \cdots u_m b$$

and

$$s(u_1, \ldots, u_m, u; p, q) = \frac{b-u}{b-a}s(u_1, \ldots, u_m, a; p, q) + \frac{u-a}{b-a}s(u_1, \ldots, u_m, b; p, q).$$

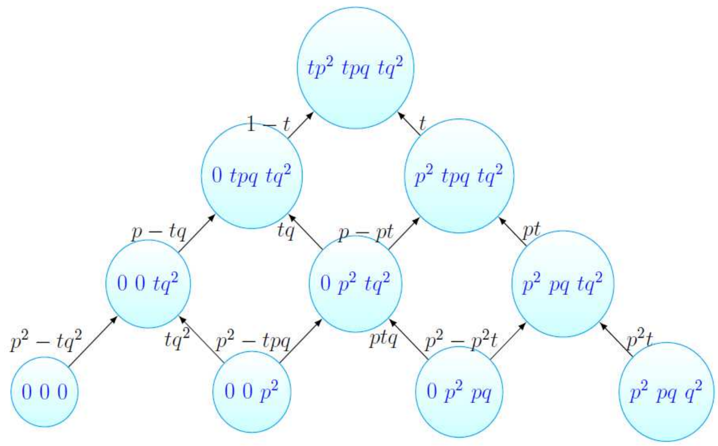

FIGURE 3. 'Computing $s(p^{m-1}t, p^{m-2}tq, \dots, tq^{m-1}; p, q) = S(t)$ recursively from the initial post-quantum blossom values $s(0, \dots, 0, p^{m-1}, p^{m-2}q, \dots, p^{m-k}q^{k-1}; p, q),\ k = 0, 1, \dots, m$.'

The diagram represents the symmetry and multiaffinity character while at same time also make the case for multiplication and post-quantum blossoming both. Therefore this multiplicative representation for the post-quantum blossom seems to be natural. Due to similarity between multiplication and post-quantum blossoming, identities corresponding to multiplication expect to give an analogous identities for the post-quantum blossom.

Using this multiplicative notation, Figure 3 shows (for $m = 3$) how to compute an arbitrary value of $s(p^{m-1}t, p^{m-2}qt, \dots, q^{m-1}t; p, q) = S(t)$ recursively from the initial post-quantum blossom values
$s(0, \dots, 0, p^{m-1}, p^{m-2}q, \dots, p^{m-k}q^{k-1}; p, q)$, with exactly $m-k$ blossom values set to 0 for $k = 0, 1, \dots, m$, by applying the multiaffine and symmetry properties at each node.

Now compare the post-quantum blossoming algorithm in Figure 3 to the de-Casteljau algorithm in Figure 1 for post-quantum Bézier curves. For arbitrary m, Figures 1 and 3 are similar, and Figure 3 is this de Casteljau algorithm for
$s(p^{m-1}t, p^{m-2}qt, \dots, q^{m-1}t; p, q) = S(t)$ with control points
$s(0, \dots, 0, p^{m-1}, p^{m-2}q, \dots, p^{m-k}q^{k-1}; p, q)$, $k = 0, 1, \dots, m$. In next three theorems, certain observations having some important consequences for arbitrary values of the

degree m has been done. standard restrictions given by 4.4 and 4.5 on the value of p, q will be applicable until mentioned otherwise.

Theorem 5.1. *Any polynomial represents post-quantum Bézier Curve. In other words, let S(t) be a polynomial of degree m with post-quantum blossom* $s(u_1, \dots, u_m; p, q)$*. Then using de Casteljau algorithm 3.2, S(t) can be generated by control points* $\mathbf{P_k} = s(0, \dots, 0, p^{m-1}, p^{m-2}q, \dots, p^{m-k}q^{k-1}; p, q),\ k = 0, 1, \dots, m.$

Proof. Let S(t) be a polynomial of degre m and let $s(u_1, \dots, u_m; p, q)$ be the post-quantum blossom of S(t). Set $\mathbf{P_i} = \tilde{\mathbf{P}}_\mathbf{i}^\mathbf{0} := s(0, \dots, 0, p^{m-1}, p^{m-2}q, \dots, p^{m-i}q^{i-1}; p, q)$, $k = 0, 1, \dots, m$, and apply the post-quantum de Casteljau algorithm 3.2. The post-quantum Bézier curve is given by

$$\tilde{\mathbf{P}}_\mathbf{0}^\mathbf{m} = \sum_{i=0}^{m} \mathbf{P}_i\ B_i^m(t; p, q). \tag{5.1}$$

On the other hand applying induction on k and using multiaffine property of the post-quantum blossom, it can be shown that points $\tilde{\mathbf{P}}_\mathbf{i}^\mathbf{k}(t)$ generated by the post-quantum de Casteljau algorithm 3.2 satisfy
$\tilde{\mathbf{P}}_\mathbf{i}^\mathbf{k}(t) = s(0, \dots, 0, p^{m-1}, p^{m-2}q, \dots, p^{m-i}q^{i-1}, tp^{k-1}q^{m-k}, \dots, tq^{m-1}; p, q)$,
$i = 0, 1, \dots, m-k,\ k = 0, 1, \dots, m.$
In particular,

$$\tilde{\mathbf{P}}_\mathbf{0}^\mathbf{m}(t) = s(p^{m-1}t, p^{m-2}qt, \dots, q^{m-1}t; p, q) = S(t). \tag{5.2}$$

The theorem now follows from 5.1 and 5.2.

Corollary 5.1. *On interval* $[0, 1]$*,* m *degree post-quantum Bernstein basis functions form the basis for* m *degree polynomial, except when* $(p = 0)$ *and* $(q = -p$ *for even* $m)$.

Proof. Result can be drawn directly from Theorem 5.1 when p and q satisfies the standard restrictions given by 4.4 and 4.5. Further, when $q = 0 \neq p$ this result can be obtained explicitly by using formula for basis in 2.1, since
$B_i^m(t; p, 0) = t^i - t^{i+1},\ i = 0, 1, \dots, m-1$
$B_m^m(t; p, 0) = t^m.$

Corollary 5.2. *On interval* $[0,1]$*, post-quantum Bézier curve's control points are unique.*

Theorem 5.2. ***(Dual Functional Property of the post-quantum Blossom)****. Let* $S(t)$ *be a post-quantum Bézier curve of degree* m *and let* $s(u_1,\dots,u_m;p,q)$ *be the post-quantum blossom of S(t). Then the post-quantum Bézier control points of S(t) are given by*

$$\mathbf{P_k} = s(0,\dots,0,p^{m-1},p^{m-2}q,\dots,p^{m-k}q^{k-1};p,q),\;\; k=0,1,\dots,m. \tag{5.3}$$

Proof. By Theorem 5.1

$$S(t) = \sum_{k=0}^{m} s(0,\dots,0,p^{m-1},p^{m-2}q,\dots,p^{m-k}q^{k-1};p,q)B_k^m(t;p,q). \tag{5.4}$$

Now 5.3 follows from 5.4 and the uniqueness of the post-quantum Bézier control points.

Figure 4 illustrates a recursive evaluation algorithm for computing an arbitrary post-quantum blossom value $s(u_1,\dots,u_m;p,q)$ from the post-quantum blossom values $s(0,\dots,0,p^{m-1},p^{m-2}q,\dots,p^{m-k}q^{k-1};p,q),\;\; k = 0,1,\dots,m$ by blossoming the de Casteljau evaluation algorithm i.e. by substituting u_k for $tp^{k-1}q^{m-k}$ on the k-th level of the de Casteljau evaluation algorithm in Figure 3.

From Figure 4, it can also be observed that for post-quantum Bézier curves the recursive evaluation algorithm is not unique, as $p^2t,\; ptq,\; tq^2$ can be substituted in any order for the values of parameters $u_1,\; u_2,\; u_3$. These observations are summarized in the next two theorems.

Theorem 5.3. *Let* $S(t) = \sum_{i=0}^{m} \mathbf{P}_i\; B_i^m(t;p,q)$ *be a post-quantum Bézier curve of degree* m *with post-quantum Blossom* $s(u_1,\dots,u_m;p,q)$*. Define recursively a set of multiaffine functions by setting* $Q_i^0 = \mathbf{P_i},\; i=0,...,m$ *and*

$$Q_i^{k+1}(u_1,\dots,u_{k+1}) = (1-u_{k+1}\,p^i\,q^{-i})Q_i^k(u_1,\dots,u_k) + u_{k+1}\,p^i\,q^{-i}Q_{i+1}^k(u_1,\dots,u_k) \tag{5.5}$$

$i=0,1,\dots,m-k-1$ *and* $k=0,1,\dots,m-1$*. Then*

$$Q_i^k(u_1,\dots,u_k) = s(0,\dots,0,p^{m-1},p^{m-2}q,\dots,p^{m-i}q^{i-1},u_1,\dots,u_k;p,q)$$

$i = 0, 1, \ldots, m-k$ and $k = 0, 1, \ldots, m$.

In particular,

$Q_0^m(u_1, \ldots, u_m) = s(u_1, \ldots, u_m; p, q)$

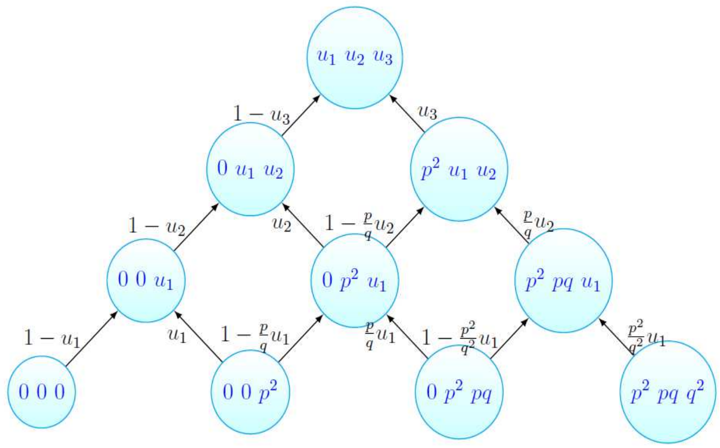

FIGURE 4. 'Recursive evaluation algorithm for the post-quantum blossom of a cubic post-quantum Bézier curve.'

Proof. By the dual functional property,

$$Q_i^0(u_1, \ldots, u_m) = s(0, \ldots, 0, p^{m-1}, p^{m-2}q, \ldots, p^{m-k}q^{k-1}; p, q)\ ,\ \ i = 0, 1, \ldots, m.$$

By applying induction on k, rest of the proof can be easily done. The case $m = 3$ is illustrated by Figure 4.

Theorem 5.4. *Let $S(t) = \sum\limits_{i=0}^{m} \mathbf{P}_i\ B_i^m(t; p, q)$ be a post-quantum Bézier curve of degree m with post-quantum Blossom $s(u_1, \ldots, u_m; p, q)$. There are $m!$ affine invariant, recursive evaluation algorithms for $S(t)$ defined as follows: Let σ be a permutation of $\{1, 2, \ldots, m\}$ and let $\mathbf{P_i^0}(t) = \mathbf{P}_i$, $i = 0, 1, \ldots, m$. Define*

$$\mathbf{P_i^{k+1}}(t) = (p^{\sigma(k+1)-1} - t\ p^i\ q^{\sigma(k+1)-1-i})\mathbf{P_i^k}(t) + p^i\ q^{\sigma(k+1)-1-i}\ t\ \mathbf{P_{i+1}^k}(t) \tag{5.6}$$

$i = 0, 1, \ldots, m-k-1$ and $k = 0, 1, \ldots, m-1$. Then

$$\mathbf{P_i^k}(t) = s(0, \ldots, 0, p^{m-1}, p^{m-2}q, \ldots, p^{m-i}q^{i-1}, tp^{\sigma(1)-1}\ q^{m-\sigma(1)}, \ldots, tp^{\sigma(k)-1}\ q^{m-\sigma(k)}; p, q) \tag{5.7}$$

$i = 0, 1, \dots, m-k$ *and* $k = 0, 1, \dots, m$.

In particular

$$\mathbf{P}_{\mathbf{0}}^{\mathbf{m}}(t) = s(tp^{\sigma(1)-1}\ q^{m-\sigma(1)}, \dots, tp^{\sigma(m)-1}\ q^{m-\sigma(m)}; p, q) = S(t). \tag{5.8}$$

Proof: Theorem 5.4 follows from Theorem 5.3 substituting value $u_k = tp^{\sigma(k)-1}\ q^{m-\sigma(k)}$.

6. Identities for Bernstein basis functions based on post-quantum blossoming

Three identities have been derived for the post-quantum Bernstein basis functions in this section. Each of these identities can be expressed into standard quantum Bernstein basis functions after putting $p = 1$. Standard restrictions on p and q given by 4.4 and 4.5. Starting from new variant of Marsden's identity.

Proposition 6.1. ***(Marsden's Identity)***

$$\prod_{i=1}^{m}(p^{i-1}x - q^{i-1}t) = \sum_{j=0}^{m} \frac{(-1)^j\ p^{\frac{(m-j)(m-j-1)}{2}}\ q^{\frac{j(j-1)}{2}}\ B_{m-j}^{m}(x; \frac{1}{p}, \frac{1}{q})\ B_j^m(t; p, q)}{\begin{bmatrix} m \\ j \end{bmatrix}_{\frac{1}{p}, \frac{1}{q}}}. \tag{6.1}$$

Proof. Let S(t) denote the left hand side of Eq. 6.1. The post-quantum blossom of S(t) is given by

$$\begin{aligned} s(u_1, \dots, u_m; p, q) &= \prod_{n=1}^{m}\left(p^{\frac{-(m-1)(m-2)}{2}}\ x\ -\ p^{\frac{-m(m-1)}{2}}\ u_n\right) \\ &= p^{\frac{-m(m-1)}{2}}\ \prod_{n=1}^{m}\left(p^{m-1}\ x\ -\ u_n\right). \end{aligned}$$

Thus by the dual functional property 5.2,

$$\begin{aligned} \prod_{i=1}^{m}\left(p^{i-1}x - q^{i-1}t\right) &= \sum_{j=0}^{m}\ s(0, \dots, 0, p^{m-1}, p^{m-2}q, \dots, p^{m-j}q^{j-1}; p, q)\ B_j^m(t; p, q) \\ &= p^{\frac{-m(m-1)}{2}} \sum_{j=0}^{m}\ p^{m-1}x\ p^{m-1}x\ \cdots p^{m-1}x\ (p^{m-1}x - p^{m-1}) \\ &\times (p^{m-1}x - p^{m-2}q)\ \cdots\ (p^{m-1}x - p^{m-j}q^{j-1})\ B_j^m(t; p, q) \end{aligned}$$

$$= p^{\frac{m(m-1)}{2}} \sum_{j=0}^{m} x^{m-j} \prod_{n=0}^{j-1} (x - p^{-n} q^{n}) \ B_j^m(t; p, q)$$

$$= p^{\frac{m(m-1)}{2}} \sum_{j=0}^{m} (-1)^j \ q^{\frac{j(j-1)}{2}} \ x^{m-j}$$

$$\times \prod_{n=0}^{j-1} (\ (\frac{1}{p})^n \ - \ (\frac{1}{q})^n \ x \ ) \ B_j^m(t; p, q)$$

which after factoring out powers of p and q gives the right hand side of 6.1.

Monomials can also be expressed in terms of the post-quantum Bernstein basis functions.

Proposition 6.2. ***(Monomial Representation)***

$$t^i = \sum_{k=i}^{m} p^{i(m-k)} \frac{\begin{bmatrix} k \\ i \end{bmatrix}_{p,q}}{\begin{bmatrix} m \\ i \end{bmatrix}_{p,q}} \ B_k^m(t; p, q), \ \ i = 0, 1, \ldots, m. \tag{6.2}$$

Proof. Let the monomial $M_i^m(t) = t^i$ (considered as a polynomial of degree m) Now by using dual functional property 5.2, Eqs. 4.2, 4.3 and the fact that

$$\phi_{m,i}(0, \ldots, 0, p^{m-1}, p^{m-2}q, \ldots, p^{m-k}q^{k-1}) = \phi_{k,i}(p^{m-1}, p^{m-2}q, \ldots, p^{m-k}q^{k-1}) \text{ for } k \geq i.$$

as follows:

$$t^i = M_i^m(t)$$

$$= \sum_{k=0}^{m} R_i^m(0, \ldots, 0, p^{m-1}, p^{m-2}q, \ldots, p^{m-k}q^{k-1}; p, q) \ B_k^m(t; p, q)$$

$$= \sum_{k=0}^{m} \frac{\phi_{m,i}(0, \ldots, 0, p^{m-1}, p^{m-2}q, \ldots, p^{m-k}q^{k-1})}{\phi_{m,i}(p^{m-1}, p^{m-2}q, \ldots, q^{m-1})} \ B_k^m(t; p, q)$$

$$= \sum_{k=i}^{m} \frac{\phi_{k,i}(p^{m-1}, p^{m-2}q, \ldots, p^{m-k}q^{k-1})}{\phi_{m,i}(p^{m-1}, p^{m-2}q, \ldots, q^{m-1})} \ B_k^m(t; p, q)$$

$$= \sum_{k=i}^{m} \frac{(p^{m-k})^i \ \phi_{k,i}(p^{k-1}, p^{k-2}q, \dots, q^{k-1})}{\phi_{m,i}(p^{m-1}, p^{m-2}q, \dots, q^{m-1})} \ B_k^m(t; p, q)$$

$$= \sum_{k=i}^{m} \frac{p^{i(m-k)} \ (pq)^{\frac{i(i-1)}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_{p,q}}{(pq)^{\frac{i(i-1)}{2}} \begin{bmatrix} m \\ i \end{bmatrix}_{p,q}} \ B_k^m(t; p, q)$$

$$= \sum_{k=i}^{m} p^{i(m-k)} \ \frac{\begin{bmatrix} k \\ i \end{bmatrix}_{p,q}}{\begin{bmatrix} m \\ i \end{bmatrix}_{p,q}} \ B_k^m(t; p, q).$$

Reparametrization formula for post-quantum Bernstein basis functions is last identity of this section which has its use in subdivision algorithms for Bézier curves. Before proceeding to proof this change of basis formula, first there is a need to know a lemma.

Lemma 6.1. *Let $b_i^m(u_1, \dots, u_m; p, q)$ denote the (p, q)-blossom of $B_i^m(t; p, q)$, $i = 0, 1, \dots, m$. Then*

$$b_i^m(u_1, \dots, u_{m-1}, 0; p, q) = b_i^{m-1}(u_1, \dots, u_{m-1}; p, q), \ i = 0, 1, \dots, m-1.$$

Proof. First apply the post-quantum blossom algorithm from Theorem 5.3 to the polynomials $B_i^m(t; p, q)$ and $B_i^{m-1}(t; p, q)$. Through dual functional property, the first $m-1$ initial post-quantum blossom values for these two polynomials as defined in Theorem 5.3 are the same: $Q_j^0 = 0, \ j = 0, 1, \dots, m-1, \ j \neq i$ and $Q_i^0 = 1$. Therefore the functions $Q_j^k, \ j = 0, 1, \dots, m-1-k, \ k = 0, 1, \dots, m-1$, generated by Eq. 5.5 of the recursive evaluation algorithms for the post-quantum blossoms of $B_i^m(t; p, q)$ and $B_i^{m-1}(t; p, q)$ coincide. Thus the function Q_0^{m-1} for $B_i^m(t; p, q)$ is the same as the post-quantum blossom of $B_i^{m-1}(t; p, q)$. On the other hand by 5.5, substituting $u_m = 0$ in the function Q_0^m for $B_i^m(t; p, q)$, which is precisely the post-quantum blossom of $B_i^m(t; p, q)$, also gives exactly the function Q_0^{m-1} for $B_i^m(t; p, q)$.

Proposition 6.3. ***(Reparametrization Formula)***

$$B_k^m(rt;p,q)=\sum_{i=k}^{m}B_k^i(r;p,q)B_i^m(t;p,q).$$

Proof. Let F and G be polynomials of degree m with post-quantum blossoms f and g. If $F(t)=G(rt)$, then $f(u_1,\ldots,u_m;p,q)=g(ru_1,\ldots,ru_m;p,q)$. This property holds because the three post-quantum blossoming axioms for $f(u_1,\ldots,u_m;p,q)$ are satisfied by $g(ru_1,\ldots,ru_m;p,q)$. Therefore, $b_k^m(ru_1,\ldots,ru_m;p,q)$ is the post-quantum blossom of $B_k^m(rt;p,q)$. Hence by the dual functional property, Lemma 6.1, and the post-quantum diagonal property

$$\begin{aligned}B_k^m(rt;p,q)&=\sum_{i=0}^{m}b_k^m(0,\ldots,0,p^{m-1}r,p^{m-2}rq,\ldots,rp^{m-i}q^{i-1};p,q)\ B_i^m(t;p,q)\\&=\sum_{i=k}^{m}b_k^i(p^{m-1}r,p^{m-2}rq,\ldots,rp^{m-i}q^{i-1};p,q)B_i^m(t;p,q)\\&=\sum_{i=k}^{m}B_k^i(r;p,q)B_i^m(t;p,q).\end{aligned}$$

References

[1] T. Acar, A. Aral, S. A. Mohiuddine, On Kantorovich modification of (p,q)-Baskakov operators, *J. Inequal. Appl.* 2016, 98 (2016). https://doi.org/10.1186/s13660-016-1045-9

[2] T. Acar, A. Aral, S. A. Mohiuddine, Approximation by bivariate (p,q)-Bernstein-Kantorovich operators, *Iran. J. Sci. Technol. Trans. A Sci.* 42 (2018), 655-662.

[3] F.A.M. Ali, S.A.A. Karim, A. Saaban, M.K. Hasan, A. Ghaffar, K.S. Nisar, and D. Baleanu, Construction of Cubic Timmer Triangular Patches and its Application in Scattered Data Interpolation, *Mathematics,* 8(2),(2020) 159.

[4] A. Aral, V. Gupta, R. P. Agarwal, *Applications of q-Calculus in Operator Theory*, Springer-Verlag New York (2013).

[5] R.A. DeVore, G.G. Lorentz, *Constructive Approximation*, Springer, Berlin, 1993.

[6] G. Farin, *Curves and Surfaces for CAGD-A Practical Guide,* 5ed, *Elsevier,* 2013.

[7] S.A.A. Karim, A. Saaban, V. Skala, A. Ghaffar, K.S. Nisar, and D. Baleanu, Construction of new cubic Bzier-like triangular patches with application in scattered data interpolation, *Advances in Difference Equations* 2020, 1-22. Article number: 151 (2020).

[8] R. T. Farouki, V. T. Rajan, Algorithms for polynomials in Bernstein form, *Computer Aided Geometric Design,* 5(1) (1988), 1-26.

[9] R. Goldman, *Pyramid Algorithms: A Dynamic Programming Approach to Curves and Surfaces for Geometric Modelling,* Elsevier, 2003.

[10] R. Goldman and P. Simeonov, Formulas and algorithms for quantum differentiation of quantum Bernstein bases and quantum Bézier curves based on quantum blossoming,*Graphical Models,* 74(6) (2012), 326-334.

[11] L.W. Hana, Y. Chua, Z. Y. Qiu, Generalized Bézier curves and surfaces based on Lupaş q-analogue of Bernstein operator, *Jour. Comput. Appl. Math.* 261 (2014), 352-363.

[12] Kh. Khan, D.K. Lobiyal, Bézier curves based on Lupaş (p, q)-analogue of Bernstein functions in CAGD, *Jour. Comput. Appl. Math.* 317 (2017), 458-477.

[13] Kh. Khan, D. K. Lobiyal and Adem Kilicman, A de Casteljau Algorithm for Bernstein type Polynomials based on (p, q)-integers, *Appl. Appl. Math.* 13(2) (2018).

[14] A. Lupaş, A q-analogue of the Bernstein operator, *Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca,* 9 (1987), 85-92.

[15] H. N. Mhaskar, Devidas V. Pai, Fundamentals of Approximation Theory, Narosa Publ. House New Delhi, 2000.

[16] S. A. Mohiuddine, T. Acar, and A. Alotaib, Construction of a new family of Bernstein Kantorovich operators, *Mathematical Methods in the Applied Sciences.* 40(18) (2017), 7749-59.

[17] S. A. Mohiuddine, T. Acar. (eds). *Advances in Summability and Approximation Theory.* Singapore: Springer, 2018.

[18] S. A. Mohiuddine, F. Özger, Approximation of functions by Stancu variant of Bernstein-Kantorovich operators based on shape parameter α, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 114(2)(2020)70.

[19] M. Mursaleen, K. J. Ansari, Asif Khan, On (p, q)-analogue of Bernstein Operators, *Appl. Math. Comput.* 266 (2015), 874-882 [Erratum: 278 (2016), 70-71].

[20] M. Mursaleen, F. Khan and Asif Khan, Approximation by (p, q)-Lorentz polynomials on a compact disk, *Complex Anal. Oper. Theory*, DOI: 10.1007/s11785-016-0553-4.

[21] H. Oruç, G. M. Phillips, q-Bernstein polynomials and Bézier curves, *Jour. Comput. Appl. Math.* 151 (2003), 1-12.

[22] F. Özger, H. M. Srivastava, S. A. Mohiuddine, Approximation of functions by a new class of generalized Bernstein-Schurer operators, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 114:173(2020) .

[23] A. Rababah, S. Manna, Iterative process for G2-multi degree reduction of Bézier curves, *Appl. Math. Comput.* 217 (2011), 8126-8133.

[24] N. Rao, A. Wafi, (p, q)-bivariate-Bernstein-Chlodowsky operators, *Filomat,* 32 (2018), 369-378.

[25] P. Simeonova, V. Zafirisa, R. Goldman, q-Blossoming: A new approach to algorithms and identities for q-Bernstein bases and q-Bézier curves, *J. Approx. Theory,* 164(1) (2012), 77-104.

[26] H. M. Srivastava, F. Özger, S. A. Mohiuddine, Construction of Stancu-type Bernstein operators based on Bézier bases with shape parameter λ, Symmetry 11(3) (2019), Article 316.

(1,4) Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
Email address: allaobad4@gmail.com; asifjnu07@gmail.com

(2,3) School of Computer and System Sciences, SC & SS, J.N.U., New Delhi-110067, India
Email address: khalidga1517@gmail.com; dklobiyal@gmail.com