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UNIQUENESS OF ENTIRE FUNCTIONS CONCERNING PRODUCT OF DIFFERENCE POLYNOMIALS

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Abstract. In this paper, using the concept of weakly weighted sharing and relaxed weighted sharing we investigate the uniqueness of product of difference polynomials that share a small function. The results of the paper improve and extend the recent results due to Chao Meng [9].

1. Introduction and Definitions

A meromorphic function f means meromorphic in the complex plane. If no poles occur, then f is called an entire function. The fundamental results and the standard basics of the Nevanlinna value distribution theory of entire functions are used (see [4],[11],[14]). For a meromorphic function f, $S(r,f)$ denotes any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside a possible exceptional set of the finite logarithmic measure.

Let a be a finite complex number, and l be a positive integer. We denote by $N_{l)}(r,\frac{1}{f-a})$ the counting function for the zeros of $f(z)-a$ with multiplicity $\leq l$, and by $\overline{N}_{l)}(r,\frac{1}{f-a})$ the corresponding one for which multiplicity is not counted.

Let $N_{(l}(r,\frac{1}{f-a})$ be the counting function for the zeros of $f(z)-a$ with multiplicity $\geq l$ and $\overline{N}_{(l}(r,\frac{1}{f-a})$ be the corresponding one for which multiplicity is not counted. Moreover, we set $N_l(r,\frac{1}{f-a})=\overline{N}(r,\frac{1}{f-a})+\overline{N}_{(2}(r,\frac{1}{f-a})+...+\overline{N}_{(l}(r,\frac{1}{f-a})$. In the same

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way, we can define $N_l(r, f)$.

Recently, A.Banerjee and S. Mukherjee [1] introduced another sharing notion which is also a scaling between IM and CM but weaker than weakly weighted sharing.

Definition 1.[1] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N_E(r, a; f, g)(\overline{N}_E(r, a; f, g))$ the counting function(reduced counting function) of all common zeros of $f - a$ and $g - a$ with same multiplicities and by $N_0(r, a; f, g)(\overline{N}_0(r, a; f, g))$ the counting function(reduced counting function) of all common zeros of $f - a$ and $g - a$ ignoring multiplicities. If

$$\overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) - 2\overline{N}_E(r, a; f, g) = S(r, f) + S(r, g),$$

then we say that f and g share the value a "CM". If

$$\overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) - 2\overline{N}_0(r, a; f, g) = S(r, f) + S(r, g),$$

then we say that f and g share the value a "IM".

Definition 2.[7] Let f and g share the value a "IM" and k be a positive integer or infinity. Then $\overline{N}^E_{k)}(r, a; f, g)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, and both of their multiplicities are not greater than k. $\overline{N}^0_{(k}(r, a; f, g)$ denotes the reduced counting function of those a-points of f which are a-points of g and both of their multiplicities are not less than k.

Definition 3.[7] For $a \in \mathbb{C} \cup \{\infty\}$, if k is a positive integer or ∞ and

$$\overline{N}_{k)}\left(r, \frac{1}{f-a}\right) - \overline{N}^E_{k)}(r, a; f, g) = S(r, f),$$

$$\overline{N}_{k)}\left(r, \frac{1}{g-a}\right) - \overline{N}^E_{k)}(r, a; f, g) = S(r, g),$$

$$\overline{N}_{(k+1}\left(r, \frac{1}{f-a}\right) - \overline{N}^0_{(k+1}(r, a; f, g) = S(r, f),$$

$$\overline{N}_{(k+1}\left(r, \frac{1}{g-a}\right) - \overline{N}^0_{(k+1}(r, a; f, g) = S(r, g),$$

or if $k = 0$ and

$$\overline{N}\left(r, \frac{1}{f-a}\right) - \overline{N}_0(r, a; f, g) = S(r, f),$$

$$\overline{N}\left(r, \frac{1}{g-a}\right) - \overline{N}_0(r, a; f, g) = S(r, g),$$

then we say f and g weakly share a with weight k. Here we write f, g share "(a, k)" to mean that f, g weakly share a with weight k.

Definition 4.[1] We denote by $\overline{N}(r, a; f \mid= p; g \mid= q)$ the reduced counting function of common a-points of f and g with multiplicities p and q, respectively.

Definition 5.[1] Let f, g share a "IM." Also let k be a positive integer or ∞ and $a \in \mathbb{C} \cup \{\infty\}$. If $\sum_{p,q \leq k} \overline{N}(r, a; f \mid= p; g \mid= q) = S(r)$, then we say f and g share a with weight k in a relaxed manner. Here we write f and g share $(a, k)^*$ to mean that f and g share a with weight k in a relaxed manner.

In 1997, Yang and Hua [12], studied the unicity of differential monomials and obtained the following theorem.

Theorem 1.1.[12] Let $f(z)$ and $g(z)$ be two non-constant entire functions, $n \geq 6$ a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, where c_1, c_2, c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f(z) \equiv t g(z)$ for a constant t such that $t^{n+1} = 1$.

In 2001, Fang and Hong studied the unicity of differential polynomials of the form $f^n(f-1)f'$ and proved the following uniqueness theorem.

Theorem 1.2.[3] Let $f(z)$ and $g(z)$ be two transcendental entire functions, $n \geq 11$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f \equiv g$.

In 2004, Lin and Yi extended the above theorem as to the fixed point. They proved the following result.

Theorem 1.3.[6] Let $f(z)$ and $g(z)$ be two transcendental entire functions, $n \geq 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then $f \equiv g$.

Theorem 1.4.[15] Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small functio n with respect to both $f(z)$ and $g(z)$. Suppose that c is a nonzero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.

In 2014, Chao Meng [9] proved the following results.

Theorem 1.5.[9] Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share "$(\alpha(z), 2)$", then $f(z) \equiv g(z)$.

Theorem 1.6.[9] Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is non-zero complex constant and $n \geq 10$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $(\alpha(z), 2)^*$ then $f(z) \equiv g(z)$.

Theorem 1.7.[9] Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geq 16$ is an integer. If $\overline{E}_{2)}(\alpha(z), f^n(z)(f(z)-1)f(z+c)) = \overline{E}_{2)}(\alpha(z), g^n(z)(g(z)-1)g(z+c))$ then $f(z) \equiv g(z)$.

Question 1. What can be said about the relationship between two entire functions f and g if we consider the difference polynomials of the form $f^n(z)(f(z)-$

$1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j}$ where $n(\geq 1), m(\geq 1)$ and $d \geq 1$ are integers?

In this paper, our main aim is to find the possible answer to above question. We assume, $c_j \in \mathbb{C} \setminus \{0\}(j = 1, 2, ..., d)$ are distinct constants, $n, m, s_j(j = 1, 2, ..., d)$ are positive integers and $\sigma = \sum_{j=1}^{d} s_j = s_1 + s_2 + ... + s_d$.

We prove the following results which improve and extend Theorem 1.5 - 1.7. The following theorems are the main results of the paper.

2. Main Results

Theorem 2.1. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Let $c_j(j = 1, 2, ..., d)$ be complex constants and $s_j(j = 1, 2, ..., d)$ be non-negative integers. Suppose $n(\geq 1)$ and $m(\geq 1)$ are integers satisfying $n \geq \sigma + m + 5$. If $f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j}$ and $g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z+c_j)^{s_j}$ share "$(\alpha(z), 2)$", then $f(z) \equiv g(z)$.

Theorem 2.2. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Let $c_j(j = 1, 2, ..., d)$ be complex constants and $s_j(j = 1, 2, ..., d)$ be non-negative integers. Suppose $n(\geq 1)$ and $m(\geq 1)$ are integers satisfying $n \geq 2\sigma + 2m + 6$. If $f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j}$ and $g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{s_j}$ share $(\alpha(z), 2)^*$, then $f(z) \equiv g(z)$.

Theorem 2.3. Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Let $c_j(j = 1, 2, ..., d)$ be complex constants and $s_j(j = 1, 2, ..., d)$ be non-negative integers. Suppose $n(\geq 1)$ and $m(\geq 1)$ are integers satisfying $n \geq 4\sigma + 4m + 8$. If $\overline{E}_{2)}(\alpha(z), f^n(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j}) = \overline{E}_{2)}(\alpha(z), g^n(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{s_j})$, then $f(z) \equiv g(z)$.

Remark 2.1. Since Theorems 1.5 - 1.7 can be obtained from Theorems 2.1 - 2.3 respectively by putting $m = 1$ and $\sigma = 1$, Theorems 2.1 - 2.3 improve and extend Theorems 1.5 - 1.7 respectively.

3. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$$

Lemma 3.1.[1] Let H be defined as above. If F and G share "$(1,2)$" and $H \not\equiv 0$, then

$$T(r,F) \leq N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)-\sum_{p=3}^{\infty}\overline{N}_{(p}(r,\frac{G}{G'})+S(r,F)+S(r,G),$$

and the same inequality holds for $T(r,G)$.

Lemma 3.2.[1] Let H be defined as above. If F and G share $(1,2)^*$ and $H \not\equiv 0$, then

$$T(r,F) \leq N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) - m(r,\frac{1}{G-1}) + S(r,F) + S(r,G),$$

and the same inequality holds for $T(r,G)$.

Lemma 3.3.[14] Let H be defined as above. If $H \equiv 0$ and

$$\limsup_{r\to\infty} \frac{\overline{N}(r,\frac{1}{F}) + \overline{N}(r,F) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,G)}{T(r)} < 1, r \in I$$

where $T(r) = max\{T(r,F), T(r,G)\}$ and I is a set with infinite linear measure, then $F \equiv G$ or $FG \equiv 1$.

Lemma 3.4.[8] Let F and G be two non-constant entire functions, and $p \geq 2$ an integer. If $\overline{E}_{p)}(1,F) = \overline{E}_{p)}(1,G)$ and $H \not\equiv 0$, then

$$T(r,F) \leq N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + 2\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + S(r,F) + S(r,G).$$

Lemma 3.5.[2] Let $f(z)$ be a meromorphic function in the complex plane of finite order $\rho(f)$, and let η be a fixed non-zero complex number. Then for each $\epsilon > 0$ one has
$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\rho(f)-1+\epsilon}) + O(logr)$.

Lemma 3.6.[10] Let $f(z)$ be a entire function of finite order $\rho(f)$, c a fixed non-zero complex number, and $P(z) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + ... + a_1 f(z) + a_0$ where $a_j (j = 0, 1, ..., n)$ are constants. If $F(z) = P(z) f(z+c)$, then $T(r, F) = (n+1) T(r, f) + O(r^{\rho(f)-1+\epsilon}) + O(logr)$.

Lemma 3.7. Let f be meromorphic function of finite order and c be a non-zero complex constant. Then,

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O\{r^{\rho(f)-1+\varepsilon}\}.$$

Lemma 3.8. Let f be an entire function of order $\rho(f)$ and $F(z) = f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j}$ where $n\,(\geq 1)$ and $m\,(\geq 1)$ are integers. Then,

$$T(r, F) = (n+m+\sigma) T(r, f) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r, f),$$

for all r outside of a set of finite linear measure where $\sigma = s_1 + s_2 + ... + s_d = \sum_{j=1}^{d} s_j$.

Proof. Since f is an entire function of finite order, from Lemma 3.7 and standard Valiron-Mohon'ko theorem we have

$$\begin{aligned}
(n+m+\sigma) T(r, f(z)) &= T(r, f^{n+\sigma}(z)(f(z)-1)^m) + S(r, f) \\
&= m\left(r, f^{n+\sigma}(z)(f(z)-1)^m\right) + S(r, f) \\
&\leq m\left(r, \frac{f^{n+\sigma}(z)(f(z)-1)^m}{F(z)}\right) + m(r, F(z)) + S(r, f) \\
&\leq m\left(r, \frac{f^{\sigma}(z)}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + m(r, F(z)) + S(r, f) \\
&\leq T(r, F(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r, f). \qquad (3.1)
\end{aligned}$$

On the other hand, from Lemma 3.5, we have

$$\begin{aligned}
T(r,F(z)) &\le m(r,f^n(z)) + m(r,(f(z)-1)^m) + m\left(r, f^{\sigma}(z)\cdot\prod_{j=1}^{d}\frac{f(z+c_j)^{s_j}}{f(z)^{s_j}}\right) + S(r,f)\\
&\le (n+m)\, m(r,f(z)) + \sigma m(r,f(z)) + \sum_{j=1}^{d} s_j\cdot m\left(r,\frac{f(z+c_j)}{f(z)}\right) + S(r,f)\\
&\le (n+m+\sigma)\, m(r,f(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r,f)\\
&\le (n+m+\sigma)\, T(r,f(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r,f). \qquad (3.2)
\end{aligned}$$

From 3.1 and 3.2, we can prove this lemma easily.

4. Proof of the Theorems

Proof of Theorem 2.1.
Let $F(z)=\frac{[f(z)^n(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{s_j}]}{\alpha(z)}$, $G(z)=\frac{[g(z)^n(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{s_j}]}{\alpha(z)}$.
Then $F(z)$ and $G(z)$ share "$(1,2)$" except the zeros or poles of $\alpha(z)$. By Lemma 3.6, we have

$$T(r,F(z)) = T(r,f(z)^n(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{s_j}) + S(r,f) \qquad (4.1)$$

$$T(r,G(z)) = T(r,g(z)^n(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{s_j}) + S(r,g) \qquad (4.2)$$

Also, we have

$$\begin{aligned}
N_2(r,\frac{1}{F}) &= N_2(r,\frac{1}{f^n(f-1)^m\prod_{j=1}^{d}f(z+c_j)^{s_j}}) + S(r,f)\\
&= N_2(r,\frac{1}{f^n}) + N(r,\frac{1}{(f-1)^m}) + N(r,\frac{1}{\prod_{j=1}^{d}f(z+c_j)^{s_j}}) + S(r,f) \qquad (4.3)\\
&\le (2+m+\sigma)T(r,f) + S(r,f)
\end{aligned}$$

and

$$N_2(r,\frac{1}{G}) \le (2+m+\sigma)T(r,g) + S(r,g) \qquad (4.4)$$

Suppose $H \not\equiv 0$, then by Lemmas $3.1, 3.5$ and Lemma 3.8, we have

$$\begin{aligned}
T(r,F)+T(r,G) &\leq 2N_2(r,\frac{1}{F})+2N_2(r,\frac{1}{G})+S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,\frac{1}{f})+2N(r,\frac{1}{(f-1)^m})+2N(r,\frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}})\\
&+4\overline{N}(r,\frac{1}{g})+2N(r,\frac{1}{(g-1)^m})+2N(r,\frac{1}{\prod_{j=1}^{d} g(z+c_j)^{s_j}})\\
&+S(r,f)+S(r,g)
\end{aligned}$$

$$\begin{aligned}
(n+m+\sigma)[T(r,f)+T(r,g)] \leq&(4+2m+2\sigma)[T(r,f)+T(r,g)]+O(r^{\rho(f)-1+\epsilon})\\
&+O(r^{\rho(g)-1+\epsilon})+S(r,f)+S(r,g)
\end{aligned}$$

(4.5)

$$(n-\sigma-m-4)[T(r,f)+T(r,g)] \leq O(r^{\rho(f)-1+\epsilon})+O(r^{\rho(g)-1+\epsilon})+S(r,f)+S(r,g)$$

which contradicts with $n \geq \sigma+m+5$. Thus we have $H \equiv 0$. Note that $\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G}) \leq (1+m+\sigma)T(r,f)+(1+m+\sigma)T(r,g)+S(r,f)+S(r,g) \leq T(r)$. where $T(r)=max\{T(r,F),T(r,G)\}$. By Lemma 3.3, we deduce that either $F \equiv G$ or $FG \equiv 1$. Next we will consider the following two cases, respectively.

Let $FG=1$. Then

$$[f^n(z)(f(z)-1)^m\prod_{j=1}^{d} f(z+c_j)^{s_j}][g^n(z)(g(z)-1)^m\prod_{j=1}^{d} g(z+c_j)^{s_j}]=\alpha^2$$

$[f^n(z)(f(z)-1)(f^{m-1}(z)+f^{m-2}(z)+...+1)\prod_{j=1}^{d} f(z+c_j)^{s_j}][g^n(z)(g(z)-1)^m(g^{m-1}(z)+g^{m-2}(z)+...+1)\prod_{j=1}^{d} g(z+c_j)^{s_j}]=\alpha^2$.

It can be easily viewed from above that

$$N(r,\frac{1}{f})=S(r,f) \;\; and \;\; N(r,\frac{1}{f-1})=S(r,f)$$

Thus,

$\delta(0,f)+\delta(1,f)+\delta(\infty,f)=3$, which is not possible. Therefore we must have $F \equiv G$.

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. Let

$$F(z)=\frac{[f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{s_j}]}{\alpha(z)},\quad G(z)=\frac{[g^n(z)(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{s_j}]}{\alpha(z)}.$$

Then $F(z)$ and $G(z)$ share $(1,2)^*$ except the zeros or poles of $\alpha(z)$. Obviously

$$\begin{aligned} T(r,F)+T(r,G) &\leq 2N_2(r,\frac{1}{F})+2N_2(r,\frac{1}{G})+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+S(r,F)+S(r,G)\\ (n+m+\sigma)[T(r,f)+T(r,g)] &\leq (5+3m+3\sigma)\{T(r,f)+T(r,g)\}+O(r^{\rho(f)-1+\epsilon})\\ &\quad +O(r^{\rho(g)-1+\epsilon})+S(r,f)+S(r,g). \end{aligned}$$

(4.6)

$$(n-2m-2\sigma-5)[T(r,f)+T(r,g)]\leq O(r^{\rho(f)-1+\epsilon})+O(r^{\rho(g)-1+\epsilon})+S(r,f)+S(r,g).$$

According to (4.6) and Lemma 3.2, we can prove Theorem 2.2 in a similar way as in proof of Theorem 2.1.

Proof of Theorem 2.3. Let

$$F(z)=\frac{f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{s_j}}{\alpha(z)},\quad G(z)=\frac{g^n(z)(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{s_j}}{\alpha(z)}.$$

Then $\overline{E}_{2)}\left(1,f^n(z)(f-1)^m\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)=\overline{E}_{2)}\left(1,g^n(z)(g-1)^m\prod_{j=1}^{d}g(z+c_j)^{s_j}\right)$ except the zeros or poles of $\alpha(z)$.

Obviously

$$\begin{aligned} T(r,F)+T(r,G) \leq& 2N_2(r,\frac{1}{F})+2N_2(r,\frac{1}{G})+3\overline{N}(r,\frac{1}{F})+3\overline{N}(r,\frac{1}{G})\\ &+S(r,F)+S(r,G)\\ (n+m+\sigma)[T(r,f)+T(r,g)] &\leq (7+5m+5\sigma)[T(r,f)+T(r,g)]+O(r^{\rho(f)-1+\epsilon})\\ &\quad +O(r^{\rho(g)-1+\epsilon})+S(r,f)+S(r,g) \end{aligned}$$

(4.7)

$$(n-4m-4\sigma-7)[T(r,f)+T(r,g)]\leq O(r^{\rho(f)-1+\epsilon})+O(r^{\rho(g)-1+\epsilon})S(r,f)+S(r,g).$$

Using to (4.7) and Lemma 3.4, we can prove Theorem 2.3 in a similar way as in proof of Theorem 2.1.

5. Open Questions

Question 5.1 Can the Theorem 2.1 - 2.3 be extend to meromorphic functions?

Question 5.2 Can the difference polynomials in Theorem 2.1 - 2.3 be replaced by difference polynomials of the form $f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j} \prod_{j=1}^{k} f^{(i)}(z)$?

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