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FRAME SYSTEMS IN NON-LOCALLY CONVEX BANACH SPACES

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Abstract. In this paper, we define atomic decompositions in a non- locally convex Banach space $l^p(0 < p \leq 1)$ and discuss its existence through examples. Also, a sufficient condition for its existence is given and it is observed that if a p-Banach space has an atomic decomposition, then the space is isomorphic to its associated p-Banach sequence space. Further, necessary and sufficient conditions for an atomic decomposition in a p-Banach space is given. Finally, we define shrinking atomic decomposition and gave a necessary and sufficient condition for it.

1. Introduction

Let X be a vector space over a field $\mathbb{F}$. A p-norm $\|.\|_p$ for $0 < p \leq 1$ on X is a mapping from $X \longrightarrow \mathbb{R}$ satisfying the following properties:

(1) $\|x\|_p \geq 0$, for all $x \in X$.
(2) $\|x\|_p = 0 \iff x = 0$.
(3) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$, for all $x \in X$ and $\alpha \in \mathbb{F}$.
(4) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$, for all $x, y \in X$.

The pair $(X, \|.\|_p)$ is called a p-normed linear space.
If $p = 1$, then the p-norm is equal to norm on X.
A p-normed linear space X over a field $\mathbb{F}$ is called a p-Banach space if it is complete.
A linear operator $T : (X, \|.\|_p) \longrightarrow (Y, \|.\|_q)$ is said to be bounded if there exists a

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real number $M > 0$ such that

$$\|T(x)\|_q^{\frac{1}{q}} \leq M \|x\|_p^{\frac{1}{p}}, \text{ for all } x \in X.$$

The collection of all bounded linear operators from the p-Banach space X to the q-Banach space Y is denoted by $B(X,Y)$ which is a Banach space with norm given by

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|T(x)\|_q^{\frac{1}{q}}}{\|x\|_p^{\frac{1}{p}}}.$$

Let X be a p-Banach space. A linear functional $f : X \longrightarrow \mathbb{F}$ is said to be bounded on X if there exists a real number $M > 0$ such that

$$|f(x)| \leq M \|x\|^{\frac{1}{p}}, \text{ for all } x \in X.$$

The collection of all bounded linear functionals on the p-Banach space X is denoted by X^* which is also a Banach space with norm given by $\|f\| = \sup_{\|x\|_p \leq 1} |f(x)|$ and is called the conjugate space of X.

The concept of frame was first defined by Duffin and Schaeffer [7] in 1952. Frames were reintroduced in 1986 by Daubechies, Grossmann and Y. Meyer [6]. Coifman and Weiss [5], introduced the notion of atomic decomposition for certain Function spaces. Later, the notion of atomic decomposition to certain Banach spaces was extended by Feichtinger and Gröchenig [8]. Atomic decompositions were further studied by Kaushik and Sharma [10, 11, 12, 13]

In this paper, we define atomic decompositions in a non- locally convex Banach space $l^p (0 < p \leq 1)$. Also, the existence of atomic decomposition is exhibited through examples. Further, a sufficient condition for its existence is given and it is proved that if a p-Banach space has an atomic decomposition, then the space is isomorphic to its associated p-Banach sequence space. Furthermore, necessary and sufficient conditions for an atomic decomposition in a p-Banach space is given. Finally, shrinking atomic decomposition is defined and a necessary and sufficient condition for it is given.

2. Atomic decompositions in p-Banach spaces

In this section, we define atomic decomposition in a p-Banach space and give examples for its existence. Then, we give a sufficient condition, a necessary condition, and a necessary and sufficient condition for its existence. We begin with the following definition:

Definition 1. *Let X be a p-Banach space. A sequence $\{f_n\} \subset X^*$ is said to be fundamental over X if $\{x \in X : f_n(x) = 0, \text{ for all } n \in \mathbb{N}\} = \{0\}$.*

Example 2.1. *Consider the p-Banach space $X = l_p$, $0 < p < 1$ with p-norm given by*

$$\|x\|_p = \sum_{n=1}^{\infty} |x_n|^p \,, \text{ for all } x \in X.$$

Define $f_n : X \longrightarrow \mathbb{F}$ by

$$f_n(x) = f_n(\{x_i\}) = x_n, \text{ for all } x \in l_p \text{ and } n \in \mathbb{N}.$$

Then, each f_n is linear. Also, f_n is bounded for each n because

$$|f_n(x)| = |x_n| \leq (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} = \|x\|_p^{\frac{1}{p}} \,, \quad \text{for all } x \in X.$$

Also, for $x \in X$, $f_n(x) = 0$, for all $n \in \mathbb{N} \implies x = 0$.
Hence, $\{f_n\}$ is a fundamental sequence over $X = l_p$.

Next, we give a necessary condition for a fundamental sequence in X^*.

Theorem 2.1. *If X is a p-Banach space and $\{f_n\} \subset X^*$ is fundamental over X. Then there exists an associated p-Banach sequence space $X_d = \{\{f_n(x)\} : x \in X\}$ with p-norm $\|\{f_n(x)\}\|_{X_d} = \|x\|_p$, for all $x \in X$.*

Proof. Let $x \in X$. If $x = 0$, then $f_n(0) = 0$, for all $n \in \mathbb{N}$.
Let $x \neq 0$. Then there exist at least one $f_i \in X^*$ such that $f_i(x) \neq 0$.
Define the set of sequences associated with the p-Banach space X by

$$X_d = \{\{f_n(x)\} : x \in X\} .$$

Then, it is easy to verify that X_d is a p-normed linear space with the p-norm defined by

$$\|\{f_n(x)\}\|_{X_d} = \|x\|_p \,, \quad \text{for all } x \in X.$$

Now, we prove that X_d is a p-Banach space.

Let $\{\{f_n(x_i)\}_n\}_i$ be a p-Cauchy sequence in X_d, which using the definition of p-Cauchy sequence and p-norm on X_d implies that $\{x_i\}$ is a p-Cauchy in X which is p-Banach space and so it is p-convergent in X.

One can easily verify that $\{\{f_n(x_i)\}_n\}_i$ will converge to $\{f_n(x)\}$ in $(X_d, \|.\|_{X_d})$ and hence $(X_d, \|.\|_{X_d})$ is a p-Banach space. □

Corollary 2.1. *If X is a p-Banach space and $\{f_n\} \subset X^*$ is fundamental over X. Then, X is isomorphic to the p-Banach sequence space $X_d = \{\{f_n(x)\} : x \in X\}$ with p-norm $\|\{f_n(x)\}\|_{X_d} = \|x\|_p$, for all $x \in X$.*

Proof. Existence of the associated p-Banach sequence space X_d is proved in Theorem 2.1.

Now, define $T : X \longrightarrow X_d$ by

$$T(x) = \{f_n(x)\}, \text{ for all } x \in X.$$

Clearly, T is linear, bijective, and an isometry. □

Next, we define the notion of atomic decomposition for p-Banach spaces. We give the following definition.

Definition 2. *Let $(X, \|.\|_p)$ be a p-Banach space and let X_d be a p-Banach sequence space associated with X. Let $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$. Then the pair $(\{x_n\}, \{f_n\})$ is an atomic decomposition of X with respect to X_d if*

(i) $\{f_n(x)\}_n \in X_d$, *for all* $x \in X$.

(ii) There exists constants A,B > 0 such that

$$A\|x\|_p \leq \|\{f_n(x)\}_n\|_{X_d} \leq B\|x\|_p, \text{ for all } x \in X.$$

(iii) $x = \sum_{n=1}^{\infty} f_n(x)x_n$, *for all* $x \in X$.

In the following example, we discuss the existence of an atomic decomposition in a p-Banach space.

Example 2.2. *Consider the p-Banach space $(l_p, \|.\|_p)$. Let $\{e_n\}$ be the sequence of unit vectors. Then $\{e_n\}$ is a Schauder basis of l_p.*

Hence for every $x \in X$, there exists a sequence $\{x_n\}$ of scalars such that $x = \sum_{n=1}^{\infty} x_n e_n$.
Let $\{f_n\} \subset X^*$ be a sequence of bounded linear functionals such that

$$f_n(e_m) = \delta_{nm}, \text{ for all } m, n \in \mathbb{N}.$$

Then, for each $n \in \mathbb{N}$, we have

$$f_n(x) = \sum_{i=1}^{\infty} x_i f_n(e_i) = x_n, \quad x \in X$$

Thus $x = \sum_{n=1}^{\infty} f_n(x) e_n$, where $\{f_n(x)\} = \{x_n\} \in l_p = X_d$. Hence property (ii) holds trivially in view of the definition of p-norm of x.

Remark 1. *Let X be a p-Banach space, X_d be a p-Banach sequence space such that $(\{x_n\}, \{f_n\})$ is an atomic decomposition for X with respect to X_d. Then X is isomorphic to a subspace of X_d.*
Indeed, if $(\{x_n\}, \{f_n\})$ is an atomic decomposition for X with respect to X_d, then there exists an isomorphism $T : X \longrightarrow T(X) \subset X_d$ defined by

$$T(x) = \{f_n(x)\}, \text{ for all } x \in X.$$

Definition 3. *Let $(X, \|.\|_p)$ be a p-Banach space and let X_d be a p-Banach sequence space associated with X. Then $\{f_n\} \subset X^*$ is said to be an X_d- frame of X with respect to X_d if*

(i) $\{f_n(x)\}_n \in X_d$, for all $x \in X$.
(ii) There exists constants A,$B > 0$ such that

$$A \|x\|_p \leq \|\{f_n(x)\}_n\|_{X_d} \leq B \|x\|_p, \text{ for all } x \in X.$$

In the following result, we give a sufficient condition for the existence of an X_d-frame for X.

Theorem 2.2. *Let X be a p-Banach space, X_d be a p-Banach sequence space such that X is isomorphic to a subspace of X_d, then there exists a sequence $\{f_n\} \subset X^*$ such that $\{f_n\}$ is an X_d-frame for X.*

Proof. Let $X \cong Z_d$ under the isomorphism T, where Z_d is a subspace of X_d. Then for every $\{a_n\} \in Z_d$ there exists $x \in X$ such that if f_n is the coordinate functionals, then

(i) $\{f_n(x)\}_n \in X_d$, for all $x \in X$.

(ii) There exists constants A,$B > 0$ such that

$$A \|x\|_p \leq \|\{f_n(x)\}_n\|_{X_d} \leq B \|x\|_p \,, \text{ for all}\, x \in X.$$

□

Next, we give some characterizations of atomic decompositions of X with respect to X_d.

Theorem 2.3. *Let X be a p-Banach space, $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$. If $X = \bar{A}$ where $A = \{\sum_{i=1}^{n} f_i(x)x_i\}$. Then the following conditions are equivalent:*

(i) $(\{x_n\}, \{f_n\})$ is an atomic decomposition for X with respect to the associated p-Banach sequence space $X_d = \{\{a_i\} : \sum_{i=1}^{\infty} a_i x_i < \infty\}$ with p-norm

$$\|\{a_i\}\|_{X_d} = \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} a_i x_i \right\|_p .$$

(ii) The sequence $\{P_n\}$ of projections defined by

$$P_n(x) = \sum_{i=1}^{n} f_i(x)x_i, \textit{ for all}\, x \in X$$

is uniformly bounded.

(iii) $\lim_{n\to\infty} P_n(x) = x$, for all $x \in X$.

In this case $\sup \|P_n\|^p$ *is called the norm of the* $\{x_n\}$.

Proof. **(i) $\Longrightarrow$ (iii)** Suppose that $(\{x_n\}, \{f_n\})$ is an atomic decomposition for X. Then every $x \in X$ can be written as $x = \sum_{n=1}^{\infty} f_n(x)x_n$. Hence $x = \lim_{n\to\infty} P_n(x)$.

(iii) $\Longrightarrow$ (ii) Suppose that $\lim_{n\to\infty} P_n(x) = x$, for all $x \in X$.

Then the sequence $\left\{\|P_n x\|_p\right\}_n$ of real numbers is convergent and hence bounded for each $x \in X$. Therefore, using Banach-Steinhaus Theorem we conclude that $\{P_n\}$ is uniformly bounded on X.

(ii) $\Longrightarrow$ (i) Suppose that there exists a $K > 0$ such that

$$\|P_n\| \leq K, \text{ for all } n \in \mathbb{N}.$$

If $x \in A$, then $P_m(x) = x$, for all $m \geq n$. Therefore

$$\lim_{n\to\infty} P_n(x) = x, \text{ for all } x \in A.$$

If $x \in X$, then for every $\epsilon > 0$ there exists $y \in A$ such that

$$\|x - y\|_p < \frac{\epsilon}{3}$$

Since $y \in A$, $P_n(y) \to y$ and so for $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that

$$\|P_n y - y\|_p < \frac{\epsilon}{3}, \text{ for all } n \geq M \tag{2.1}$$

This yields $\lim_{n\to\infty} P_n(x) = x$, for all $x \in X$ and so $x = \sum_{i=1}^{\infty} f_i(x)x_i$. Then $\{f_i(x)\} \in X_d$.

Also for every $x \in X$, we compute

$$\begin{aligned} \|x\|_p &= \left\| \lim_{n\to\infty} P_n x \right\|_p \\ &\leq \sup \left\| \sum_{i=1}^{n} f_i(x)x_i \right\|_p \\ &\leq K \|x\|_p . \end{aligned}$$

Hence (i) holds. □

Lemma 2.1. *Let $(X, \|.\|_p)$ be a p-Banach space, $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$ such that $(\{x_n\}, \{f_n\})$ is an atomic decomposition for X with respect to X_d. Then $\|x_n\|_p \|f_n\|^p \leq 2K$, where K is the norm of $\{x_n\}_n$.*

Proof. Since $\|f_n(x)x_n\|_p = \|P_n(x) - P_{n-1}(x)\|_p \leq 2K \|x\|_p$, for all $x \in X$. This implies $|f_n(x)|^p \|x_n\|_p \leq 2K \|x\|_p$ and so we obtain

$$\|x_n\|_p \|f_n\|^p \leq 2K.$$

□

In the following result, we discuss construction of an associated p-Banach space and its basis.

Lemma 2.2. *Let $(X, \|.\|_p)$ be a p-Banach space and $\{x_n\} \subset X\setminus\{0\}$. Then the sequence space $X_d = \{\{a_n\} : \sum_{n=1}^{\infty} a_n x_n$ converges in $X\}$ is a p-Banach space with the p-norm*

$$\|\{a_i\}\|_{X_d} = \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} a_i x_i \right\|_p$$

for which the canonical unit vectors form a basis.

Proof. It can be easily proved that X_d is a p-normed linear space using the fact that $\|.\|_p$ is a p-norm on X.

For $(X_d, \|.\|_{X_d})$ to be a p-Banach space let $\{c^n\}$be a p-Cauchy sequence in X_d which implies for every $\epsilon > 0$ there exist $K(\epsilon) \in \mathbb{N}$ such that for all $n, m \geq K$, we have

$$\|c^n - c^m\|_{X_d} = \|\{c_i^n - c_i^m\}\|_{X_d} = \sup_{k\in\mathbb{N}} \left\| \sum_{i=1}^{k} (c_i^n - c_i^m) x_i \right\|_p < \frac{\epsilon}{2},$$

which implies for all $n, m \geq K$ and $k \in \mathbb{N}$ that

$$\left\| \sum_{i=1}^{k} (c_i^n - c_i^m) x_i \right\|_p < \frac{\epsilon}{2}.$$

Hence for each $k \in \mathbb{N}$ and $n, m \geq K$, we obtain

$$|c_k^n - c_k^m|^p \, \|x_k\|_p = \|(c_k^n - c_k^m) x_k\|_p = \left\| \sum_{i=1}^{k} (c_i^n - c_i^m) x_i - \sum_{i=1}^{k-1} (c_i^n - c_i^m) x_i \right\|_p < \epsilon.$$

This implies that $\{c_i^n\}$ is a Cauchy sequence of real numbers and hence convergent to say $\{c_i\}$ for each i. It follows easily now that $\{c^n\}$ converges to c in X_d. Now for $\{e_i\}$ to form a basis for X_d, it is enough to prove that $\{e_i\}$ is complete and there exists a constant $C \geq 1$ such that for every $m \geq n$ and every sequence $a_1, a_2, \cdots, a_m$ of scalars,

$$\left\| \sum_{i=1}^{n} a_i e_i \right\|_{X_d} \leq C \left\| \sum_{i=1}^{m} a_i e_i \right\|_{X_d}.$$

For every sequence $a_1, a_2, \cdots, a_m$ of scalars and $m \geq n$, we have

$$\left\| \sum_{i=1}^{n} a_i e_i \right\|_{X_d} = \sup_{\mathbb{N}\leq n} \left\| \sum_{i=1}^{N} a_i x_i \right\|_p \leq \sup_{\mathbb{N}\leq m} \left\| \sum_{i=1}^{N} a_i x_i \right\|_p = \left\| \sum_{i=1}^{m} a_i e_i \right\|_{X_d}.$$

If $\{a_i\}$ is an arbitrary sequence in X_d, then for every $\epsilon > 0$ there exists M such that for all $m > n > M$, we have

$$\left\| \sum_{i=n+1}^{m} a_i x_i \right\|_p < \frac{\epsilon}{2}.$$

This gives

$$\sup_{\mathbb{N}>n} \left\| \sum_{i=n+1}^{N} a_i x_i \right\|_p \leq \frac{\epsilon}{2} < \epsilon, \quad \text{for all } n > M.$$

Hence for all $n > M$, we obtain

$$\left\| \{a_i\} - \sum_{i=1}^{n} a_i e_i \right\|_{X_d} = \sup_{\mathbb{N}>n} \left\| \sum_{i=n+1}^{N} a_i x_i \right\|_p < \epsilon.$$

□

We conclude this section with the following result concerning atomic decompositions for p-Banach spaces.

Lemma 2.3. *Let $(X, \|.\|_p)$ be a p-Banach space and $\{f_n\} \subset X^*$. Then the following conditions are equivalent:*

(i) There exists a sequence $\{x_n\} \subset X$ such that $x = \sum_{n=1}^{\infty} f_n(x) x_n$ for all $x \in X$.

(ii) There is a p-Banach sequence space X_d with the canonical unit vectors $\{e_n\}_n$ as a basis such that $(\{x_n\}, \{f_n\})$ is an atomic decomposition for X with respect to X_d and a bounded linear operator $S : X_d \longrightarrow X$ such that $S(e_n) = x_n$.

Proof. It follows by using the techniques given in [3]. □

3. Shrinking Atomic Decomposition in p-Banach space

Shrinking Schauder frames were introduced and studied by Liu [15] while shrinking atomic decompositions in locally convex Banach spaces were studied by Carando and Lassalle [2]. In this section, we define shrinking atomic decompositions in p- Banach spaces. We begin with the following definition:

Definition 4. *Let $(X, \|.\|_p)$ be a p-Banach space and let X_d be a p-Banach sequence space associated with X. Let $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$. Then the atomic decomposition $(\{x_n\}, \{f_n\})$ is said to be shrinking if*

$$\lim_{n\to\infty} \|f \circ T_N\| = 0, \ \textit{for all} \ f \in X^*,$$

where $T_N : X \longrightarrow X$ *is defined as*

$$T_N(x) = \sum_{n=N}^{\infty} f_n(x)x_n, \text{ for all } x \in X.$$

Remark 2. *Since* $T_N = I - P_{N-1}$, *it is uniformly bounded on* X.

Theorem 3.1. *Let* $(\{x_n\}, \{f_n\})$ *be an atomic decomposition for the p-Bancah space* X *with respect to the p-Banach sequence space* X_d *and let* $\pi : X \longrightarrow X^{**}$ *be the natural embedding. Then the following statements are equivalent:*

(i) $(\{x_n\}, \{f_n\})$ *is shrinking.*

(ii) $(\{f_n\}, \{\pi_{x_n}\})$ *is an atomic decomposition for* X^* *with respect to the p-Banach sequence space* Z_d *which has canonical unit vectors as basis.*

Proof. **(i)** $\Longrightarrow$ **(ii)** Since $(\{x_n\}, \{f_n\})$ is an atomic decomposition for X with respect to X_d, we have

(i) $(\{f_n(x)\}) \in X_d$, for all $x \in X$.

(ii) There exist constants A,$B > 0$ such that

$$A\|x\|_p \leq \|\{f_n(x)\}_n\|_{X_d} \leq B\|x\|_p, \text{ for all } x \in X.$$

(iii) $x = \sum_{n=1}^{\infty} f_n(x)x_n$, for all $x \in X$.

We claim that $(\{f_n\}, \{\pi(x_n)\})$ is an atomic decomposition for X^* with respect to the p-Banach sequence space Z_d which has the canonical unit vectors as basis.
From condition (iii) it is clear that

$$f(x) = \sum_{n=1}^{\infty} \pi_{x_n}(f)f_n(x) = \sum_{n=1}^{\infty} f(x_n)f_n(x), \text{ for all } f \in X^*.$$

Hence it is enough to show that series on the right is convergent in X^*.
For $M > N$,

$$\left\|\sum_{n=N}^{M-1} f(x_n)f_n\right\| = \sup_{\|x\|_p \leq 1} |(f \circ (T_N - T_M))(x)| \leq \|f \circ T_N\| + \|f \circ T_M\|$$

which vanishes as $M, N \to \infty$ as the atomic decomposition $(\{x_n\}, \{f_n\})$ is shrinking. Therefore there exists a p-Banach sequence space Z_d with the canonical unit vectors $\{e_n\}$ as a basis such that $(\{f_n\}, \{\pi_{x_n}\})$ satisfy conditions (i) and (ii) of the atomic decomposition for X^* with respect to Z_d.

(ii) $\Longrightarrow$ **(i)** Clearly for each $f \in X^*$, we have

$$\begin{aligned}\|f \circ T_N\| &= \sup_{\|x\|_p \le 1} |(f \circ T_N)(x)| \\ &= \sup_{\|x\|_p \le 1} \left| \sum_{n=N}^{\infty} f_n(x) f(x_n) \right| \\ &= \sup_{\|x\|_p \le 1} |f(x) - S_N(x)| \\ &= \|f - S_N\|,\end{aligned}$$

where $S_N(x) = \sum_{n=1}^{N-1} f_n(x) f(x_n)$.

Hence using the fact that $(\{f_n\}, \{\pi_{x_n}\})$ is an atomic decomposition for X^*, we conclude that $\|f \circ T_N\| \to 0$. $\square$

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