Jordan Journal of Mathematics and Statistics (JJMS), 15(2), 2022, pp 243 - 254
DOI: https://doi.org/10.47013/15.2.6

ON STAR COLORING OF DEGREE SPLITTING OF CARTESIAN PRODUCT GRAPHS

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Abstract. A star coloring of a graph G is a proper vertex coloring with the condition that no path on four vertices in G can be labelled by two colors. The star chromatic number $\chi_s(G)$ of G is the least number of colors that is required to star color G. In this paper, we determine the star chromatic number of the degree splitting graph of the Cartesian product of any two simple graphs G and H denoted by $G \square H$ and also we portray the star chromatic number for the degree splitting graph of the Cartesian product of prism graphs, toroidal graphs and grid graphs.

1. Introduction

Throughout this paper, the graphs considered are to be finite, simple, connected and undirected, [2, 3, 7]. The concept of star chromatic number was introduced by Branko Grünbaum in 1973. A star coloring [1, 5, 6] is a proper vertex coloring so that no path on four vertices of a graph G can be colored by only two colors. The star chromatic number of G denoted by $\chi_s(G)$ is the least number of colors needed to star color G. In a star coloring, the vertices having any two colors form an induced subgraph which has connected components in the form of star graphs. For a detailed study on star chromatic number and its complexity, see [1, 4, 5].

We consider graphs $G = (V, E)$ with $V(G) = S_1 \cup S_2 \cup S_3 \cup \dots S_t \cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T = V(G) - \bigcup_{i=1}^{t} S_i$. Thus to construct the degree splitting graph of G, [8, 9], add new

2010 *Mathematics Subject Classification.* 05C15, 05C75.

Key words and phrases. Star coloring, Cartesian product, degree splitting.

Received: Oct. 19, 2020 Accepted: Feb. 7, 2021 .

vertices $w_1, w_2, \dots w_t$ and join w_i to each vertex of S_i for $1 \leq i \leq t$. The degree splitting graph of G is denoted by $DS(G)$.

Given two graphs G and H, the Cartesian product $G \Box H$ is the simple graph with vertex set $V_1 \times V_2$ so that two vertices (u, v) and (u', v') are adjacent in $G \Box H$ if and only if either $u = u'$ and v, v' are adjacent in H or u, u' are adjacent in G and $v = v'$, [10].

2. Observations

The following are easy results and their proofs are omitted:

(1) If $G = C_n$, then $DS(G) = W_{n+1}$.
(2) For any graph G, $\chi_s(G) \leq \chi_s(DS(G))$.

3. Main Results

We now obtain the star chromatic number of degree splitting of Cartesian product of two graphs G and H, where G and H are either path or cyclic graphs giving the grid graph $G_{m,n}$, toroidal graph $TG_{m,n}$ or prism graph $Y_{m,n}$.

3.1. Star coloring of the degree splitting of the grid graph.

Theorem 3.1. *Let $m \geq 3$ and $n \geq 4$. Let P_m and P_n be path graphs. Then the star chromatic number of degree splitting of the Cartesian product of two path graphs is* 6.

Proof. Let P_m and P_n be two path graphs of order m and n, respectively. Let $P_m \Box P_n$ be the Cartesian product of these graphs which is also called as grid graph and denoted by $G_{m,n}$. Let the vertex set of P_m be $\{u_1, u_2, \dots, u_m\}$ and the vertex set of P_n be $\{v_1, v_2, \dots, v_n\}$. Then the graph $G_{m,n}$ is of order

$$|V(G_{m,n})| = mn$$

and size

$$|E(G_{m,n})| = 2mn - (m + n).$$

We have

$$V(G_{m,n}) = \left\{ \begin{matrix} (u_1, v_1) & (u_1, v_2) & \dots & (u_1, v_n) \\ (u_2, v_1) & (u_2, v_2) & \dots & (u_2, v_n) \\ \vdots & \vdots & & \vdots \\ (u_m, v_1) & (u_m, v_2) & \dots & (u_m, v_n) \end{matrix} \right\} = S_1 \cup S_2 \cup S_3$$

where

$$S_1 = \{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\},$$

$$S_2 = \{(u_i, v_j) : \ i = 1 \ or \ m; \ 2 \le j \le n-1\} \cup \{(u_i, v_j) : \ 2 \le i \le m-1; j = 1 \ or \ n\}$$

and

$$S_3 = \{(u_i, v_j) : \ 2 \le i \le m-1; \ 2 \le j \le n-1\}.$$

To construct $DS(P_m \Box P_n)$ from $P_m \Box P_n$, we additionally include three vertices w_1, w_2 and w_3 corresponding to the sets S_1, S_2 and S_3, respectively. Hence $V(DS(G_{m,n})) = V(P_m \Box P_n) \cup \{w_1, w_2, w_3\}$. The proof of this theorem is demonstrated in the following cases:

For $i \equiv 1 \pmod 4$,

$$c(u_i, v_j) = \begin{cases} 1 \ if & j \equiv 1 \pmod 3 \\ 2 \ if & j \equiv 2 \pmod 3 \\ 3 \ if & j \equiv 0 \pmod 3 \end{cases}.$$

For $i \equiv 2 \pmod 4$,

$$c(u_i, v_j) = \begin{cases} 2 \ if & j \equiv 1 \pmod 3 \\ 4 \ if & j \equiv 2 \pmod 3 \\ 5 \ if & j \equiv 0 \pmod 3 \end{cases}.$$

For $i \equiv 3 \pmod 4$,

$$c(u_i, v_j) = \begin{cases} 3 \ if & j \equiv 1 \pmod 3 \\ 1 \ if & j \equiv 2 \pmod 3 \\ 2 \ if & j \equiv 0 \pmod 3 \end{cases}.$$

For $i \equiv 0 \pmod 4$,

$$c(u_i, v_j) = \begin{cases} 5\ if & j = 1 \pmod 3 \\ 3\ if & j = 2 \pmod 3 \\ 4\ if & j = 0 \pmod 3 \end{cases}$$

and also assign $c(w_1) = c(w_2) = c(w_3) = 6$. Therefore, the star chromatic number of degree splitting graph of $G_{m,n}$ is 6. So we conclude that the star chromatic number of the degree splitting of $P_m \Box P_n$ is 6, when $m \geq 3$ and $n \geq 4$. This completes the proof. □

As a special case, for $n = 2$, the graph $P_m \Box P_2$ is known as the ladder graph L_m. We have the following immediate result:

Corollary 3.1. *We have*

$$\chi_s(DS(L_m)) = \begin{cases} 4, & when\ m = 2 \\ 5, & otherwise. \end{cases}$$

Proof. **Case (i)** Let $m = n = 2$. The graph is the cycle C_4. By observation (1), we get $DS(C_4) = 4$. Thus the star chromatic number of $DS(L_2) = 4$.

Case (ii) Let $n = 2$ and $m \geq 3$ be arbitrary. Since the star coloring of $G_{3,2}$ requires at least 4 colors and also $G_{3,2}$ is a subgraph of $G_{m,2}$ for any $m \geq 3$, we have $\chi_s(G_{m,2}) \geq 4$ for any $m \geq 3$. Clearly $\chi_s(DS(G_{m,2}))$ is 5. Hence the star chromatic number of the degree splitting of the ladder graph is 5 when $n = 2$ and for any m. □

3.2. Star coloring of the degree splitting of the toroidal graphs.

Theorem 3.2. *For $m \geq 3$ and $n \geq 3$, let C_m and C_n be the cycle graphs. Let the Cartesian product $C_m \Box C_n$ be named as toroidal (or torus grid) graph denoted by*

$TG_{m,n}$. *Then*

$$\chi_s(DS(TG_{m,n})) = \begin{cases} 7, & \text{if} \quad m \equiv 0\ (mod\ 3)\ \&\ n \equiv 0\ (mod\ 3) \\ & \quad\quad m \equiv 1\ (mod\ 3)\ \&\ n \equiv 0\ (mod\ 3) \\ & \quad\quad m \equiv 1\ (mod\ 3)\ \&\ n \equiv 1\ (mod\ 3) \\ & \quad\quad m \equiv 2\ (mod\ 3)\ \&\ n \equiv 0\ (mod\ 3) \\ 8, & if \quad m \equiv 2\ (mod\ 3)\ \&\ n \equiv 2\ (mod\ 3) \\ & \quad\quad m \equiv 2\ (mod\ 3)\ \&\ n \equiv 1\ (mod\ 3) \end{cases}$$

Proof. Let $TG_{m,n} = C_m \square C_n$ be the torus grid graph with m and n vertices, respectively. Let the vertices of $TG_{m,n}$ be $V(T_{m,n}) = \{(u_i, v_j) :\ 1 \leq i \leq m;\ 1 \leq j \leq n\}$. The graph $TG_{m,n}$ is of order $|V(TG_{m,n})| = mn$ and of size $|E(TG_{m,n})| = 2mn$. Hence we have

$$V(TG_{m,n}) = \left\{ \begin{matrix} (u_1, v_1) & (u_1, v_2) & \dots & (u_1, v_n) \\ (u_2, v_1) & (u_2, v_2) & \dots & (u_2, v_n) \\ \vdots & \vdots & & \vdots \\ (u_m, v_1) & (u_m, v_2) & \dots & (u_m, v_n) \end{matrix} \right\} = S_1$$

where $S_1 = \{(u_i, v_j) :\ 1 \leq i \leq m; 1 \leq j \leq n\}$.

To construct $V(DS(TG_{m,n}))$ from $V(TG_{m,n})$, include one additional vertex w_1 corresponding to S_1. Therefore $V(DS(TG_{m,n})) = V(TG_{m,n}) \cup \{w_1\}$. The proof of this theorem is demonstrated in the following cases:

Case (i) Let $m = 3k$, $n = 3k$ for $k \geq 1$. For $1 \leq i \leq m$, if $i \equiv 1$ (mod 3), then

$$c(u_i v_j) = \begin{cases} 1, & if\ j \equiv 1 \text{ (mod 3)} \\ 2, & if\ j \equiv 2 \text{ (mod 3)} \\ 3, & if\ j \equiv 0 \text{ (mod 3)} \end{cases}.$$

If $i \equiv 2$ (mod 3), then

$$c(u_i v_j) = \begin{cases} 2, & if\ j \equiv 1 \text{ (mod 3)} \\ 4, & if\ j \equiv 2 \text{ (mod 3)} \\ 5, & if\ j \equiv 0 \text{ (mod 3)} \end{cases}.$$

If $i \equiv 0 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 3, \ if \ j \equiv 1 \pmod 3 \\ 5, \ if \ j \equiv 2 \pmod 3 \\ 6, \ if \ j \equiv 0 \pmod 3 \end{cases}$$

and therefore $c(w_1) = 7$.

Case (ii) Let $m = 3k + 1$, $n = 3k$ for $k \geq 1$. For $1 \leq i \leq m - 1$, if $i \equiv 1 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 1, \ if \ j \equiv 1 \pmod 3 \\ 2, \ if \ j \equiv 2 \pmod 3 \\ 3, \ if \ j \equiv 0 \pmod 3). \end{cases}$$

If $i \equiv 2 \pmod 3$,

$$c(u_iv_j) = \begin{cases} 2, \ if \ j \equiv 1 \pmod 3 \\ 4, \ if \ j \equiv 2 \pmod 3 \\ 5, \ if \ j \equiv 0 \pmod 3). \end{cases}$$

If $i \equiv 0 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 5, \ if \ j \equiv 1 \pmod 3 \\ 3, \ if \ j \equiv 2 \pmod 3 \\ 6, \ if \ j \equiv 0 \pmod 3). \end{cases}$$

Similarly

$$c(u_nv_j) = \begin{cases} 4, \ if \ j \equiv 1 \pmod 3 \\ 6, \ if \ j \equiv 2 \pmod 3 \\ 1, \ if \ j \equiv 0 \pmod 3 \end{cases}$$

and hence $c(w_1) = 7$.

Case (iii) Let $m = 3k + 2$, $n = 3k$ for $k \geq 1$. Let $1 \leq i \leq m - 1$. If $i \equiv 1 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 1, \ if \ j \equiv 1 \pmod 3 \\ 2, \ if \ j \equiv 2 \pmod 3 \\ 3, \ if \ j \equiv 0 \pmod 3). \end{cases}$$

If $i \equiv 2 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 2, \ if \ j \equiv 1 \pmod 3 \\ 4, \ if \ j \equiv 2 \pmod 3 \\ 5, \ if \ j \equiv 0 \pmod 3). \end{cases}$$

If $i \equiv 0 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 5, \ if \ j \equiv 1 \pmod 3 \\ 3, \ if \ j \equiv 2 \pmod 3 \\ 6, \ if \ j \equiv 0 \pmod 3). \end{cases}$$

Also

$$c(u_nv_j) = \begin{cases} 6, \ if \ j \equiv 1 \pmod 3 \\ 5, \ if \ j \equiv 2 \pmod 3 \\ 4, \ if \ j \equiv 0 \pmod 3 \end{cases}$$

and therefore $c(w_1) = 7$.

Case (iv) Let $m = 3k+1$, $n = 3k+1$ for $k \geq 1$. Let $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$.

If $i \equiv 1 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 1, \ if \ j \equiv 1 \pmod 3 \\ 2, \ if \ j \equiv 2 \pmod 3 \\ 3, \ if \ j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_iv_n) = 4.$$

If $i \equiv 2 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 3, \ if \ j \equiv 1 \pmod 3 \\ 4, \ if \ j \equiv 2 \pmod 3 \\ 5, \ if \ j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_iv_n) = 2.$$

If $i \equiv 0 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 6, & if\ j \equiv 1 \pmod 3 \\ 5, & if\ j \equiv 2 \pmod 3 \\ 1, & if\ j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_iv_n) = 3.$$

Also

$$c(u_mv_j) = \begin{cases} 4, & if\ j \equiv 1 \pmod 3 \\ 3, & if\ j \equiv 2 \pmod 3 \\ 6, & if\ j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_mv_n) = 5.$$

Hence

$$c(w_1) = 7$$

as no path on four vertices are bi-colored. Therefore it is a proper star coloring. We get the star chromatic number of the degree splitting of the torus grid graph is 7.

Case (v) Let $m = 3k+2$, $n = 3k+1$ for $k \geq 1$. Let $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. If $i \equiv 1 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 1, & if\ j \equiv 1 \pmod 3 \\ 2, & if\ j \equiv 2 \pmod 3 \\ 3, & if\ j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_iv_n) = 2.$$

If $i \equiv 2 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 3, & if\ j \equiv 1 \pmod 3 \\ 4, & if\ j \equiv 2 \pmod 3 \\ 5, & if\ j \equiv 0 \pmod 3), \end{cases}$$

and

$$c(u_i v_n) = 4.$$

If $i \equiv 3 \pmod 3$, then

$$c(u_i v_j) = \begin{cases} 5, & if\ j \equiv 1 \pmod 3 \\ 6, & if\ j \equiv 2 \pmod 3 \\ 7, & if\ j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_i v_n) = 6.$$

Also

$$c(u_m v_j) = \begin{cases} 4, & if\ j \equiv 1 \pmod 3 \\ 5, & if\ j \equiv 2 \pmod 3 \\ 6, & if\ j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_m v_n) = 5$$

and hence $c(w_1) = 8$.

Case (vi) Let $m = 3k+2$, $n = 3k+2$ for $k \neq 1$. For $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$, if $i \equiv 1 (\bmod\ 3)$, then

$$c(u_i v_j) = \begin{cases} 1, & if\ j \equiv 1 \pmod 3 \\ 2, & if\ j \equiv 2 \pmod 3 \\ 3, & if\ j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_i v_n) = 4.$$

If $i \equiv 2 \pmod 3$, then

$$c(u_i v_j) = \begin{cases} 3, & if\ j \equiv 1 \pmod 3 \\ 4, & if\ j \equiv 2 \pmod 3 \\ 5, & if\ j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_iv_n) = 6.$$

If $i \equiv 3 \pmod 3$, then

$$c(u_iv_j) = \begin{cases} 7, \ if \ j \equiv 1 \pmod 3 \\ 5, \ if \ j \equiv 2 \pmod 3 \\ 6, \ if \ j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_iv_n) = 2$$

$$c(u_mv_j) = \begin{cases} 5, \ if \ j \equiv 1 \pmod 3 \\ 6, \ if \ j \equiv 2 \pmod 3 \\ 4, \ if \ j \equiv 0 \pmod 3 \end{cases}$$

and

$$c(u_mv_n) = 3$$

and also assign $c(w_1) = 8$ where no path on four vertices are bi-colored. Therefore it is a proper star coloring. Hence, the star chromatic number of the degree splitting of the torus grid graph is 8. Thus, the proof is completed. ☐

3.3. **Star coloring of the degree splitting of the prism graph.** Let $m \geq 3$, $n \geq 4$ and let C_m be the cycle graph of order m and P_n be the path graph of order n. Then the Cartesian product $C_m \square P_n$ is known as the prism graph and is denoted by $Y_{m,n}$.

Theorem 3.3. *We have*

$$\chi_s(DS(Y_{m,n})) = 6.$$

Proof. Let $Y_{m,n}$ be the prism graph and let its set of vertices be

$$V(Y_{m,n}) = \{(u_i, v_j) : \ 1 \leq i \leq m; \ 1 \leq j \leq n)\}.$$

Then the order and the size of the prism graph are $|V(Y_{m,n})| = mn$ and $|E(Y_{m,n})| = 2mn - n$, respectively. We have

$$V(Y_{m,n}) = \begin{Bmatrix} (u_1, v_1) & (u_1, v_2) & \dots & (u_1, v_n) \\ (u_2, v_1) & (u_2, v_2) & \dots & (u_2, v_n) \\ \vdots & \vdots & & \vdots \\ (u_m, v_1) & (u_m, v_2) & \dots & (u_m, v_n) \end{Bmatrix} = S_1 \cup S_2$$

where

$$S_1 = \{(u_i, v_1) : \ 1 \leq i \leq m\} \cup \{(u_i, v_n) : \ 1 \leq i \leq m\}$$

and

$$S_2 = \{(u_i, v_j) : \ 1 \leq i \leq m; \ 2 \leq j \leq n-1\}.$$

To obtain the $DS(Y_{m,n})$ from $Y_{m,n}$, we introduce two additional vertices w_1 and w_2 corresponding to the sets S_1 and S_2, respectively.
Now, the star coloring of $DS(Y_{m,n})$ is as follows: Assign the colors 1, 2, 3, 4 alternatively to the vertices of the innermost cycle of the graph $Y_{m,n}$ and next assign colors 1, 2, 3, 4, 5 to all the vertices of the remaining copies of $Y_{m,n}$. Finally, color the leftover vertices as $c(w_1) = c(w_2) = 6$, in which no path on four vertices are bi-colored. Thus it is a proper star coloring. Hence $\chi_s(DS(Y_{m,n})) = 6$ which completes the proof. □

Corollary 3.2. *For $m \geq 3$, $n = 1$, the graph $C_m \square P_1$ is the cycle graph C_m. Then, by observation (1), the degree splitting of the cycle graph is a wheel graph. Therefore*

$$\chi_s(DS(C_m \square P_1)) = \begin{cases} 5, \ when \ m = 5 \\ 4, \ otherwise. \end{cases}$$

4. Conclusion

In this work, we obtained exact results for the star chromatic number of degree splitting of the Cartesian product of any two simple graphs G and H. Also we illustrate the exact results for the star chromatic number of the degree splitting of the Cartesian product of prism graphs, toroidal graphs and grid graphs.

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