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CONSTRUCTION OF A RIESZ WAVELET BASIS ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract. We have explored the concept of Riesz multiresolution analysis on a locally compact Abelian group G, and have extensively studied the methods of construction of a Riesz wavelet from the given Riesz MRA. We have proved that, if δ_α is the order of dilation, then precisely $\delta_\alpha - 1$ functions are required to construct a Riesz wavelet basis for $L^2(G)$. An example, supporting our theory and illustrating the methods developed, has also been discussed in detail.

1. Introduction

An orthonormal basis $\mathcal{B} = \{x_\beta\}$ of a Hilbert space $\mathcal{H}$ is one of its most important subsets. In terms of its basis elements, any element $y \in \mathcal{H}$ has a unique representation of the form

$$y = \sum c_\beta x_\beta.$$

The advantage of working with orthonormal bases is that the coefficients appearing in the above equation are easy to derive and the above equation has the exact form:

$$y = \sum \langle y, x_\beta \rangle x_\beta. \tag{1.1}$$

The orthonormal bases which consist of scaled and integer-translated versions of a single element are called *wavelet bases*. A central tool dedicated to the study, construction and analysis of wavelet bases is *multiresolution analysis* (henceforth abbreviated as MRA). The concept of MRA was first introduced by Mallat and

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Meyer in their paper [9]. The Hilbert space they worked in, was the space $L^2(\mathbb{R})$, where $\mathbb{R}$ is the usual Euclidean space.

Due to S. Dahlke, [4], in the year 1994, this concept of MRA was generalized to the separable space $L^2(G)$, where G stands for an arbitrary locally compact Abelian (henceforth abbreviated as LCA) group. Another generalization, not with respect to the space under consideration but with respect to the structure of MRA, was proposed in [15, 10]. Both these papers used Riesz basis instead of an orthonormal basis in the definition of MRA.

In the paper [12], the concept of Riesz MRA has been extended to the space $L^2(G)$ using some methods as developed in [4]. Our aim in this paper is to contruct a Riesz basis from the given Riesz MRA on an LCA group G. This paper has been structured as follows. Some preliminaries and notations have been listed in section two. A detailed method for the construction of Riesz basis via Riesz MRA has been presented in section three. To support our theory, we have also given an elaborated example there.

2. Preliminaries and Notations

2.1. **LCA Groups.** A topological group G is called an LCA group if

- it is locally compact, Hausdorff and metrizable in its topology; and,
- it can be written as a countable union of compact sets.

The symbols $'+'$ and $'0'$ have been used to denote the group composition and the identity element of G, respectively. The groups of real numbers $\mathbb{R}$, the circle group $\mathbb{T}$ and the group of integers $\mathbb{Z}$ are some of the frequently used LCA groups.

The dual group of G, denoted by $\hat{G}$, is the set of all continuous homomorphisms from the group G to the circle group $\mathbb{T}$. The elements of this group $\hat{G}$ are called *characters*. Moreover, with a suitable topology and a suitable group operation, the dual group $\hat{G}$ is also an LCA group. We refer [5, Chapter 3] for more details.

The double dual of G (or the dual of $\hat{G}$), denoted by $\hat{\hat{G}}$, can be identified with the group G itself and thus we can write $\hat{\hat{G}} = G$. This identification of an LCA group and its double dual is presented in the *Pontryagin duality theorem*, proof of which may be found in [5, Chapter 3]. Due to this identification, the notation (γ, x) will be used by us in this paper which can be interpreted as either the action of $\gamma \in \hat{G}$ on $x \in G$; or action of $x \in \hat{\hat{G}} = G$ on $\gamma \in \hat{G}$.

The group G is now given a translation invariant Radon measure μ_G, i.e. a measure for which the following equality holds for all compactly supported functions f on G:

$$\int_G f(x+y)d\mu_G(x) = \int_G f(x)d\mu_G(x), \ \ \forall\, y \in G.$$

This measure is unique up to a constant and is called the *Haar measure*. We refer [6, Chapter 2] for the existence and uniqueness of Haar measure. Throughout this paper, the Haar measure μ_G has been kept fixed. Further, based on this Haar measure, the spaces $L^1(G), L^2(G)$ and $L^\infty(G)$ have been defined in the usual way. Moreover, due to our assumptions of G being metrizable and being a countable union of compact sets, $L^2(G)$ becomes a separable Hilbert space. See [8] for a detailed proof.

We now define the operator of *Fourier transform* on $L^1(G)$ by:

$$\mathcal{F} : L^1(G) \to C_0(\hat{G}), \quad \mathcal{F}(f)(\gamma) = \int_G f(x)(\gamma, -x)d\mu_G(x). \tag{2.1}$$

Here, $C_0(\hat{G})$ is the space of all continuous functions on $\hat{G}$ vanishing at infinite.

The Haar measure $\mu_{\hat{G}}$ on $\hat{G}$ can be appropriately normalized so that for a specific class of functions, the following *inversion formula* holds, (see[11, Chapter 1]);

$$f(x) = \int_{\hat{G}} \hat{f}(\gamma)(\gamma, x)d\mu_{\hat{G}}(\gamma), x \in G. \tag{2.2}$$

In this paper, we shall always choose a normalized Haar measure $\mu_{\hat{G}}$ for $\hat{G}$ so that the inversion formula holds. Once this is done, the Fourier transform can be extended to a surjective isometry $\mathcal{F} : L^2(G) \to L^2(\hat{G})$ exactly as in the classical case of $G = \mathbb{R}$. To simplify notations, from now onwards, in all integrals when the context is clear, we will write $d\mu_G(x) = dx$ and $d\mu_{\hat{G}}(\gamma) = d\gamma$.

Apart from the operator of Fourier transform as defined above, the operators of traslation, modulation and dilation will also be used frequently throughout this paper. The first two operators, i.e. the translation operator and the modulation operator can be extended to $L^2(G)$ without much difficulty. For instance, for any $y \in G$, the operator T_y and $\mathcal{E}_y$ given by

$$T_y f(x) = f(x-y) \quad \text{and} \quad \mathcal{E}_y F(\gamma) = (\gamma, y)F(\gamma)$$

define the generalized versions of translation and modulation operators on $L^2(G)$ and $L^2(\hat{G})$ respectively. Similarly, for any $\xi \in \hat{G}$, the operators $\mathcal{T}_\xi$ and E_ξ define generalized translation and modulation operators on $L^2(\hat{G})$ and $L^2(G)$ respectively. To define a generalized dilation operator, we proceed via a method used by Dahlke in [4], for which we first need to define dilative automorphisms. An automorphism α on G (algebraic automorphism and topological homeomorphism) is said to be dilative if, for any compact set $\mathcal{K}$ and any open neighborhood U of $0 \in G$, there exists a positive integer n_0 such that $\mathcal{K} \subseteq \alpha^n(U)$, $\forall\, n \geq n_0$. Now, let $\alpha : G \to G$ be a dilative automorphism on G, then there exists a δ_α>0 such that

$$\int_G f(x)dx = \delta_\alpha \int_G f(\alpha(x))dx$$

for any appropriate function f on G. This means that α induces a unitary operator D on $L^2(G)$ given by

$$D : L^2(G) \to L^2(G), \quad Df(x) = \delta_\alpha^{1/2} f(\alpha(x))$$

for any appropriately chosen function f on G. This D works as the dilation operator on the space $L^2(G)$ and the constant δ_α is called the *order of dilation* for the operator D. Further, using this dilation operator D, we can construct a dilation operator $\mathcal{D}$ for the space $L^2(\hat{G})$. The following lemma sums up the required information for the operator $\mathcal{D}$. We omit the striaghtforward proof.

Lemma 2.1. *Let G be an LCA group and $\hat{G}$ be its dual group. Suppose $\alpha : G \to G$ is a dilative automorphism on G. Then the following hold:*

(i) The map, $\hat{\alpha} : \hat{G} \to \hat{G}$ given by

$$(\hat{\alpha}(\gamma), x) = (\gamma, \alpha(x));\ x \in G,$$

is a dilative automorphism on $\hat{G}$.

(ii) $\displaystyle\int_{\hat{G}} F(\gamma)d\gamma = \delta_\alpha \int_{\hat{G}} F(\hat{\alpha}(\gamma))d\gamma$ for any appropriat function F on $\hat{G}$.

(iii) The operator $\mathcal{D} : L^2(\hat{G}) \to L^2(\hat{G})$ given by $\mathcal{D}F(\gamma) = \delta_\alpha^{1/2} F(\hat{\alpha}(\gamma))$ is a unitary operator on $L^2(\hat{G})$. This operator $\mathcal{D}$ works as dilation operator on $L^2(\hat{G})$. Order of this dilation operator $\mathcal{D}$ is also δ_α.

It is easy to note that all these generalized operators satisfy all the commutative relations amongst them and behave similarly under Fourier transform and inverse Fourier transform, as in the case of $G = \mathbb{R}$.

We now introduce lattices, an important class of subgroups of LCA groups. A *lattice* Λ, (sometimes called a uniform lattice), in an LCA group G, is a countable, closed and discrete subgroup Λ of G for which the quotient group G/Λ is compact in the quotient topology. The *annihilator* $\Lambda^\perp$ of a lattice Λ is defined by

$$\Lambda^\perp = \{\gamma \in \hat{G} : (\gamma, \lambda) = 1, \ \ \forall \lambda \in \Lambda\}.$$

It follows from the definition of topology on $\hat{G}$ that $\Lambda^\perp$ is also a lattice in $\hat{G}$. Further, a lattice in G can be used to obtain a splitting of groups G and $\hat{G}$ into disjoint cosets (see [3], Chapter 21):

Lemma 2.2. *Let G be an LCA group and Λ a lattice in G. Then the following hold:*

(*i*) *There exists a Borel measurable relatively compact set $\mathcal{Q} \subseteq G$ such that*

$$G = \bigcup_{\lambda \in \Lambda} (\lambda + \mathcal{Q}), \ \ (\lambda + \mathcal{Q}) \cap (\lambda' + \mathcal{Q}) = \emptyset \ \text{for} \ \lambda \neq \lambda'; \ \ \lambda, \lambda' \in \Lambda. \tag{2.3}$$

(*ii*) *There exists a Borel measurable relatively compact set $\mathcal{S} \subseteq \hat{G}$ such that*

$$\hat{G} = \bigcup_{\omega \in \Lambda^\perp} (\omega + \mathcal{S}), \ \ (\omega + \mathcal{S}) \cap (\omega' + \mathcal{S}) = \emptyset \ \text{for} \ \omega \neq \omega'; \ \ \omega, \omega' \in \Lambda^\perp.$$

Moreover, the sets $\mathcal{Q}$ and $\mathcal{S}$ are respectively in one to one correspondance with the quotient groups G/Λ and $\hat{G}/\Lambda^\perp$.

Remark 1. *In this paper, the uniform lattice Λ and the dilative automorphism α are chosen such that $\alpha(\Lambda) \subseteq \Lambda$. Moreover, any pair (Λ, α) satisfying this relation is called a scaling system on G.*

The set $\mathcal{Q} \subset G$ which appears in equation (2.3) is called a *fundamental domain* associated with the lattice Λ. Also note that the sets of the form of $\mathcal{Q}$, which satisfies the two conditions of the equation (2.3), have been called *tiles* in [7]. In this paper we will use the term *fundamental domain* for such sets. We now further refine the fundamental domain $\mathcal{Q}$ and thus give the definition of a *self-similar fundamental* *domain*. The fundamental domain $\mathcal{Q}$ is said to be *self-similar* if for some finite subset Λ_0 of Λ, we have the following representation (see [7]):

$$\mathcal{Q} = \bigcup_{\lambda \in \Lambda_0} (\alpha^{-1}(\lambda) + \alpha^{-1}(\mathcal{Q})); \tag{2.4}$$

Throughout this paper, we will assume that the fundamental domain $\mathcal{Q}$ associated with the lattice Λ is self-similar. Thus for $\mathcal{Q}$, we have a representation of the form (2.4). Naturally, the immediate problem we face now is to find a precise representation of the set Λ_0 which appears in (2.4). The following lemma, proved in [13], gives us the required insight to this problem.

Lemma 2.3. *Let G be an LCA group with a uniform lattice Λ and an automorphism α. If $\mathcal{Q}$ is a self-similar fundamental domain associated to the lattice Λ, then the following hold:*

- *(i) The set Λ_0, which appears in* (2.4)*, is a complete set of coset representatives for $\alpha(\Lambda)$ in Λ.*
- *(ii)* $|\Lambda/\alpha(\Lambda)| = \delta_\alpha$.

Throughout this paper we shall also assume that $\mathcal{S}$ is a self-similar fundamental domain of $\hat{G}$ associated to the lattice Λ. All the results, which we have stated for $\mathcal{Q}$, hold analogously for $\mathcal{S}$. Thus $\mathcal{S}$ has a representation of the form:

$$\mathcal{S} = \bigcup_{\lambda \in \Lambda_0^\perp} (\hat{\alpha}^{-1}(\omega) + \alpha^{-1}(\mathcal{S})); \tag{2.5}$$

where $\Lambda_0^\perp \subset \Lambda^\perp$ is finite.

Now, all the above information presented above and the fact $|\Lambda/\alpha(\Lambda)| = |\Lambda^\perp/\hat{\alpha}(\Lambda^\perp)|$ can be used to write

$$\Lambda/\alpha(\Lambda) = \{\lambda_0 + \alpha(\Lambda), \lambda_1 + \alpha(\Lambda), \cdots, \lambda_{\delta_\alpha - 1} + \alpha(\Lambda)\} \tag{2.6}$$

and

$$\Lambda^\perp/\hat{\alpha}(\Lambda^\perp) = \{\omega_0 + \hat{\alpha}(\Lambda^\perp), \omega_1 + \hat{\alpha}(\Lambda^\perp), \cdots, \omega_{\delta_\alpha - 1} + \hat{\alpha}(\Lambda^\perp)\}. \tag{2.7}$$

Further, we will use a result proved by K. Gröchenig and W. R. Madych to make the choice (see [7, Lemma 4])

$$\lambda_0 = 0 \in G \quad \text{and} \quad \omega_0 = 0 \in \hat{G}.$$

To simplify the calculations in section three, we make one last assumption here. From here onwards, we assume that the group $\Lambda^\perp/\hat{\alpha}(\Lambda^\perp)$ is cyclic. Moreover, let $\omega_1+\hat{\alpha}(\Lambda^\perp)$ be the generator of this group. From here onwards

$$\omega_j + \hat{\alpha}(\Lambda^\perp) = j\omega_1 + \hat{\alpha}(\Lambda^\perp), \ \forall\ 0 \le j \le \delta_\alpha - 1.$$

The following lemma and the subsequent corollary gives us a relation between $\lambda_i's$ and $\omega_i's$ which appear in (2.6) and (2.7) respectively. We refer [14] for a direction of their proofs.

Lemma 2.4. *Let G be an LCA group and let (Λ, α) be a scaling system defined on G. if, for each $j \in \{0, 1, \cdots, \delta_\alpha - 1\}$, we write $\gamma_j = \hat{\alpha}^{-1}(\omega_j)$, then we have*

$$\sum_{j=0}^{\delta_\alpha - 1} (\gamma_j, \lambda_k) = 0; \ \forall\ k \in \{1, 2, 3, \cdots, \delta_\alpha - 1\}. \tag{2.8}$$

Corollary 2.1. *Let G be an LCA group and let (Λ, α) be a scaling system defined on G. If all the notations are same as used in Lemma 2.4, then for any $k, l \in \Lambda$, we have*

$$\sum_{j=0}^{\delta_\alpha - 1} (\gamma_j, \lambda_k - \lambda_l) = 0; \ \forall\ k, l \in \{1, 2, 3, \cdots, \delta_\alpha - 1\}, k \neq l. \tag{2.9}$$

Remark 2. *Due to the assumption of $\Lambda^\perp/\hat{\alpha}(\Lambda^\perp)$ being a cyclic group, we get the following relation:*

$$\gamma_j + \Lambda_0 = j\gamma_1 + \Lambda_0.$$

We will now list some more notations corresponding to the group G, which shall also be applicable to the group $\hat{G}$ in a similar way.

- For any $f, g \in L^2(G)$, the term fg, a member of $L^1(G)$, is used to denote the point-wise product of f and g.
- If $H \subset G$, then the function $\mathcal{X}_H$ is given by:

$$\mathcal{X}_H(x) = \begin{cases} 1 & , x \in H \\ 0 & , x \notin H \end{cases}$$

  is called the *indicator function of* H or *the characteristic function of* H.
- For any $H \subset G$, we say that a function $f : G \to \mathbb{C}$ is H*-periodic*, if

$$f(x+h) = f(x); \;\; \forall\, x \in G \text{ and } \forall\, h \in H.$$

Now, since, the quotient G/Λ is in one to one correspondence with the fundamental domain $\mathcal{Q} \subset G$ associated to the lattice $\Lambda \subset G$, therefore we are here tempted to assert a relation between the spaces $L^p(G/\Lambda)$ and $L^p(\mathcal{Q})$ (p=1 or 2 or ∞). But before that, we define the spaces $L^p(\mathcal{Q})$:

$$L^p(\mathcal{Q}) = \{f \in L^p(G) : f = 0 \text{ a.e. } G/\mathcal{Q}\}; \quad \text{p=1, 2 or } \infty.$$

Further, for the fundamental domain $\mathcal{S}$ associated with the lattice $\Lambda^\perp \subset \hat{G}$, we define the space $L^p(\mathcal{S})$ (p=1,2 or ∞) in a similar fashion. In both the above cases, the fundamental domains can be chosen as self similar. The following remark provides us an orthonormal family in $L^2(\mathcal{S})$ which is also its basis. For more details, we refer [2].

Remark 3. *Let, for each* $\lambda \in \Lambda$, $\eta_\lambda : \hat{G} \to \mathbb{C}$ *be defined by* $\eta_\lambda(\gamma) = (\gamma, \lambda)\mathcal{X}_{\mathcal{S}}(\gamma)$. *Then the family,*

$$\left\{ \frac{1}{\sqrt{\mu_{\hat{G}}(\mathcal{S})}} \eta_\lambda \right\}_{\lambda \in \Lambda}$$

forms an orthonormal basis for $L^2(\mathcal{S})$.

Using the above-given notation of the periodic functions, we note that there is a one to one correspondence between $L^2(G/\Lambda)$ and the set of all $\Lambda-$periodic functions f such that $f\mathcal{X}_{\mathcal{Q}} \in L^2(\mathcal{Q})$. So, with a slight abuse of notation, we write $f \in L^2(G/\Lambda)$ whenever f is a $\Lambda-$periodic function on G satisfying $f\mathcal{X}_{\mathcal{Q}} \in L^2(\mathcal{Q})$. Analogously, we give the definition for the space $L^2(\hat{G}/\Lambda^\perp)$.

We end our discussion about LCA groups by giving a lemma which helps us in explicitly representating the elements of the space $L^2(\hat{G}/\Lambda^{\perp})$. Proof of this lemma uses the information provided in Remark 3 and it follows without much complications.

Lemma 2.5. *If, for each $\lambda \in \Lambda$, the functions η_λ are defined as in Remark 3, then the following are equivalent :*

(*i*) $F \in L^2(\hat{G}/\Lambda^{\perp})$.

(*ii*) *There exists a sequence $\{c_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$ such that*

$$F = \sum_{\lambda\in\Lambda} c_\lambda \varepsilon_\lambda;$$

where $\varepsilon_\lambda : \hat{G} \to \mathbb{C}$ is given by, $\varepsilon_\lambda(\gamma) = (\gamma, \lambda)$.

2.2. **Riesz Bases.** We will now have a brief discussion on Riesz bases in an arbitrary separable Hilbert space. For a detailed study on Riesz bases and their properties, we refer [3].

Definition 2.1. Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{I}$ be a countable index set. A sequence of elements $\{f_\beta\}_{\beta\in\mathbb{I}}$ is called a Riesz basis for $\mathcal{H}$ if there exist a bounded bijective operator $U : \mathcal{H} \to \mathcal{H}$ and an orthonormal basis $\{e_\beta\}_{\beta\in\mathbb{I}}$ of $\mathcal{H}$ such that, for each $\beta \in \mathbb{I}$, $f_\beta = Ue_\beta$.

In the lemma below, we give one of the most used implications of the Riesz bases. For more details, we refer [3].

Lemma 2.6. *If $\{f_\beta\}_{\beta\in\mathbb{I}}$ is a Riesz basis for $\mathcal{H}$, then there exist constants $A, B > 0$ such that*

$$A||f||^2 \leq \sum_{\beta\in\mathbb{I}} |\langle f, f_\beta\rangle|^2 \leq B||f||^2. \tag{2.10}$$

The numbers A and B are called the Riesz bounds. Precisely, A is the lower Riesz bound and B is the upper Riesz bound. Moreover, the largest possible value of A is called the *optimal lower Riesz bound* and the smallest possible value of B is called the *optimal upper Riesz bound.*

The following lemma gives us one of the main characterizations of the Riesz bases in a separable Hilbert space. It does not involve any knowledge of the Riesz bounds. Proof of this lemma may be deduced using various results given in [3].

Lemma 2.7. *Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{I}$ be a countable index set. Then, a sequence $\{f_\beta\}_{\beta\in\mathbb{I}}$ in $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if and only if the map $T : l^2(\mathbb{I}) \to \mathcal{H}$, given by*

$$T(\{c_\beta\}) = \sum_{\beta\in\mathbb{I}} c_\beta f_\beta,$$

is well defined and bijective.

When we are studying the wavelet theory, more often than not, we encounter a family of the type $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ where $\phi \in L^2(G)$. We wish to see the condtions under which such a family is Riesz basis. For that, we first introduce a function Φ corresponding to this given function ϕ.

Definition 2.2. Let G be an LCA group with dual group $\hat{G}$ and let (Λ, α) be the scaling system defined on $\hat{G}$. If $\phi \in L^2(G)$ is given then corresponding to this function ϕ, the function Φ is given by

$$\Phi(\gamma) = \sum_{\omega\in\Lambda^\perp} |\hat{\phi}(\gamma + \omega)|^2, \ \ \gamma \in \hat{G}. \tag{2.11}$$

It is easy to note that according to the notations used previously in this paper, $\Phi \in L^1(\hat{G}/\Lambda^\perp)$. We now give an equivalent condition for the family $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ to be a Riesz sequence, i.e. a Riesz basis for its closed linear span. A detailed proof of the following lemma for the case of $G = \mathbb{R}$ has already been proved in [3].

Lemma 2.8. *Let $\phi \in L^2(G)$ be given. Then the family $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a Riesz sequence with bounds A and B if and only if*

$$A \le \Phi(\gamma) \le B,$$

for all $\gamma \in \hat{G}$.

We shall use above lemma to verify whether a family of the form $\{T_\lambda\phi\}_{\lambda\in\Lambda}$ is a Riesz sequence or not.

Now we wish to generalize the above given Lemma, i.e. we wish to seek an alternate characterization for the family of the type

$$\{T_\lambda f_i : \lambda \in \Lambda, 1 \le i \le n\}$$

to be a Riesz sequence; $f_i' s$ being appropriately chosen function. The following lemma gives us the required result. Its detailed and extensive proof may be found in [15].

Lemma 2.9. *Let $f_1, f_2, \cdots f_n \in L^2(G)$ and let $L(\gamma)$, $\gamma \in \hat{G}$, denote the $n \times n$ matrix*

$$L(\gamma) = \left[\sum_{\omega \in \Lambda^\perp} \hat{f}_i(\gamma + \omega) \overline{\hat{f}_j(\gamma + \omega)} \right]_{1 \le i,j \le n}.$$

If θ and Θ respectively denote the smallest and the largest eigenvalue of $L(\gamma)$, then the family

$$\{T_\lambda f_i : \lambda \in \Lambda, 1 \le i \le n\}$$

is a Riesz basis if and only if there exist some constants $C, D > 0$ such that

$$0 < C \le \theta \le \Theta \le D.$$

2.3. **Riesz Multiresolution Analysis.** The concept of MRA with the structure of Riesz bases for the space $L^2(\mathbb{R})$ has been given [10] and the concept of classical MRA for $L^2(G)$ has been presented in [4]. We combine the definitions in these two papers to give the definition of Riesz MRA on the space $L^2(G)$. Note that we have already given this definition in our paper [12].

Definition 2.3. A Riesz multiresolution analysis for $L^2(G)$ consists of a sequence of closed subspaces $\{V_j\}_{j\in\Lambda}$ of $L^2(G)$ and a function $\phi \in V_0$ such that

(i) the subspaces V_j are nested, i.e.

$$\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots ;$$

(ii) the subspaces V_j have a dense union and a trivial intersection, i.e.

$$\overline{\underset{j\in\mathbb{Z}}{\cup} V_j} = L^2(G) \quad \text{and} \quad \underset{j\in\mathbb{Z}}{\cap} V_j = \{0\};$$

(iii) they are related by the dilation property: $V_j = D^j V_0$;

(iv) the subspaces V_j are translation invariant, i.e.

$$f \in V_j \implies T_\lambda f \in V_j, \forall\ \lambda \in \Lambda \text{ and } \forall\ j \in \mathbb{Z};$$

(v) $\{T_\lambda \phi\}_{\lambda\in\Lambda}$ is a Riesz basis for V_0.

The subspaces V_j are called the *approximation subspaces* or the *multiresolution subspaces* and the function ϕ is called the scaling function.
In the theorem below, we sum up all the conditions which need to be imposed on the scaling function ϕ to generate a Riesz MRA for the space $L^2(G)$. All these conditions have been extensively investigated in our paper [12].

Theorem 2.1. *Let G be an LCA group with the dual group $\hat{G}$ and let (Λ, α) be a scaling system defined on G. A function $\phi \in L^2(G)$ is said to generate a Riesz MRA if and only if the following conditions are satisfied:*

(i) The family $\{T_\lambda \phi\}_{\lambda\in\Lambda}$ is a Riesz sequence.

(ii) The subspaces V_j are defined by

$$V_j = D^j(\overline{span}\{T_k\phi\}_{k\in\Lambda}) = \overline{span}\{D^jT_k\phi\}_{k\in\Lambda}, \quad j \in \mathbb{Z}. \tag{2.12}$$

(iii) The function $\hat{\phi}$ is nonzero on a neighbourhood of $0 \in \hat{G}$.

(iv) There exists a function $m_0 \in L^\infty(\hat{G}/\Lambda^\perp)$ such that

$$\hat{\phi}(\hat{\alpha}(\gamma)) = m_0(\gamma)\hat{\phi}(\gamma) \tag{2.13}$$

holds for all $\gamma \in \hat{G}$.

Equation (2.13) is called *the refinement equation* and if a function ϕ satisfies this equation, then it is called *refinable*. Further, the function m_0 appearing in (2.13) is called the *refinement mask*. Also note that this function m_0 is unique.

3. Constructing Riesz Wavelet From Given Riesz MRA

Throughout this section, we will assume that we have a function ϕ which generates a Riesz MRA, i.e. all the conditions of Theorem 2.1 are satisfied for this function. Here, using Lemma 2.8, we also get existence of positive numbers $A, B > 0$ such that

$$0 < A \leq \Phi(\gamma) \leq B; \ \forall \ \gamma \in \hat{G}.$$

We begin the construction of Riesz wavelet by writing an orthogonal decomposition of the space $L^2(G)$. For each $j \in \mathbb{Z}$, let W_j denote the orthogonal complement of V_j in V_{j+1}. Then it is easy to see that

$$V_{j+1} = V_j \oplus W_j$$

and hence

$$L^2(G) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

Next, we show that the spaces W_j are related to each other by the same dilation property as the subspaces V_j. This property of the subspaces $W_j's$ reduces our work and now we only need to find some functions whose family of Λ-translates form a Riesz basis for W_0. All this information is presented in the following lemma. We refer [3] for its detailed proof.

Lemma 3.1. *Assume that $\phi \in L^2(G)$ generates an Riesz MRA. Then the following holds:*

(i) $W_j = D^j W_0, \quad \forall j \in \mathbb{Z}.$

(ii) If the functions $\psi_1, \psi_2 \cdots \psi_n \in W_0$ are such that the family of its Λ-translates, $\{T_\lambda \psi_i : \lambda \in \Lambda, 1 \leq i \leq n\}$, is a Riesz for W_0, then for all $j \in \mathbb{Z}$, the family $\{D^j T_\lambda \psi_i : \lambda \in \Lambda, 1 \leq i \leq n\}$ is a Riesz basis for W_j, and the family

$$\{D^j T_\lambda \psi_i : \lambda \in \Lambda, 1 \leq i \leq n, j \in \mathbb{Z}\}$$

is a Riesz basis for $L^2(G)$. Moreover, all these bases have exactly the same Riesz bounds.

From the above lemma, we see that the space W_0 is of utmost importance to us and thus we give its characterization in the following lemma.

Lemma 3.2. *Assume that $\phi \in L^2(G)$ generates a Riesz MRA with order of dilation δ_α and two-scale symbol $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$. Let $F \in L^2(\hat{G}/\Lambda^\perp)$ and define $f \in V_1$ by:*

$$\hat{f}(\hat{\alpha}(\gamma)) = F(\gamma)\hat{\phi}(\gamma). \tag{3.1}$$

Then the following hold:

(i) If we write $\mathcal{S}' = \hat{\alpha}^{-1}(\mathcal{S})$, then

$$\langle f, T_\lambda \phi\rangle = \delta_\alpha \int_{\mathcal{S}'} \left(\sum_{j=0}^{\delta_\alpha - 1} \mathcal{T}_{\gamma_j}(F\Phi\overline{H_0})(\gamma) \right)(\alpha(\gamma), \lambda) d\gamma \tag{3.2}$$

(*ii*) *A function $f \in V_1$ belongs to W_0 if and only if*

$$\sum_{j=0}^{\delta_\alpha - 1} \mathcal{T}_{\gamma_j}(F\Phi\overline{H_0}) = 0 \tag{3.3}$$

a.e. on $\hat{G}$.

The proof of this lemma is on similar lines as given in [1]. Further, we mention it here that the LCA group which has been considered in [1] is the Euclidean group $\mathbb{R}$. As pointed out before, we now only need to find some functions in W_0 such that the family consisting of their translated versions forms a Riesz basis for W_0. From the previous works on MRA and Riesz MRA on the Euclidean group $\mathbb{R}$, we get a hint that $\delta_\alpha - 1$ functions should suffice. So our focus now shifts to find the existance of $\delta_\alpha - 1$ functions $\psi_1, \psi_2, \cdots, \psi_{\delta_\alpha - 1}$ such that the family

$$\{T_\lambda \psi_i : \lambda \in \Lambda, 1 \le i \le \delta_\alpha - 1\} \tag{3.4}$$

forms a Riesz basis for W_0. We intend to achieve this in two steps:

- We find $\delta_\alpha - 1$ functions $\psi_1, \psi_2, \cdots, \psi_{\delta_\alpha - 1}$ such that Λ-transalates of these functions generate the space W_0, i.e.

$$W_0 = \overline{\text{span}}\{T_\lambda \psi_i : \lambda \in \Lambda, 1 \le i \le \delta_\alpha - 1\}. \tag{3.5}$$

- We will then show that these functions, as obtained in above step, are such that the family (3.4) forms a Riesz basis for the space W_0.

In the lemma below, we give a sufficient condition, in terms of solvability of a system of linear equation, for the family $\{T_\lambda \psi_i : \lambda \in \Lambda, 1 \le i \le \delta_\alpha - 1\}$ to generate the space W_0. This alternate characterization will be of much use to us.

Lemma 3.3. *Let G be an LCA group and let $\phi \in L^2(G)$ generates a Riesz MRA of δ_α order of dilations and with two scale symbol $H_0 \in L^\infty(\hat{G}/\Lambda^\perp)$. Suppose there exist functions $m_1, m_2, \cdots, m_{\delta_\alpha} \in L^\infty(\hat{G}/\Lambda^\perp)$ and the functions $\psi_1, \psi_2, \cdots, \psi_{\delta_\alpha - 1}$ are defined via*

$$\hat{\psi}_i(\hat{\alpha}(\gamma)) = m_i(\gamma)\hat{\phi}(\gamma). \tag{3.6}$$

Further assume that there exist functions $G_0, G_1, \cdots, G_{\delta_\alpha - 1} \in L^\infty(\hat{G}/\Lambda^\perp)$ *such that the following system of equations*

$$\sum_{j=0}^{\delta_\alpha-1} \mathcal{T}_{\gamma_j}(\overline{m_0}\Phi m_i)(\gamma) = 0; \qquad i \in \{1, 2, \cdots, \delta_\alpha - 1\}; \tag{3.7}$$

$$\mathcal{T}_{\gamma_i}\Phi(\gamma) \sum_{j=0}^{\delta_\alpha-1} \mathcal{T}_{\gamma_i} m_j(\gamma) G_j(\gamma) = \delta_{i,0}(\gamma)\Phi(\gamma); \;\; i \in \{0, 1, \cdots, \delta_\alpha - 1\} \tag{3.8}$$

is satisfied for all $\gamma \in \hat{G}$*, then*

$$W_0 = \overline{span}\{T_\lambda \psi_i : \lambda \in \Lambda, 1 \le i \le \delta_\alpha - 1\}.$$

Using some earlier known results, Lemma 2.4, Corollary 2.1 and some of their easy manipulations, we see that the above Lemma is easy to prove. Therefore, we skip its proof here.

Making use of the above lemma, we now explicitly construct $\delta_\alpha - 1$ functions $\psi_1, \psi_2, \cdots, \psi_{\delta_\alpha - 1}$ which generate W_0 in the sense of (3.5). Again, we will not give a detailed proof of the following theorem, but will briefly give the directions for the same.

Theorem 3.1. *Let* G *be an LCA group and let* $\phi \in L^2(G)$ *generate a Riesz MRA with order of dilations* δ_α *and two scale symbol* $m_0 \in L^\infty(\hat{G}/\Lambda^\perp)$*. Then there always exist* $\delta_\alpha - 1$ *functions* $\psi_1, \psi_2, \cdots, \psi_{\delta_\alpha - 1}$ *in* W_0 *generating* W_0*.*

Proof. It is only required to prove that the equation sets given by (3.7) and (3.8) hold true on $\mathcal{S}$, the self-similar fundamental domain associated to the lattice $\Lambda^\perp \subset \hat{G}$. Next we define the set $\mathcal{S}_0$ by:

$$\mathcal{S}_0 = \{\gamma \in \mathcal{S} : |m_0(\gamma)| \ge |(\mathcal{T}_{\gamma_i} m_0)(\gamma)|, \forall\ 0 \le i \le \delta_\alpha - 1\} \tag{3.9}$$

and then for each $j \in \{1, 2, \cdots, \delta_\alpha - 1\}$ further define

$$\mathcal{S}_j = \gamma_j + \mathcal{S}_0 \tag{3.10}$$

These sets $\mathcal{S}_j$ satisfy the following equality:

$$\bigcup_{j=0}^{\delta_\alpha-1} \mathcal{S}_j = \mathcal{S}.$$

We note that, on any set $\mathcal{S}_j$, the set of equations given by (3.7) gives us essentially the same information and that is

$$m_i \text{ on } \mathcal{S}_1 = \frac{1}{(m_0\Phi) \text{ on } \mathcal{S}_1} \sum_{j=1}^{\delta_\alpha - 1} (m_i \text{ on } \mathcal{S}_j \times (m_0\Phi) \text{ on } \mathcal{S}_j)$$

In the following table, we list one of the many possible choices for the values of these functions $m_i's$. We have also used Remark 2 to obtain these entries and to ensure the consistency between them.

TABLE 1. Table for the functions $m_j's$

	$\mathcal{S}_0$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\cdots$	$\cdots$	$\mathcal{S}_{\delta_\alpha-1}$
m_1	$-\frac{\overline{H_0(\gamma-\gamma_1)}\Phi(\gamma-\gamma_1)}{\overline{H_0(\gamma)}\Phi(\gamma)}$	1	0	$\cdots$	$\cdots$	0
m_2	$-\frac{\overline{H_0(\gamma-\gamma_2)}\Phi(\gamma-\gamma_2)}{\overline{H_0(\gamma)}\Phi(\gamma)}$	0	1	$\cdots$	$\cdots$	0
m_3	$-\frac{\overline{H_0(\gamma-\gamma_3)}\Phi(\gamma-\gamma_3)}{\overline{H_0(\gamma)}\Phi(\gamma)}$	0	0	$\cdots$	$\cdots$	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$m_{\delta_\alpha-1}$	$-\frac{\overline{H_0(\gamma-\gamma_{(\delta_\alpha-1)})}\Phi(\gamma-\gamma_{(\delta_\alpha-1)})}{\overline{H_0(\gamma)}\Phi(\gamma)}$	0	0	$\cdots$	$\cdots$	1

Further note that finding the functions $G_0, G_1, \cdots, G_{\delta_\alpha-1}$, which satisfy equation set (3.8), is a routine exercise and it can be done quite easily once we use the function $m_i's$ as given in Table 1. □

This completes our quest for $\delta_\alpha - 1$ functions which generate the space W_0. We now show that the family of the type (3.4), constructed using the functions obtained in above theorem, is indeed a Riesz basis for W_0.

Theorem 3.2. *Assume that $\phi \in L^2(G)$ generates a Riesz MRA with order of dilations δ_α and two scale symbol $m_0 \in L^\infty(\hat{G}/\Lambda^\perp)$. Further assume that the functions $\psi_1, \psi_2, \cdots, \psi_{\delta_\alpha-1}$ are defined by* (3.6) *and the functions $F_1, F_2, \cdots, F_{\delta_\alpha-1}$ are assumed to be as they appear in Theorem 3.1. Then the family* (3.4) *generates a Riesz basis for the space $L^2(G)$.*

Proof. We intend to use Lemma 2.9 to prove our assertion. Moreover, here we will consider the functions $m_1, m_2, \cdots, m_{\delta_\alpha-1}$ as obtained in Table 1. Instead of the matrix $L(\gamma)$, we will use the matrix $L(\hat{\alpha}(\gamma))$. Now, θ and Θ will respectively denote the smallest and largest eigenvalues of the matrix $L(\hat{\alpha}(\gamma))$. We now proceed to show that there exist some constants $C, D > 0$ such that

$$0 < C < \theta \leq \Theta < D.$$

It is enough to prove the above relation for $\gamma \in \mathcal{S}' = \hat{\alpha}^{-1}(\mathcal{S})$ only. First note that, just like the previous case, we devide the set $\mathcal{S}'$ into δ_α disjoint parts. We let

$$\mathcal{S}'_0 = \{\gamma \in \mathcal{S}' : |m_0(\gamma)| \geq |(\mathcal{T}_{\gamma_i} m_0)(\gamma)|, \forall\ 0 \leq i \leq \delta_\alpha - 1\}$$

and then for each $j \in \{1, 2, \cdots, \delta_\alpha - 1\}$ further define

$$\mathcal{S}'_j = \gamma_j + \mathcal{S}'_0.$$

It is easy to note that, for any $j \in \{0, 1, \cdots, \delta_\alpha - 1\}$, $\mathcal{S}'_j \subseteq \mathcal{S}_j$.
As noted earlier, $L(\hat{\alpha}(\gamma))$ is $(\delta_\alpha - 1) \times (\delta_\alpha - 1)$ matrix. Below, we write a general expression for the matrix $L(\hat{\alpha}(\gamma))$.

$$L(\hat{\alpha}(\gamma)) = [a_{ij}]_{1 \leq i,j \leq \delta_\alpha - 1};$$

where a_{ij}, the entry at the intersection of i^{th} row and j^{th} column, has the explicit expression:

$$a_{ij} = \sum_{k=0}^{\delta_\alpha - 1} \left(m_i(\gamma - \gamma_k)\overline{m_j(\gamma - \gamma_k)}\Phi(\gamma - \gamma_k) \right).$$

The proof after this is a routine exercise to verify that the eigenvalues of this matrix $L(\hat{\alpha}(\gamma))$ are bounded below away from zero, as well as bounded above. This step has to be repeated for γ in each $\mathcal{S}'_j$. □

We conclude our paper with an illustrative example as given below.

Example 3.1. *Let $G = \mathbb{R}$ denote the Euclidean group of real numbers under multiplication. For any Borel set $\mathcal{B}$ in G, a Haar measure μ_G on G is given by:*

$$\mu_G(\mathcal{B}) = \int_{\mathcal{B}} d\mu_G(t); \ \textit{where } d\mu_G(t) = dt.$$

The set $\Lambda = \mathbb{Z}$ works as a uniform lattice in G, and the map $\alpha : x \mapsto 3x$ works as a dilative automorphism of G. One representation for the quotient group $\Lambda/\alpha(\Lambda)$ is

$$\frac{\Lambda}{\alpha(\Lambda)} = \frac{\mathbb{Z}}{3\mathbb{Z}} = \{3\mathbb{Z}, 1 + 3\mathbb{Z}, 2 + 3\mathbb{Z}\}.$$

Now, if we let $\mathcal{Q} = [0,1)$, then it is easy to note that

$$\begin{aligned}\mathcal{Q} = &\left[0, \frac{1}{3}\right) \bigcup \left[\frac{1}{3}, \frac{2}{3}\right) \bigcup \left[\frac{2}{3}, 1\right) \\ =&\alpha^{-1}(\mathcal{Q}) \cup (\alpha^{-1}(1) + \alpha^{-1}(\mathcal{Q})) \cup (\alpha^{-1}(2) + \alpha^{-1}(\mathcal{Q}))\end{aligned}$$

Thus $\mathcal{Q}$ is a self-similar fundamental domain associated with Λ and α. Further note that for the chosen measure μ_G, we have $\mu_G(\mathcal{Q}) = 1$.
For any $x, \xi \in G$, the map $x \mapsto e^{2\pi i x\xi}$ is a continuous homomorphism from G to $\mathbb{T}$. Defining the characters of G in this way, i.e. by writing $(\xi, x) = \phi_\xi(x)$, we get that the Pontryagin dual group of $\mathbb{R}$ is $\mathbb{R}$ i.e., $\hat{G} = G$. The measure $\mu_{\hat{G}}$ is normalized appropriately so that the inversion formula and the Parseval formula hold. Further, the annihilator $\Lambda^\perp$ of Λ and the automorphism $\hat{\alpha}$ of $\hat{G}$(corresponding to the automorphism α of G) can be derived accordingly. We choose the following epresentation for the quotient group $\Lambda^\perp/\hat{\alpha}(\Lambda^\perp)$

$$\frac{\Lambda^\perp}{\hat{\alpha}(\Lambda^\perp)} = \frac{\mathbb{Z}}{3\mathbb{Z}} = \{3\mathbb{Z}, 1 + 3\mathbb{Z}, 2 + 3\mathbb{Z}\}.$$

A self-similar fundamental domain $\mathcal{S}$ of $\hat{G}$ is given by $\mathcal{S} = [0,1)$. It is easy to see that $\mu_{\hat{G}}(\mathcal{S}) = 1$.
We now define a function ϕ on G via its Fourier transform by:

$$\hat{\phi}(\gamma) = \mathcal{X}_{A_1}(\gamma) + 2\mathcal{X}_{A_2}(\gamma), \ \ \gamma \in \hat{G},$$

where

$$A_1 = \left[\tfrac{-1}{3}, \tfrac{1}{3}\right) \quad \textit{and} \quad A_2 = \quad \left[\tfrac{-1}{2}, \tfrac{-1}{3}\right) \bigcup \left[\tfrac{1}{3}, \tfrac{1}{2}\right).$$

It is easy to see that if the we define the subspaces V_j by $V_j = \overline{\ span}\{T_\lambda\phi : \lambda \in \Lambda\}$ and chose the two-scale symbol m_0 such that its restriction on the set $\mathcal{S} = [0,1)$ is given by:

$$\mathcal{X}_{A_3}(\gamma) + 2\mathcal{X}_{A_4}(\gamma);$$

where

$$A_3 = \left[0, \tfrac{1}{9}\right) \cup \left[\tfrac{8}{9}, 1\right) \quad \textit{and} \quad A_4 = \left[\tfrac{1}{9}, \tfrac{1}{6}\right) \cup \left[\tfrac{5}{6}, \tfrac{8}{9}\right);$$

then the function ϕ generates a Riesz MRA for the space $L^2(\mathbb{R})$.
Now, with the notations taken from last two theorems, we note that

$$\mathcal{S}_0 = \left[0, \tfrac{1}{6}\right) \bigcup \left[\tfrac{5}{6}, 1\right), \quad \mathcal{S}_1 = \left[\tfrac{1}{6}, \tfrac{1}{2}\right) \quad \text{and} \quad \mathcal{S}_2 = \left[\tfrac{1}{2}, \tfrac{5}{6}\right).$$

Applying a procedure as mentioned in Theorem 3.1, we get that

$$m_1(\gamma) = \mathcal{X}_{A_5}(\gamma) \quad \text{and} \quad m_2(\gamma) = \mathcal{X}_{A_6}(\gamma);$$

where

$$A_5 = \left[\tfrac{1}{6}, \tfrac{1}{2}\right) \quad \text{and} \quad A_6 = \left[\tfrac{1}{2}, \tfrac{5}{6}\right).$$

Now we define the functions ψ_1 and ψ_2 as mentioned in (3.6)*. We finally obtain*

$$\begin{aligned} \hat{\psi_1}(\gamma) =& \mathcal{X}_{A_7}(\gamma) + 2\mathcal{X}_{A_8}(\gamma) \\ \hat{\psi_2}(\gamma) =& \mathcal{X}_{A_9}(\gamma) + 2\mathcal{X}_{A_{10}}(\gamma); \end{aligned}$$

where

$$A_7 = \left[\tfrac{1}{2}, 1\right), \quad A_8 = \left[1, \tfrac{3}{2}\right), \quad A_9 = \left[-1, \tfrac{-1}{2}\right) \text{ and } \quad A_{10} = \left[\tfrac{-3}{2}, -1\right). \tag{3.11}$$

At last, we note that the smallest eigenvalue of the matrix $L(\gamma)$ is 1 *and the largest eigenvalue is* 2*. So using Lemma 2.9, it is now easy to conclude for us that the family $\{D^j T_k \psi_i : k \in \mathbb{Z}$ and $i = 1, 2\}$ is a Riesz basis for the space $L^2(\mathbb{R})$.*

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