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A STUDY ON FUZZY WEAK ENRICHED VECTOR SOFT TOPOLOGY

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Abstract. In this paper, the notions of fuzzy enriched soft topology, fuzzy weak enriched soft topology and fuzzy weak enriched vector soft topology are introduced. The relation between fuzzy vector soft topology and fuzzy enriched soft topology is obtained. In this connection, several properties are discussed. This paper also discusses extensions of ideas of convexity and balance to the fuzzy soft case.

1. Introduction

The most appropriate theory, for dealing with uncertainties is the theory of fuzzy sets developed by L.Zadeh [9]. L.Zadeh writes : "The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly, in the fields of pattern classification and information processing. Subsequently, several mathematical structures have been developed using fuzzy set theory and soft set theory or a combination of these two theories. Maji et al [4] who worked on some mathematical aspects of soft sets and fuzzy soft sets. Fuzzy topological vector spaces are introduced by A. K. Katsaras and D. B. Liu [2]. In 1981, Katsaras [3], changed the definition of fuzzy topological vector spaces. In view of this and also considering the importance of fuzzy topological vector space with fuzzy soft theory in developing the theory of functional analysis, we have introduced

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in this paper a notion of fuzzy vector soft topology which is a combined structure of a fuzzy soft topological space and a vector space. The notions of fuzzy enriched soft topology, fuzzy weak enriched soft topology and fuzzy weak enriched vector soft topology are introduced. The relation between fuzzy vector soft topology and fuzzy weak enriched vector soft topology is obtained. In this connection, several properties are discussed. This paper also discusses extensions of the algebraic and geometric ideas of convexity and balance to the fuzzy soft case.

2. Preliminaries

In this section, some basic concepts of fuzzy soft sets have been recalled. Also related results and propositions are studied from various research articles. Some definitions and preliminary results are presented in the context.

Definition 2.1. ([4], Definition 1, p. 2) Let X be an initial universe, E is a set of parameters, $\Sigma \subseteq E$ and $P(X)$ is the set of all fuzzy sets of X. A pair (Λ, Σ) is called a fuzzy soft set over X, where $\Lambda : \Sigma \to P(X)$.

Definition 2.2. ([4], Definition 1, p. 2) For any two fuzzy soft sets (F, A) and (G, B) over a common universe R, we say that (F, A) is a fuzzy soft subset of (G, B) if

(i) $A \subseteq B$.
(ii) $F(a) \leq G(a)$ for all $a \in A$.In this case, we write $(F, A) \sqsubseteq (G, B)$.

Definition 2.3. ([7], Definition 2.2, p. 73) Let $\{(F_i, A); i \in \Delta\}$ be a nonempty family of fuzzy soft sets over a common universe X. Then their

(i) Intersection, denoted by $\sqcap_{i\in\Delta}$, is defined by $\sqcap_{i\in\Delta}(F_i, A) = (\sqcap_{i\in\Delta}F_i, A)$, where $(\sqcap_{i\in\Delta}F_i)(\alpha) = \wedge_{i\in\Delta}(F_i(\alpha)), \forall\alpha \in A$.
(ii) Union, denoted by $\sqcup_{i\in\Delta}$, is defined by $\sqcup_{i\in\Delta}(F_i, A) = (\sqcup_{i\in\Delta}F_i, A)$, where $(\sqcup_{i\in\Delta}F_i)(\alpha) = \vee_{i\in\Delta}(F_i(\alpha)), \forall\alpha \in A$.

Definition 2.4. ([7], Definition 2.3, p. 73) Let X and Y be two nonempty sets and $f : X \to Y$ is a mapping and $FS(X, A)$ denote the set of all fuzzy soft sets over X. Then

(i) the image of a fuzzy soft set $(F, A) \in FS(X, A)$ under the mapping f is defined by $f(F, A) = (f(F), A)$, where $[f(F)](\alpha) = f[F(\alpha)]$, $\forall\alpha \in A$.

(ii) the inverse image of a fuzzy soft set $(G, A) \in FS(Y, A)$ under the mapping f is defined by $f^{-1}(G, A) = (f^{-1}(G), A)$, where $[f^{-1}(G)](\alpha) = f^{-1}[G(\alpha)], \forall \alpha \in A$.

Definition 2.5. ([1], Definition 2.8, p. 409) Let f_A, g_B be any two fuzzy soft sets over X. The union of f_A and g_B, denoted by $f_A \sqcup g_B$, is the fuzzy soft set $h_{A\cup B}$ defined by $h_{A\cup B}(e) = f_A(e) \wedge g_B(e), \forall e \in E$. That is, $h_{A\cup B} = f_A \sqcup g_B$.

Definition 2.6. ([1], Definition 2.9, p. 409) Let f_A, g_B be any two fuzzy soft sets over X. The intersection of f_A and g_B, denoted by $f_A \sqcap g_B$, is the fuzzy soft set $h_{A\cap B}$ defined by $h_{A\cap B}(e) = f_A(e) \wedge g_B(e), \forall e \in E$. That is, $h_{A\cap B} = f_A \sqcap g_B$.

Definition 2.7. ([8], Definition 2.15, p. 456) Let R be the universe, E be the set of parameters and $A \subseteq E, B \subseteq E$. Let (F, A)and (G, B) be two fuzzy soft sets over a common universe R. Then,

(i) The product of two fuzzy soft sets (F, A)and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\odot}(G, B)$ over R, where $C = A\times B$ and $H(a, b) = F(a)\cdot G(b)$ for all $(a, b) \in A \times B$.

(ii) The sum of two fuzzy soft sets (F, A)and (G, B) is defined as the fuzzy soft set $(H, C) = (F, A)\widetilde{\oplus}(G, B)$ over R, where $C = A \times B$ and $H(a, b) = F(a)+G(b)$ for all $(a, b) \in A \times B$.

Definition 2.8. ([10], Definition 4.1, p. 939) Let X be the universe and E be the parameter set. The fuzzy soft topology τ over X is a collection of fuzzy soft subsets over X with fixed parameter set E such that:

(i) $\widetilde{X}, \Phi \in T$,

(ii) The union of any numbers of fuzzy soft sets in τ belongs to τ .

(iii) The intersection of any two fuzzy soft sets in τ belongs to τ.

The triplet (X, E, τ) is called a fuzzy soft topological space over X, say fuzzy soft topological space, and each element of τ is called a fuzzy soft open set, say fuzzy soft open set, in X. The complement of a fuzzy soft open set is called fuzzy soft closed set, say fuzzy soft closed set.

Definition 2.9. ([3], Definition 3.3, p. 87) A fuzzy vector topology on a vector space E over K is a fuzzy topology φ on E such that the two mappings

$$+ : E \times E \to E, (x, y) \mapsto x + y,$$

$$\cdot : K \times E \to E, (t, x) \mapsto tx.$$

are continuous when K has the usual fuzzy topology and $K \times E, E \times E$ the corresponding product fuzzy topologies.

Definition 2.10. ([6], Definition 2.9, p. 648) Let $S(X, A)$ denote the collection soft set over X. A soft set $(F, A) \in S(X, A)$ is said to be pseudo constant soft set if $F(\alpha) = X or \phi, \forall \alpha \in A$. Let $CS(X, A)$ denote the set of all pseudo constant soft sets over X under the parameter set A.

Definition 2.11. ([6], Definition 2.11, p. 648) A sub collection τ of $S(X, A)$ is said to be an enriched soft topology on X if

(1) $(F, A) \in \tau$, $\forall (F, A) \in CS(X, A)$.
(2) the intersection of any two soft sets in τ belongs to τ;
(3) the union of any number of soft sets in τ belongs to τ;

Then (X, A, τ) is called an enriched soft topological space over X.

Definition 2.12. ([1], Definition 2.20, p. 412) Let $f_A \in F(X, E)$ and $g_B \in F(Y, K)$. The fuzzy product $f_A \times g_B$ is defined by $(f \times g)_{A\times B}$ where

$$(f \times g)_{A\times B}(e, k) = f_A(e) \times g_B(k) \in I^X \times I^Y, \forall (e, k) \in A \times B,$$

and for all $(x, y) \in X \times Y$, $(f_A(e) \times g_B(k))(x, y) = f_A(e)(x) \wedge g_B(k)(y)$. According to this definition the fuzzy soft set $f_A \times g_B$ is a fuzzy soft set over $X \times Y$ and its parameter universe is $E \times K$.

Definition 2.13. ([7], Definition 3.9, p. 82) The fuzzy soft topology in $X \times Y$ induced by the open base F is said to be the product fuzzy soft topology of the fuzzy topologies τ and γ. It is denoted by $\tau \widetilde{\times} \gamma$. The fuzzy soft topological spaces $[X \times Y, A, \tau \widetilde{\times} \gamma]$ is said to be the fuzzy soft topological product of the fuzzy soft topologies (X, A, τ) and (Y, A, γ).

Definition 2.14. ([10], Definition 5.5, p. 943) Let $X = X_1 \times X_2$ and $E = E_1 \times E_2$. Let $\pi_i^X : X \to X_i$ and $\pi_i^E : E \to E_i$ be projection maps. Suppose (X_1, E_1, τ_1) and (X_2, E_2, τ_2) be F.S topological spaces. The fuzzy soft map

$$\pi_i^{X,E} : X_{1E_1} \widetilde{\bigotimes} X_{2E_2} \to X_{iE_i}$$

is called the fuzzy soft projection map where $\pi_i^{X,E}(f_{1E_1} \widetilde{\bigotimes} f_{2E_2}) = f_{iE_i}$.

Proposition 2.1. *([7], Proposition 3.13, p. 84) Let (X, A, τ) be the product space of two fuzzy soft topological spaces (X_1, A, τ_1) and (X_2, A, τ_2) and $\pi_i : (X, A, \tau) \to (X_i, A, \tau_i), i = 1, 2$ be the projection mappings. If (Y, A, γ) be any fuzzy soft topological space, then the mapping $f : (Y, A, \gamma) \to (X, A, \tau)$ is fuzzy soft continuous iff the mapping $\pi_i f : (Y, A, \gamma) \to (X_i, A, \tau_i), i = 1, 2$ are fuzzy soft continuous.*

Proposition 2.2. *([7], Proposition 3.18, p. 86) Let (X, A, τ) be fuzzy soft topological space. Then the mapping $f : (X, A, \tau) \to (X, A, \tau)$ defined by $f(x) = x, \forall x \in X$ is fuzzy soft continuous.*

Proposition 2.3. *([7], Proposition 3.19, p. 86) Let (X, A, τ) and (Y, A, γ) be any two fuzzy soft topological spaces. Then the constant mapping $f : (X, A, \tau) \to (Y, A, \gamma)$ defined by $f(x) = y_0, \forall x \in X$ where y_0 is a fixed element of Y, is fuzzy soft continuous.*

Proposition 2.4. *([7], Proposition 3.12, p. 83) Let (X, A, τ) and (Y, A, γ) be any two fuzzy soft topological spaces. Then the projection mapping $\pi_X : (X \times Y, A, \tau \widetilde{\times} \gamma) \to (X, A, \tau)$ and $\pi_Y : (X \times Y, A, \tau \widetilde{\times} \gamma) \to (Y, A, \gamma)$ are fuzzy soft continuous and fuzzy soft open. Also $\tau \widetilde{\times} \gamma$ is the smallest fuzzy soft topology in $X \times Y$ for which the projection mappings are fuzzy soft continuous.*

3. Fuzzy Soft Convex, Fuzzy Soft Balanced, Fuzzy Soft Absorbing

Throughout this paper, V is a vector space over the field K, the field of real or complex numbers, E is a set of parameters and $\Sigma \subseteq E$. Then (λ, Σ) is fuzzy soft set in V over the field K under the parameter Σ and $\lambda : \Sigma \to P(V)$. From now on we will use $\mathfrak{F}(V, \Sigma)$ instead of the family of all fuzzy soft sets over V. In this section, some definitions related to fuzzy vector soft topology are introduced.

Definition 3.1. Let V be a vector space over the field K, E be a set of parameters and $\Sigma \subseteq E$. Let $(\lambda, \Sigma), (\mu, \Sigma) \in \mathfrak{F}(V, \Sigma)$ be any two fuzzy soft sets over V. Then

(i) $(\lambda, \Sigma) \oplus (\mu, \Sigma) = (\lambda + \mu, \Sigma)$ where $(\lambda + \mu)(\alpha) = \lambda(\alpha) + \mu(\alpha), \forall \alpha \in \Sigma$.

(ii) $\xi(\lambda, \Sigma) = (\xi\lambda, \Sigma)$ where $(\xi\lambda)(\alpha) = \{\xi\sigma : \sigma \in \lambda(\alpha)\}, \forall \alpha \in \Sigma, \forall \xi \in K$.

(iii) If (ξ, Σ) is any fuzzy soft set over K, then $(\xi, \Sigma) \odot (\lambda, \Sigma) = (\xi \cdot \lambda, \Sigma)$ where $(\xi \cdot \lambda)(\alpha) = \xi(\alpha) \cdot \lambda(\alpha), \forall \alpha \in \Sigma$.

Definition 3.2. Let V be a vector space over the field K, E be a set of parameters and $\Sigma \subseteq E$. Then the fuzzy soft set $(\lambda, \Sigma) \in \mathfrak{F}(V, \Sigma)$ is said to be

(i) Fuzzy soft convex if $[(\xi(\lambda, \Sigma)) \oplus ((1 - \xi)(\lambda, \Sigma))] \sqsubseteq (\lambda, \Sigma), \forall \xi \in [0, 1]$.

(ii) Fuzzy soft balanced if $[\xi(\lambda, \Sigma)] \sqsubseteq (\lambda, \Sigma)$ for all scalar $\xi \in K$ with $|\xi| \leq 1$.

(iii) Fuzzy soft absolutely convex if it is both fuzzy soft convex and fuzzy soft balanced.

Note 3.1. (i) (λ, Σ) *is a fuzzy soft convex (fuzzy soft balanced) set iff for each* $\alpha \in \Sigma$, *the fuzzy set* $\lambda(\alpha)$ *is fuzzy soft convex (fuzzy soft balanced).*

(ii) *If* $(\lambda, \Sigma), (\mu, \Sigma) \in \mathfrak{F}(V, \Sigma)$ *are two fuzzy soft convex (fuzzy soft balanced) sets in a vector space* V *over the field* K, *then* $[\xi_1 \odot (\lambda, \Sigma)] \oplus [\xi_2 \odot (\mu, \Sigma)]$ *is a fuzzy soft convex (fuzzy soft balanced) set in* V *for all scalars* $\xi_1, \xi_2 \in K$.

(iii) *If* $\{(\lambda_i, \Sigma)\}_{i \in J}$ *where* $(\lambda_i, \Sigma) \in \mathfrak{F}(V, \Sigma), i \in J$ *is a family of fuzzy soft convex (fuzzy soft balanced) sets in a vector space* V *over the field* K, *then* $(\lambda, \Sigma) = \sqcap_{i \in I}(\lambda_i, \Sigma)$ *is a fuzzy soft convex (fuzzy soft balanced) set in* V.

Definition 3.3. Let V and W be two vector spaces over the field K. A function $T : V \to W$ is called a fuzzy soft linear function if for any $x, y \in V$ and $\xi \in K$,

(i) $T(x + y) = T(x) + T(y)$,

(ii) $T(\xi x) = \xi T(x)$.

Proposition 3.1. *Let* V *and* W *be any two vector spaces over the field* K *and let* $f : V \to W$ *be a fuzzy soft linear function. Then*

(i) *If* $(\lambda, \Sigma) \in \mathfrak{F}(V, \Sigma)$ *is a fuzzy soft convex (fuzzy soft balanced) set in* V, *then* $f[(\lambda, \Sigma)]$ *is a fuzzy soft convex (fuzzy soft balanced) set in* W.

(ii) *If* $(\mu, \Sigma) \in \mathfrak{F}(W, \Sigma)$ *is a fuzzy soft convex (fuzzy soft balanced) set in* W*, then* $f^{-1}[(\mu, \Sigma)]$ *is a fuzzy soft convex (fuzzy soft balanced) set in* V*.*

Proof. (i) Let $\xi \in [0, 1]$ and $(\lambda, \Sigma) \in \mathfrak{F}(V, \Sigma)$ be a fuzzy soft convex set. Then

$$\begin{aligned}\{\xi f[(\lambda,\Sigma)] \oplus (1-\xi) f[(\lambda,\Sigma)]\}(\alpha) &= \{\xi f[(\lambda,\Sigma)]\}(\alpha) + \{(1-\xi) f[(\lambda,\Sigma)]\}(\alpha)\\ &= (\{(\xi f(\lambda(\alpha))) + ((1-\xi) f(\lambda(\alpha)))\}, \Sigma)\\ &\sqsubseteq (f(\lambda(\alpha)), \Sigma) \sqsubseteq f[(\lambda,\Sigma)](\alpha), \forall \alpha \in \Sigma.\end{aligned}$$

$$\{\xi f[(\lambda,\Sigma)]\} \oplus \{(1-\xi) f[(\lambda,\Sigma)]\} \sqsubseteq f[(\lambda,\Sigma)]$$

(ii) Let $(\mu, \Sigma), (\sigma, \Sigma) \in \mathfrak{F}(V, \Sigma)$ be a fuzzy soft convex set in W and let $\xi \in [0, 1]$. Let $(\sigma, \Sigma) = (\xi f^{-1}[(\mu, \Sigma)]) \oplus ((1-\xi) f^{-1}[(\mu, \Sigma)])$. Then $\forall \alpha \in \Sigma$,

$$\begin{aligned}f[(\sigma,\Sigma)](\alpha) &= f[(\xi f^{-1}[(\mu,\Sigma)]) \oplus ((1-\xi) f^{-1}[(\mu,\Sigma)])](\alpha)\\ [f(\sigma)](\alpha) &= \xi f([f^{-1}(\mu)](\alpha)) + (1-\xi) f([f^{-1}(\mu)](\alpha))\\ &= \xi f[f^{-1}(\mu(\alpha))] + (1-\xi) f[f^{-1}(\mu(\alpha))]\\ &\leq \xi\mu(\alpha) + (1-\xi)\mu(\alpha) \leq \mu(\alpha), \forall \alpha \in \Sigma.\end{aligned}$$

Hence $\sigma(\alpha) \leq f^{-1}(\mu(\alpha))$. Therefore $(\sigma, \Sigma) \leq f^{-1}[(\mu, \Sigma)]$. Thus $f^{-1}[(\mu, \Sigma)]$ is a fuzzy soft convex set in V.

□

Notation 3.1. *For simplicity of notation, we use* xy *instead of* $x \cdot y$*.*

4. Fuzzy Vector Soft Topology

In this section, fuzzy vector soft topology and fuzzy enriched soft topology are introduced and some properties are discussed.

Proposition 4.1. *Let* (X, Σ, τ) *be a fuzzy soft topological space. Then the collection* $\tau_\alpha = \{\lambda(\alpha) : (\lambda, \Sigma) \in \tau, \lambda \in I^V\}$ *for each* $\alpha \in \Sigma$ *is a fuzzy topology on* X*.*

Proof. Let $\tau_\alpha = \{\lambda(\alpha) : (\lambda, \Sigma) \in \tau\}$ for each $\alpha \in \Sigma$. Then

(i) $\widetilde{0_X}, \widetilde{1_X} \in \tau$ implies that $\lambda(\alpha) = 0 \in \tau_\alpha, \lambda(\alpha) = 1 \in \tau_\alpha$ for each $\alpha \in \Sigma$.

(ii) Let $\{\lambda_i(\alpha) : i \in J\}$ be a collection of fuzzy sets in τ_α. Then there exists $(\lambda_i, \Sigma) \in \tau$, for all $i \in J$. Since τ is a fuzzy soft topology, $\sqcup_{i\in J}(\lambda_i, \Sigma) = (\sqcup_{i\in J}\lambda_i, \Sigma) \in \tau$. Thus $(\sqcup_{i\in J}\lambda_i)(\alpha) = \vee_{i\in J}\lambda_i(\alpha) \in \tau_\alpha$.

(iii) Let $\lambda(\alpha), \mu(\alpha) \in \tau_\alpha$. Then there exists fuzzy soft sets $(\lambda, \Sigma), (\mu, \Sigma) \in \tau$. Since τ is a fuzzy soft topology, $(\lambda \sqcap \mu, \Sigma) \in \tau$. Thus $(\lambda \sqcap \mu)(\alpha) = \lambda(\alpha) \wedge \mu(\alpha) \in \tau_\alpha$ implies that $\lambda \wedge \mu \in \tau_\alpha$.

Therefore τ_α is a fuzzy topology on X for each $\alpha \in \Sigma$. □

Definition 4.1. A fuzzy soft set $(\lambda, \Sigma) \in \mathfrak{F}(X, \Sigma)$ is said to be a fuzzy pseudo constant soft set if $\lambda(\alpha) = 1$ or $\lambda(\alpha) = 0, \forall\alpha \in \Sigma$. Let $FPC(X, \Sigma)$ denote the set of all fuzzy pseudo constant soft sets over X under the parameter set Σ.

Definition 4.2. A fuzzy enriched soft topology τ is a family of fuzzy soft sets over X satisfying the following properties:

(1) $(\lambda, \Sigma) \in \tau$, $\forall(\lambda, \Sigma) \in FPC(X, \Sigma)$.

(2) If $(\lambda, \Sigma), (\mu, \Sigma) \in \tau$, then $(\lambda, \Sigma) \sqcap (\mu, \Sigma) \in \tau$

(3) If $(\lambda, \Sigma)_i \in \tau, \forall i \in \Delta$, then $\sqcup_{i\in\Delta}(\lambda, \Sigma)_i \in T$.

Then the triplet (X, Σ, τ) is called fuzzy enriched soft topological space.

Example 4.1. *Let $X = \{a, b\}$ be a universal set and $E = \{\alpha, \beta, \gamma\}$ be a set of parameters. Let $\Sigma = \{\alpha, \beta\} \subseteq E$ and $(\lambda_1, \Sigma), (\lambda_2, \Sigma), (\lambda_3, \Sigma) \in \mathfrak{F}(X, \Sigma)$ where*

$$\lambda_1(\alpha) = (1/a, 1/b),\ \ \lambda_1(\beta) = (0/a, 0/b),$$

$$\lambda_2(\alpha) = (0/a, 0/b),\ \ \lambda_2(\beta) = (1/a, 1/b),$$

$$\lambda_3(\alpha) = (0.2/a, 0.3/b),\ \ \lambda_3(\beta) = (0.4/a, 0.4/b),$$

$$\lambda_4(\alpha) = (1/a, 1/b),\ \ \lambda_4(\beta) = (0.4/a, 0.4/b),$$

$$\lambda_5(\alpha) = (0.2/a, 0.3/b),\ \ \lambda_5(\beta) = (0/a, 0/b),$$

$$\lambda_6(\alpha) = (0.2/a, 0.3/b),\ \ \lambda_6(\beta) = (1/a, 1/b),$$

$$\lambda_7(\alpha) = (0/a, 0/b),\ \ \lambda_7(\beta) = (0.4/a, 0.4/b).$$

Hence $\tau = \{\tilde{0}_X, \tilde{1}_X, (\lambda_1, \Sigma), (\lambda_2, \Sigma), (\lambda_3, \Sigma), (\lambda_4, \Sigma), (\lambda_5, \Sigma), (\lambda_6, \Sigma), (\lambda_7, \Sigma)\}$. Therefore (X, Σ, τ) is fuzzy enriched soft topological space.

Proposition 4.2. *Let (X, Σ, τ) be a fuzzy soft topological space and τ_α be a fuzzy topology on X. If $\tau^* = \{(\lambda, \Sigma) \in \mathfrak{F}(X, \Sigma) : \lambda(\alpha) \in \tau_\alpha, \forall \alpha \in \Sigma\}$, then τ^* is a fuzzy enriched soft topology on X.*

Proof.

(i) Since $0_\alpha \in \tau_\alpha$, $\lambda(\alpha) = 0, \forall \alpha \in \Sigma$ and also since $1_\alpha \in \tau_\alpha$, $\lambda(\alpha) = 1, \forall \alpha \in \Sigma$. Thus $(\lambda, \Sigma) \in FPC(X, \Sigma)$. Hence $\forall (\lambda, \Sigma) \in FPC(X, \Sigma), (\lambda, \Sigma) \in \tau^*$.

(ii) Let $(\lambda_1, \Sigma), (\lambda_2, \Sigma) \in \tau^*$. Then $\lambda_1(\alpha), \lambda_2(\alpha) \in \tau_\alpha, \forall \alpha \in \Sigma$. Thus $\lambda_1(\alpha) \wedge \lambda_2(\alpha) \in \tau_\alpha, \forall \alpha \in \Sigma$. Hence $(\lambda_1 \sqcap \lambda_2)(\alpha) \in \tau_\alpha, \forall \alpha \in \Sigma$. Therefore $(\lambda_1, \Sigma) \sqcap (\lambda_2, \Sigma) = ((\lambda_1 \sqcap \lambda_2), \Sigma) \in \tau^*$.

(iii) Let $(\lambda_i, \Sigma) \in \tau^*, \forall i \in J$. Then $\lambda_i(\alpha) \in \tau_\alpha, \forall i \in J, \forall \alpha \in \Sigma$ which implies $\bigvee_{i \in J} \lambda_i(\alpha) \in \tau_\alpha, \forall \alpha \in \Sigma$. So $(\vee_{i \in J} \lambda_i)(\alpha) \in \tau_\alpha, \forall \alpha \in \Sigma$. Hence $\sqcup_{i \in J}(\lambda_i, \Sigma) = (\sqcup_{i \in J} \lambda_i, \Sigma) \in \tau^*$.

Therefore τ^* is fuzzy enriched soft topology on X. □

Definition 4.3. Let V be a vector space over the field K, K be a field of real or complex numbers, $\Sigma \subseteq E$ be the set of parameter and γ_α be the usual fuzzy topology on K, $\forall \alpha \in \Sigma$. Then the fuzzy enriched soft topology γ defined as in Proposition 4.2 is called the usual fuzzy soft topology on K.

Definition 4.4. Let (X, Σ, τ) be a fuzzy soft topological space and let $(\lambda_1, \Sigma), (\lambda_2, \Sigma) \in \mathfrak{F}(X, \Sigma)$. The product of (λ_1, Σ) and (λ_2, Σ) is defined as $(\lambda_1, \Sigma) \widetilde{\times} (\lambda_2, \Sigma) = (\lambda_1 \widetilde{\times} \lambda_2, \Sigma)$ where $(\lambda_1 \widetilde{\times} \lambda_2)(\alpha) = \lambda_1(\alpha) \times \lambda_2(\alpha), \forall \alpha \in \Sigma$. Also $(\lambda_1 \widetilde{\times} \lambda_2, \Sigma)$ is a fuzzy soft set over $X \times X$ under the parameter Σ.

Proposition 4.3. *Let (X, Σ, τ) be a fuzzy soft topological space. Then for each $\alpha \in \Sigma$, $(\tau \widetilde{\times} \tau)_\alpha = \tau_\alpha \times \tau_\alpha$.*

Proof. Let $\alpha \in \Sigma$ and let $\mu \in (\tau \widetilde{\times} \tau)_\alpha$. Then there exists $(\lambda, \Sigma) \in (\tau \widetilde{\times} \tau)$ such that $\lambda(\alpha) = \mu$. Since $(\lambda, \Sigma) \in (\tau \widetilde{\times} \tau)$, it follows that there exist $\{(\sigma_i, \Sigma), (\beta_i, \Sigma) \in \tau, i \in J\}$ such that $(\lambda, \Sigma) = \sqcup_{i \in J}[(\sigma_i, \Sigma) \widetilde{\times} (\beta_i, \Sigma)]$. So $\mu = \lambda(\alpha) = \vee_{i \in I}[\sigma_i(\alpha) \times \beta_i(\alpha)] \in (\tau_\alpha \times \tau_\alpha)$. Therefore, $(\tau \widetilde{\times} \tau)_\alpha \subseteq (\tau_\alpha \times \tau_\alpha)$.

Next, let $\mu \in (\tau_\alpha \times \tau_\alpha)$. Then there exist $\theta_i, \eta_i \in \tau_\alpha, i \in J$ where J is an indexed set, such that $\mu = \vee_{i \in J}[\theta_i \times \eta_i]$ and there exists $(\theta_i, \Sigma), (\eta_i, \Sigma) \in \tau$ such that $\sigma_i(\alpha) = \theta_i$ and $\beta_i(\alpha) = \eta_i, i \in J, \alpha \in \Sigma$. Thus $\sqcup_{i \in J}[(\sigma_i, \Sigma) \widetilde{\times} (\beta_i, \Sigma)] \in (\tau \widetilde{\times} \tau)$ and $\mu = \vee_{i \in J}[\sigma_i(\alpha) \times \beta_i(\alpha)] = \vee_{i \in I}[(\sigma_i \widetilde{\times} \beta_i)(\alpha)] \in (\tau \widetilde{\times} \tau)_\alpha$. So $(\tau_\alpha \times \tau_\alpha) \subseteq (\tau \widetilde{\times} \tau)_\alpha$.

Therefore, for each $\alpha \in \Sigma$, $(\tau \widetilde{\times} \tau)_\alpha = (\tau_\alpha \times \tau_\alpha)$. □

Proposition 4.4. *Let (X, Σ, τ) and (Y, Σ, γ) be any two fuzzy soft topological spaces. Then for each $\alpha \in \Sigma$, $(\tau \widetilde{\times} \gamma)_\alpha = \tau_\alpha \times \gamma_\alpha$.*

Proof. The Proof is similar to the Proposition 4.3. □

Notation 4.1. *Let (X, Σ, τ) and (Y, Σ, γ) be any two fuzzy soft topological spaces. Then $\mathfrak{F}(X \times Y, \Sigma)$ be the family of fuzzy soft sets over $X \times Y$ under the parameter Σ.*

Proposition 4.5. *Let (X, Σ, τ) and (Y, Σ, γ) be any two fuzzy soft topological spaces. Let $\mathbb{T}^* = \{(\lambda, \Sigma) \in \mathfrak{F}(X \times Y, \Sigma) : \lambda(\alpha) \in \tau_\alpha \times \gamma_\alpha, \forall \alpha \in \Sigma\}$. Then $\mathbb{T}^*$ is a fuzzy soft topology over $X \times Y$ and $\mathbb{T}^* = \tau^* \widetilde{\times} \gamma^*$ where $\tau^* = \{(\mu, \Sigma) \in \mathfrak{F}(X, \Sigma) : \mu(\alpha) \in \tau_\alpha, \forall \alpha \in \Sigma\}$ and $\gamma^* = \{(\sigma, \Sigma) \in \mathfrak{F}(Y, \Sigma) : \sigma(\alpha) \in \gamma_\alpha, \forall \alpha \in \Sigma\}$.*

Proof.
(i) $0_{X \times Y}, 1_{X \times Y} \in \tau_\alpha \times \gamma_\alpha, \forall \alpha \in \Sigma$, then $(\widetilde{0_{X \times Y}}, \Sigma), (\widetilde{1_{X \times Y}}, \Sigma) \in \mathbb{T}^*$.
(ii) Let $(\lambda_1, \Sigma), (\lambda_2, \Sigma) \in \mathbb{T}^*$. Then $\lambda_1(\alpha), \lambda_2(\alpha) \in \tau_\alpha \times \gamma_\alpha, \forall \alpha \in \Sigma$. So $[(\lambda_1, \Sigma) \sqcap (\lambda_2, \Sigma)](\alpha) = ((\lambda_1 \sqcap \lambda_2)(\alpha), \Sigma) = ((\lambda_1(\alpha) \wedge \lambda_2(\alpha)), \Sigma)$. Since $\lambda_1(\alpha) \wedge \lambda_2(\alpha) \in \tau_\alpha \times \gamma_\alpha, \forall \alpha \in \Sigma$, $(\lambda_1, \Sigma) \sqcap (\lambda_2, \Sigma) \in \mathbb{T}^*$.
(iii) Let $(\lambda_i, \Sigma) \in \mathbb{T}^*, i \in J$. So $\sqcup_{i \in J}(\lambda_i, \Sigma)(\alpha) = (\sqcup_{i \in J}\lambda_i, \Sigma)(\alpha) = (\vee_{i \in J}\lambda_i(\alpha), \Sigma)$. Since $\vee_{i \in J}\lambda_i(\alpha) \in \tau_\alpha \times \gamma_\alpha, \forall \alpha \in \Sigma$, $\sqcup_{i \in J}(\lambda_i, \Sigma) \in \mathbb{T}^*$.

Therefore $\mathbb{T}^*$ is a fuzzy soft topology over $X \times Y$.
Now let $(\lambda, \Sigma) \in \mathbb{T}^*$ and $\alpha \in \Sigma$. Then $\lambda(\alpha) \in \tau_\alpha \times \gamma_\alpha$ and hence there exist $\mu_i \in \tau_\alpha, \sigma_i \in \gamma_\alpha, i \in J$ such that $\lambda(\alpha) = \vee_{i \in I}(\mu_i \times \sigma_i)$. For each pair $\mu_i \in \tau_\alpha$ and $\sigma_i \in \gamma_\alpha$, there are fuzzy soft sets (θ_i, Σ) and (η_i, Σ) such that $\theta_i(\alpha) = \mu_i, \theta_i(\beta) = 0_X, \forall \alpha, \beta \in \Sigma$ and $\alpha \neq \beta$ and $\eta_i(\alpha) = \sigma_i, \eta_i(\beta) = 0_Y, \forall \alpha, \beta \in \Sigma$ and $\alpha \neq \beta$.
So $(\theta_i, \Sigma) \in \tau^*$ and $(\eta_i, \Sigma) \in \gamma^*$. Hence $(\theta_i \widetilde{\times} \eta_i, \Sigma) \in \tau^* \widetilde{\times} \gamma^*$. Also $(\theta_i \widetilde{\times} \eta_i)(\alpha) = \theta_i(\alpha) \times \eta_i(\alpha) = \mu_i \times \sigma_i$ and $(\mu_i \widetilde{\times} \sigma_i)(\beta) = \mu_i(\beta) \times \sigma_i(\beta) = 0_{X \times Y}, \forall \alpha, \beta \in \Sigma$ and $\alpha \neq \beta$.
Let $(\rho_\alpha, \Sigma) = \sqcup_{i \in J}(\theta_i \widetilde{\times} \eta_i, \Sigma)$. Then $(\rho_\alpha, \Sigma) \in \tau^* \widetilde{\times} \gamma^*$ and $\rho_\alpha(\alpha) = \sqcup_{i \in J}(\theta_i \widetilde{\times} \eta_i)(\alpha) = \vee_{i \in J}\theta_i(\alpha) \widetilde{\times} \eta_i(\alpha) = \vee_{i \in J}\mu_i \times \sigma_i = \lambda(\alpha), \rho_\alpha(\beta) = 0_{X \times Y}, \forall \alpha, \beta \in \Sigma$ and $\alpha \neq \beta$.
Again let $(\rho, \Sigma) = \sqcup_{\alpha \in \Sigma}(\rho_\alpha, \Sigma)$. Then $(\rho, \Sigma) \in \tau^* \widetilde{\times} \gamma^*$ and $\rho(\alpha) = \vee_{\alpha \in \Sigma}\rho_\alpha(\alpha) = \vee_{\alpha \in \Sigma}\lambda(\alpha), \forall \alpha \in \Sigma$. Thus $(\lambda, \Sigma) = (\rho, \Sigma) \in \tau^* \widetilde{\times} \gamma^*$. Therefore $\mathbb{T}^* \subseteq \tau^* \widetilde{\times} \gamma^*$.
Conversely, let $(\lambda, \Sigma) \in \tau^* \widetilde{\times} \gamma^*$. Then there exist $\{(\mu_i, \Sigma) \in \tau^*, (\sigma_i, \Sigma) \in \gamma^*, i \in I\}$ such that $(\lambda, \Sigma) = \sqcup_{i \in I}[(\mu_i, \Sigma) \widetilde{\times} (\sigma_i, \Sigma)]$. Also, $\lambda(\alpha) = \vee_{i \in J}(\mu_i(\alpha) \times \sigma_i(\alpha)) \in \tau_\alpha \times \gamma_\alpha, \forall \alpha \in \Sigma$. Hence $(\lambda, \Sigma) \in \mathbb{T}^*$. Therefore $\tau^* \widetilde{\times} \gamma^* \subseteq \mathbb{T}^*$.

Thus $\mathbb{T}^* = \tau^* \widetilde{\times} \gamma^*$. □

Proposition 4.6. *Let (X, Σ, τ) be a fuzzy soft topological space. Let $\mathbb{S}^* = \{(\lambda, \Sigma) \in \mathfrak{F}(X \times X, \Sigma) : \lambda(\alpha) \in \tau_\alpha \times \tau_\alpha, \forall \alpha \in \Sigma\}$. Then $\mathbb{S}^*$ is a fuzzy soft topology over $X \times X$ and $\mathbb{S}^* = \tau^* \widetilde{\times} \tau^*$ where $\tau^* = \{(\mu, \Sigma) \in \mathfrak{F}(X, \Sigma) : \mu(\alpha) \in \tau_\alpha, \forall \alpha \in \Sigma\}$.*

Definition 4.5. Let V be a vector space over the field K endowed with usual fuzzy soft topology γ, Σ be the parameter set and τ be a fuzzy soft topology on V. Then τ is said to be a fuzzy vector soft topology on V if

(i) $\phi : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined as $\phi(x, y) = x + y$ and
(ii) $\psi : (K \times V, \Sigma, \gamma \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined as $\psi(\xi, x) = \xi x$

$\forall x, y \in V, \forall \xi \in K$, are fuzzy soft continuous functions.

Proposition 4.7. *Let τ be a fuzzy vector soft topology on a vector space V over the field K and let γ be the usual fuzzy soft topology on K. Then τ_α is a fuzzy vector topology on V, $\forall \alpha \in \Sigma$.*

Proof. Let $\mu \in \tau_\alpha$. Then there exists $(\lambda, \Sigma) \in \tau$ such that $\lambda(\alpha) = \mu$. Since τ is a fuzzy vector soft topology, then

(i) $\phi : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined as $\phi(x, y) = x + y$ and
(ii) $\psi : (K \times V, \Sigma, \gamma \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined as $\psi(k, x) = kx$

$\forall x, y \in V, \forall k \in K$, are fuzzy soft continuous functions.
Let $(\lambda, \Sigma) \in \tau$. Then $\lambda(\alpha) \in \tau_\alpha, \forall \alpha \in \Sigma$. Hence, $\phi^{-1}[(\lambda, \Sigma)] = (\phi^{-1}(\lambda), \Sigma) \in (\tau \widetilde{\times} \tau)$ and $\psi^{-1}[(\lambda, \Sigma)] = (\psi^{-1}(\lambda), \Sigma) \in (\gamma \widetilde{\times} \tau)$. So $\phi^{-1}(\lambda(\alpha)) \in (\tau \widetilde{\times} \tau)_\alpha$. By Proposition 4.3, $\phi^{-1}(\lambda(\alpha)) \in \tau_\alpha \times \tau_\alpha, \forall \alpha \in \Sigma$ and $\psi^{-1}[(\lambda, \Sigma)] \in (\gamma \widetilde{\times} \tau)_\alpha$. By Proposition 4.4, $\psi^{-1}[(\lambda, \Sigma)] \in \gamma_\alpha \times \tau_\alpha, \forall \alpha \in \Sigma$. Therefore

(i) $\phi : (V \times V, \tau_\alpha \times \tau_\alpha) \to (V, \tau_\alpha)$ defined as $\phi(x, y) = x + y$ and
(ii) $\psi : (K \times V, \gamma_\alpha \times \tau_\alpha) \to (V, \tau_\alpha)$ defined as $\psi(\xi, x) = \xi x$

$\forall x, y \in V, \forall \xi \in K$, are fuzzy continuous functions, $\forall \alpha \in \Sigma$. Hence τ_α is a fuzzy vector topology on V, $\forall \alpha \in \Sigma$. □

Proposition 4.8. *Let τ be a fuzzy vector soft topology on a vector space V over the field K and let γ be the usual fuzzy soft topology on K. Also $\forall \alpha \in \Sigma$, τ_α is a fuzzy* *vector topology on V. Then τ^* is a fuzzy vector soft topology on V, where τ^* is defined as in Proposition 4.2.*

Proof. Let $(\lambda, \Sigma) \in \tau^*$. Then $\lambda(\alpha) \in \tau_\alpha, \forall \alpha \in \Sigma$. Since $\forall \alpha \in \Sigma, \tau_\alpha$ is a fuzzy vector topology on V whereas γ_α is a usual fuzzy topology on K. Then the mapping $\phi : (V \times V, \tau_\alpha \times \tau_\alpha) \to (V, \tau_\alpha)$ defined by $\phi(x, y) = x + y$ and $\psi : (K \times V, \gamma_\alpha \times \tau_\alpha) \to (V, \tau_\alpha)$ defined as $\psi(\xi, x) = \xi x$ are fuzzy continuous, $\forall x, y \in V, \forall \xi \in K$ and $\forall \alpha \in \Sigma$.
So, $\phi^{-1}(\lambda(\alpha)) \in \tau_\alpha \times \tau_\alpha, \forall \alpha \in \Sigma$ and $\psi^{-1}(\lambda(\alpha)) \in \gamma_\alpha \times \tau_\alpha, \forall \alpha \in \Sigma$. Hence by Proposition 4.5 and Proposition 4.6, $\phi^{-1}(\lambda, \Sigma) \in \mathbb{S}^* = \tau^* \widetilde{\times} \tau^*$ and $\psi^{-1}(\lambda, \Sigma) \in \mathbb{T}^* = \gamma^* \widetilde{\times} \tau^*$.
Thus the mappings $\phi : (V \times V, \Sigma, \tau^* \widetilde{\times} \tau^*) \to (V, \Sigma, \tau^*)$ defined as $\phi(x, y) = x + y$ and $\psi : (K \times V, \Sigma, \gamma^* \widetilde{\times} \tau^*) \to (V, \Sigma, \tau^*)$ defined as $\psi(\xi, x) = \xi x$ are fuzzy soft continuous, $\forall x, y \in V$ and $\forall \xi \in K$. Therefore, τ^* is a fuzzy vector soft topology on V. □

Example 4.2. *Let us consider the vector space $\mathbb{R}$ over the field $K = \mathbb{R}$ where the scalar field $\mathbb{R}$ is equipped with the usual fuzzy soft topology and $\mathbb{R}$ be a universal set. Let $E = \{\alpha, \beta, \gamma, \delta\}$ be a set of parameters. If $\Sigma = \{\alpha, \beta\} \subseteq E$, then τ_α = Indiscrete fuzzy vector topology on $\mathbb{R}$, τ_β = Discrete fuzzy vector topology on $\mathbb{R}$. Then $\tau^* = \{(\lambda, \Sigma) : \lambda(\alpha) \in \tau_\alpha, \forall \alpha \in \Sigma\}$. Therefore τ^* is a fuzzy vector soft topology on the vector space $\mathbb{R}$.*

Proposition 4.9. *Let (X, A, τ) and (Y, A, γ) be any two fuzzy soft topological spaces. Then the constant mapping $f : (X, A, \tau) \to (Y, A, \gamma)$ defined by $f(x) = y_0, \forall x \in X$ where y_0 is a fixed element of Y, is fuzzy soft continuous, if τ contains all those fuzzy pseudo constant soft sets.*

Proof. The proof is similar to the Proposition 2.3 □

Remark 4.1. *Let (X, A, τ) and (Y, A, γ) be any two fuzzy soft topological spaces. If (X, Σ, τ) is fuzzy enriched soft topological space, from Proposition 4.9, the constant mapping $f : (X, A, \tau) \to (Y, A, \gamma)$ is fuzzy soft continuous.*
But the converse is not true (i.e.,) If the constant mapping $f : (X, A, \tau) \to (Y, A, \gamma)$ is fuzzy soft continuous, then (X, Σ, τ) need not be fuzzy enriched soft topological space. It is shown in the following Example 4.3

Example 4.3. *Let* $X = \{a, b, c\}, Y = \{x, y\}$ *be a universal set and* $E = \{\alpha, \beta, \eta\}$ *be the set of parameters. Let* $\Sigma = \{\alpha, \beta\} \subseteq E$ *and* $(\lambda_1, \Sigma), (\lambda_2, \Sigma), (\mu, \Sigma) \in \mathfrak{F}(X, \Sigma)$ *where* $\lambda_1(\alpha) = (0.2/a, 0.2/b, 0/c), \lambda_1(\beta) = (1/a, 1/b, 1/c), \lambda_2(\alpha) = (0/a, 0/b, 0/c), \lambda_2(\beta) = (1/a, 1/b, 1/c)$. *Then* $\tau = \{\tilde{0}_X,\ \tilde{1}_X,\ (\lambda_1, \Sigma), (\lambda_2, \Sigma)\}$ *is a fuzzy soft topology on* X. *Clearly* (X, Σ, τ) *is fuzzy soft topological space. Also let* $\mu(\alpha) = (0/x, 0.2/y), \mu(\beta) = (1/x, 0/y)$. *Therefore* $\gamma = \{\tilde{0}_Y,\ \tilde{1}_Y,\ (\mu, \Sigma)\}$ *is a fuzzy soft topology on* Y. *Clearly* (Y, Σ, γ) *is fuzzy soft topological space. Let* $f : (X, A, \tau) \to (Y, A, \gamma)$ *be defined by* $f(a) = f(b) = f(c) = x$ *where* f *is a constant mapping. For* $(\mu, \Sigma) \in \gamma$, $[f^{-1}(\mu)](\alpha) = (0/a, 0/b, 0/c)$ *and* $[f^{-1}(\mu)](\beta) = (1/a, 1/b, 1/c)$. *Thus* $f^{-1}(\mu, \Sigma) = (\lambda_2, \Sigma) \in \tau$. *Hence* f *is fuzzy soft continuous but* (X, Σ, τ) *is not fuzzy enriched soft topological space.*

Definition 4.6. Let (X, Σ, τ) and (Y, Σ, γ) be any two fuzzy soft topological spaces. Then τ is said to be fuzzy weak enriched soft topology iff any constant mapping $\varpi : (X, \Sigma, \tau) \to (Y, \Sigma, \gamma)$ is fuzzy soft continuous.

Proposition 4.10. *Let* (X, Σ, τ) *be the product space of two fuzzy soft topological spaces* (X_1, Σ, τ_1) *and* (X_2, Σ, τ_2) *respectively, where* τ_2 *(or* τ_1*) is a fuzzy weak enriched soft topology. Let* $\widetilde{a} \in X_1$ *(or* X_2*). Then the mapping* $\varphi : (X_2, \Sigma, \tau_2) \to (X, \Sigma, \tau)$ *(or* $\varphi : (X_1, \Sigma, \tau_1) \to (X, \Sigma, \tau)$*) defined by* $\varphi(x_2) = (\widetilde{a}, x_2)$ *(or* $\varphi(x_1) = (x_1, \widetilde{a})$*) is fuzzy soft continuous* $\forall x_2 \in X_2$ *(or* $\forall x_1 \in X_1$*).*

Proof. Let $\pi_i : (X, \Sigma, \tau) \to (X_i, \Sigma, \tau_i)$ for $i = 1, 2$, be the projection mapping defined by $\pi_i(x_1, x_2) = x_i$ for $i = 1, 2$, where $\forall x_1 \in X_1$ and $\forall x_2 \in X_2$. Let $\pi_1\varphi : (X_2, \Sigma, \tau_2) \to (X_1, \Sigma, \tau_1)$ be such that $\pi_1\varphi(x_2) = \pi_1(\widetilde{a}, x_2) = \widetilde{a}, \forall x_2 \in X_2$ and $\widetilde{a} \in X_1$. By Proposition 2.2, $\pi_1\varphi$ is fuzzy soft continuous. Similarly, $\pi_2\varphi : (X_2, \Sigma, \tau_2) \to (X_2, \Sigma, \tau_2)$ be such that $\pi_2\varphi(x_2) = \pi_2(\widetilde{a}, x_2) = x_2, \forall x_2 \in X_2$ and $\widetilde{a} \in X_1$. Thus by Proposition 2.3, $\pi_2\varphi$ is fuzzy soft continuous. Therefore by Proposition 2.1, φ is fuzzy soft continuous. □

Proposition 4.11. *Let* τ *be a fuzzy vector soft topology on a vector space* V *over the field* K *and let* γ *be the usual fuzzy soft topology on* K. *Also if* τ *is a fuzzy weak enriched soft topology, then the mapping* $\varpi_k : (V, \Sigma, \tau) \to (V, \Sigma, \tau)$ *be defined by* $\varpi_k(x) = kx$ *is a fuzzy soft continuous function* $\forall k \in K$.

Proof. Since τ is a fuzzy vector soft topology on a vector space V, the map $\psi : (K \times V, \Sigma, \gamma \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined as $\psi(k, x) = kx$ is fuzzy soft continuous. Also since τ is a fuzzy weak enriched topology, by Proposition 4.10, $\varphi : (V, \Sigma, \tau) \to (K \times V, \Sigma, \gamma \widetilde{\times} \tau)$ defined by $\varphi(x) = (k, x)$ for a fixed $k \in K$, is fuzzy soft continuous. Hence $\varpi_k = \psi \circ \varphi$ is also fuzzy soft continuous. □

Proposition 4.12. *Let τ be a fuzzy vector soft topology on a vector space V over the field K and let γ be the usual fuzzy soft topology on K. Also if τ is a fuzzy weak enriched soft topology, then the mapping $\vartheta_a : (V, \Sigma, \tau) \to (V, \Sigma, \tau)$ be defined by $\vartheta_a(x) = a + x$ is a fuzzy soft continuous function for any $a \in K$.*

Proof. The proof is similar as in Proposition 4.11. □

Proposition 4.13. *Let τ be a fuzzy vector soft topology on a vector space V over the field K and γ be the usual fuzzy soft topology on K. Also if τ is a fuzzy weak enriched soft topology, then the mapping $\varphi : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V \times V, \Sigma, \tau \widetilde{\times} \tau)$ defined by $\varphi(x, y) = (\kappa_1 x, \kappa_2 y)$ is fuzzy soft continuous for all scalars $\kappa_1, \kappa_2 \in K$ and $x, y \in V$.*

Proof. The projection mappings $\pi_i : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V, \Sigma, \tau), i = 1, 2$ defined as $\pi_1(x, y) = x$ and $\pi_2(x, y) = y, \forall x, y \in V$ are fuzzy soft continuous. Also $\varpi_\kappa : (V, \Sigma, \tau) \to (V, \Sigma, \tau)$ be defined by $\varpi_\kappa(x) = \kappa x$ is a fuzzy soft continuous function $\forall \kappa \in K$.

Now, $\pi_1 \varphi : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined by $(\pi_1 \varphi)(x, y) = \pi_1(\varphi(x, y)) = \pi_1(\kappa_1 x, \kappa_2 y) = \kappa_1 x = \varpi_{\kappa_1} \pi_1(x, y)$. Therefore, $\pi_1 \varphi = \varpi_{\kappa_1} \pi_1$. Since $\varpi_{\kappa_1} \pi_1$ is fuzzy soft continuous, $\pi_1 \varphi$ is also fuzzy soft continuous. Similarly, $\pi_2 \varphi = \varpi_{\kappa_2} \pi_2$ is fuzzy soft continuous. Then from Proposition 2.1, the mapping $\varphi : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V \times V, \Sigma, \tau \widetilde{\times} \tau)$ defined by $\varphi(x, y) = (\kappa_1 x, \kappa_2 y)$ is fuzzy soft continuous for all scalars $\kappa_1, \kappa_2 \in K$ and $x, y \in V$. □

Notation 4.2. *Let vector space V over the field K. Since $0 \in V$, θ is the zero element of V.*

Proposition 4.14. *Let τ be a fuzzy weak enriched soft topology on a vector space V over the field K, where K is endowed with the usual fuzzy soft topology γ. Then τ is* *fuzzy vector soft topology iff the mapping* $\vartheta_{(\kappa_1,\kappa_2)} : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ *defined by* $\vartheta_{(\kappa_1,\kappa_2)}(x, y) = \kappa_1 x + \kappa_2 y$*, is fuzzy soft continuous* $\forall \kappa_1, \kappa_2 \in K$ *and* $\forall x, y \in V$*.*

Proof. Let τ be a fuzzy vector soft topology. Therefore $\phi : (V \times V, \Sigma, \tau_\alpha \widetilde{\times} \tau_\alpha) \to (V, \Sigma, \tau_\alpha)$ defined as $\phi(x, y) = x + y, \forall x, y \in V$ is fuzzy soft continuous. Also, from Proposition 4.13, the mapping $\varphi : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V \times V, \Sigma, \tau \widetilde{\times} \tau)$ defined by $\varphi(x, y) = (\kappa_1 x, \kappa_2 y)$ is fuzzy soft continuous for all scalars $\kappa_1, \kappa_2 \in K$ and $\forall x, y \in V$. Therefore $\vartheta_{(\kappa_1,\kappa_2)} = \phi \circ \varphi : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ is defined by $\vartheta_{(\kappa_1,\kappa_2)}(x, y) = (\phi \circ \varphi)(x, y) = \phi(\varphi(x, y)) = \phi(\kappa_1 x, \kappa_2 y) = \kappa_1 x + \kappa_2 y$, is fuzzy soft continuous function $\forall \kappa_1, \kappa_2 \in K$ and $\forall x, y \in V$ $(i.e.)0 \in V$.
Conversely, let the mapping $\vartheta_{(\kappa_1,\kappa_2)} : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined by $\vartheta_{(\kappa_1,\kappa_2)}(x, y) = \kappa_1 x + \kappa_2 y$, is fuzzy soft continuous function $\forall \kappa_1, \kappa_2 \in K$ and $\forall x, y \in V$.
It is known that $\pi_1 : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined as $\pi_1(x, y) = x$ and $\pi_2 : (K \times V, \Sigma, \gamma \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined as $\pi_2(\kappa, y) = y, \forall x, y \in V$ and $\forall \kappa \in K$, are fuzzy soft continuous functions. Also, since τ is a fuzzy weak enriched soft topology, by Proposition 4.10, $\varphi_0 : (V, \Sigma, \tau) \to (V \times V, \Sigma, \tau \widetilde{\times} \tau)$ defined by $\varphi_0(x) = (x, \theta)$ where θ is the zero element of V.
Therefore, $\psi = \pi_1 \circ \vartheta_{(\kappa_1,\kappa_2)} \circ \varphi_0 \circ \pi_2 : (K \times V, \Sigma, \gamma \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined as

$$
\begin{aligned}
\psi(\kappa, x) &= (\pi_1 \circ \vartheta_{(\kappa_1,\kappa_2)} \circ \varphi_0 \circ \pi_2)(\kappa, x) \\
&= \pi_1(\vartheta_{(\kappa_1,\kappa_2)}(\varphi_0(\pi_2(\kappa, x)))) \\
&= \pi_1(\vartheta_{(\kappa_1,\kappa_2)}(\varphi_0(x))) \\
&= \pi_1(\vartheta_{(\kappa_1,\kappa_2)}(x, \theta)) \\
&= \pi_1(\kappa x, \theta) \\
&= \kappa x
\end{aligned}
$$

is fuzzy soft continuous, $\forall \kappa \in K$ and $\forall x \in V$.
Since, $\vartheta_{(\kappa_1,\kappa_2)} : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined by $\vartheta_{(\kappa_1,\kappa_2)}(x, y) = \kappa_1 x + \kappa_2 y$, is fuzzy soft continuous $\forall \kappa_1, \kappa_2 \in K$ and $\forall x, y \in V$. If $\kappa_1 = 1$ and $\kappa_2 = 1$, then $\phi = \vartheta_{(1,1)} : (V \times V, \Sigma, \tau \widetilde{\times} \tau) \to (V, \Sigma, \tau)$ defined by $\phi(x, y) = \vartheta_{(1,1)}(x, y) = x + y$. Then ϕ is fuzzy soft continuous.
Then τ is fuzzy vector soft topology on V. □

5. Conclusion

In this paper, the properties of fuzzy vector soft topology and fuzzy weak enriched soft topology are introduced. Several properties of fuzzy weak enriched soft topology on a vector space are also discussed. This is just a beginning of studying fuzzy vector soft topological spaces. There is a huge scope of further study in extending the results of fuzzy topological vector spaces in fuzzy soft setting.

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