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ON $\alpha-$ AND α^*- T_0 AND T_1 SEPARATION AXIOMS IN $I-$ FUZZY TOPOLOGICAL SPACES

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Abstract. Šostak and Kubiak introduced $I-$fuzzy topological spaces. Subspaces and products of $I-$fuzzy topological spaces have been introduced and studied by Peeters and Šostak. Srivastava et al. introduced and studied $\alpha-$ and α^*-Hausdorff $I-$fuzzy topological space. George and Veeramani improved the definition of a fuzzy metric, which was first introduced by Kramosil and Michálek. Grecova et al. constructed an $LM-$fuzzy topological space using a strong fuzzy metric, where L and M are complete sublattices of the unit interval [0,1] containing 0 and 1. This $LM-$fuzzy topological space reduces to an $I-$ fuzzy topological space if $L = M = I = [0,1]$. In this paper, we have introduced $\alpha - T_0$, $\alpha^* - T_0$, $\alpha - T_1$ and $\alpha^* - T_1$ separation axioms in $I-$fuzzy topological spaces and established several basic desirable results. In particular, it has been proved that these separation axioms satisfy the hereditary, productive and projective properties. Further, we have proved that in an $I-$fuzzy topological space, $\alpha-$Hausdorff$\Rightarrow \alpha - T_1 \Rightarrow \alpha - T_0$ and α^*-Hausdorff$\Rightarrow \alpha^* - T_1 \Rightarrow \alpha^* - T_0$. It has been also shown that an $I-$fuzzy topological space induced by a strong fuzzy metric is $\alpha-$Hausdorff, for $\alpha \in [0,1)$ and α^*-Hausdorff, for $\alpha \in (0,1]$, which further implies that this $I-$fuzzy topological space satisfies $\alpha - T_0$, $\alpha^* - T_0$, $\alpha - T_1$ and $\alpha^* - T_1$ separation axioms.

1. Introduction

Chang[1] introduced a fuzzy topology on a set X as a collection τ of fuzzy sets in X which contains the constant fuzzy sets 0_X, 1_X and is closed under arbitrary

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suprema and finite infima. Members of τ are called fuzzy open sets and their complements are called fuzzy closed sets. Later on, it was observed that there is no fuzziness involved in the openness and closedness of a fuzzy set. So, Šostak[18] and Kubiak[11] introduced an $I-$fuzzy topology on a set X, as a mapping τ from I^X to I(where, $I = [0, 1]$) satisfying the conditions:(i)$\tau(0_X) = \tau(1_X) = 1$, (ii)$\tau(A_1 \cap A_2) \geq \min\{\tau(A_1), \tau(A_2)\}, \forall A_1, A_2 \in I^X$, (iii)$\tau(\bigcup_{i\in\Omega} A_i) \geq \inf_{i\in\Omega} \tau(A_i)$, where $A_i \in I^X, \forall i \in \Omega$. The pair (X, τ) is called an $I-$fuzzy topological space($I-$fts, in short). Subspaces and products of $I-$fts' have been studied in [16, 18]. A Chang's type fuzzy topological space is called an $I-$topological space($I-$ts, in short). Rodabaugh[17] introduced $\alpha-$Hausdorff $I-$ts. Srivastava et al.[22] introduced and studied $\alpha-$ and α^*-Hausdorff $I-$fts.

A fuzzy metric was first introduced by Kramosil and Michálek[10] in 1975 and later in 1994, their definition was revised by George and Veeramani[2]. Many authors have studied topological structures induced by fuzzy metrics(cf. [2], [3], [5], [6], [7], [8], [13], etc.). Miñana[14] et al. introduced fuzzifying topologies $\tau : 2^X \to [0, 1]$ by using fuzzy pseudo metrics. In [4], authors have generated an $LM-$fuzzy topology $T^m : L^X \to M$ by using a strong fuzzy metric m, where L and M are complete sublattices of the interval [0,1] containing 0 and 1, which reduces to an $I-$fuzzy topological space if $L = M = I = [0, 1]$.

In this paper, we have introduced $\alpha - T_0$, $\alpha^* - T_0$, $\alpha - T_1$ and $\alpha^* - T_1$ separation axioms in $I-$fuzzy topological spaces and established several basic desirable results. In particular, we have shown that these separation axioms satisfy hereditary, productive and projective properties. Further, we have proved that in an $I-$fts, $\alpha-$Hausdorff$\Rightarrow \alpha - T_1 \Rightarrow \alpha - T_0$ and α^*-Hausdorff$\Rightarrow \alpha^* - T_1 \Rightarrow \alpha^* - T_0$. It has been also shown that an $I-$fuzzy topological space induced by a strong fuzzy metric is $\alpha-$Hausdorff, for $\alpha \in [0, 1)$ and α^*- Hausdorff, for $\alpha \in (0, 1]$, which further implies that this $I-$fuzzy topological space satisfies $\alpha - T_0$, $\alpha^* - T_0$, $\alpha - T_1$ and $\alpha^* - T_1$ separation axioms.

2. Preliminaries

A fuzzy set[25] in a non empty set X is a mapping from X to the unit interval $I = [0,1]$. The constant fuzzy set taking value $\alpha \in [0,1]$ is denoted by α_X. The collection of all fuzzy sets in X is denoted by I^X. For basic fuzzy set operations union, intersection, complement and the related results, we refer to [15, 25]. Chang[1] defined a fuzzy topology on a non empty set X as a collection of fuzzy sets in X which contains 0_X, 1_X and is closed under arbitrary unions and finite intersections.

Definition 2.1. [19] A fuzzy point $x_\lambda(0< \lambda <1)$ is a fuzzy set in X such that

$$x_\lambda(x') = \begin{cases} \lambda, & \text{if } x' = x \\ 0, & \text{otherwise.} \end{cases}$$

Here x and λ are respectively called the support and value of x_λ. A fuzzy point x_λ is said to belong to a fuzzy set A if $\lambda < A(x)$ and two fuzzy points x_r and y_s in X are said to be distinct if $x \neq y$.

Definition 2.2. ([18, 11]) Let X be a non empty set. Then an $I-$fuzzy topology is a mapping $\tau : I^X \to I$ such that the following conditions are satisfied:

(1) $\tau(0_X) = \tau(1_X) = 1$.

(2) $\tau(A_1 \cap A_2) \geq \min\{\tau(A_1), \tau(A_2)\}, \forall A_1, A_2 \in I^X$.

(3) $\tau(\bigcup_{i\in\Omega} A_i) \geq \inf_{i\in\Omega} \tau(A_i)$, where $A_i \in I^X, \forall i \in \Omega$.

Then (X,τ) is called an $I-$fuzzy topological space(in short, $I-$fts) and for $A \in I^X$, $\tau(A)$ is called the grade of openness of A.

Peeters[16] defined $\alpha-$ and α^*- cuts of τ in an $I-$fts as follows:

$$[\tau]_\alpha = \{A \in I^X : \tau(A) \geq \alpha\}, \quad [\tau]^*_\alpha = \{A \in I^X : \tau(A) > \alpha\}$$

and proved that $\forall \alpha \in [0,1]$, $[\tau]_\alpha$ is an $I-$topology on X. As mentioned in [22], $[\tau]^*_\alpha$ is a base for an $I-$topology, which is denoted by $\phi[\tau]^*_\alpha$.

Definition 2.3. [16, 17] Let (X,τ) be an $I-$fts and $Y \subseteq X$. Then $(Y, \tau|Y)$ is called a subspace of (X,τ), where $\tau|Y : I^Y \to I$ is defined by $(\tau|Y)(U) = \bigvee\{\tau(V) : V \in I^X, V|Y = U\}$.

Definition 2.4. [18] Let $(X_i, \tau_i)_{i\in\Omega}$ be a family of $I-$fts'. Then their product is (X, τ), where $X = \prod_{i\in\Omega} X_i$ and τ is the initial fuzzy topology on X generated by the family of projection maps, $\{p_i : X \to (X_i, \tau_i)\}_{i\in\Omega}$.

Lemma 2.1. [16] *Let $\{(X_i, \tau_i) : i \in \Omega\}$ be a family of $I-$fts' and $(X \to (X_i, \tau_i))_{i\in\Omega}$ be a fuzzy source. Consider for each $a \in I$, the fuzzy source $(X \to (X_i, [\tau_i]_a))_{i\in\Omega}$. Define $\sigma_a = \{f_i^{\leftarrow}(\xi) : i \in \Omega, \xi \in [\tau_i]_a\}$, then this is subbase for the unique initial $I-$topology on X, which is denoted by τ_a.*

Definition 2.5. [12, 20] Let (X, τ) be an $I-$ts. Then (X, τ) is said to be

(1) fuzzy T_0 if for $x, y \in X$ such that $x \neq y$, there exists $U \in \tau$ such that $U(x) \neq U(y)$.

(2) fuzzy T_1 if for two distinct fuzzy points x_r, y_s in X, there exist $U, V \in \tau$ such that $x_r \in U, x_r \notin V, y_s \notin U, y_s \in V$.

Theorem 2.1. [21] *Let $(X_i, \tau_i)_{i\in\Omega}$ be a family of $I-$ts'. Then their product (X, τ) is fuzzy T_1 iff each coordinate space (X_i, τ_i) is fuzzy T_1.*

We also have the following:

Theorem 2.2. [23] *Let $(X_i, \tau_i)_{i\in\Omega}$ be a family of $I-$ts'. Then their product (X, τ) is fuzzy T_0 iff each coordinate space (X_i, τ_i) is fuzzy T_0.*

Definition 2.6. [24] Let (X, τ) be an $I-$ fts. Then

(1) The degree to which two distinct crisp points $x, y \in X$ such that $x \neq y$ are RT_0 is defined as $RT_0(x, y) = \bigvee_{U(x)\neq U(y)} \tau(U)$. The degree to which (X, τ) is RT_0 is defined by $RT_0(X, \tau) = \bigwedge\{RT_0(x, y) : x \neq y\}$.

(2) The degree to which two distinct crisp points $x, y \in X$ such that $x \neq y$ are KT_1 is defined as $KT_1(x, y) = \bigvee_{U(x)>U(y)} \tau(U) \wedge \bigvee_{V(y)>V(x)} \tau(V)$. The degree to which (X, τ) is KT_1 is defined by $KT_1(X, \tau) = \bigwedge\{KT_1(x, y) : x \neq y\}$.

Theorem 2.3. [22] *Let $\{(X_i, \tau_i)\}_{i\in\Omega}$ be a family of $I-$fts' and (X, τ) be its product. Then $\phi([\tau]^*_\alpha) = \prod \phi([\tau_i]^*_\alpha)$.*

Definition 2.7. [22] Let (X, τ) be an $I-$fts. Then (X, τ) is called $\alpha-$Hausdorff, $\alpha \in [0, 1)$(resp. α^*-Hausdorff, $\alpha \in (0, 1]$) if for each pair of distinct fuzzy points

x_r, y_s in X, there exist $U, V \in [\tau]^*_\alpha$ such that $x_r \in U, y_s \in V$ and $U \cap V = 0_X$(resp. there exist $U, V \in [\tau]_\alpha$ such that $x_r \in U, y_s \in V$ and $U \cap V = 0_X$).

Definition 2.8. [9] A triangular norm or a t-norm is a mapping $* : [0,1] \times [0,1] \to [0,1]$ such that the following conditions are satisfied:

(1) commutativity: $x * y = y * x$, for each $x, y \in [0,1]$;
(2) monotonicity: $y \leq z$ implies that $x * y \leq x * z$, for each $x, y, z \in [0,1]$;
(3) associativity: $(x * y) * z = x * (y * z)$, for each $x, y, z \in [0,1]$;
(4) boundary condition: $x * 1 = x$, for each $x \in [0,1]$.

Definition 2.9. [2] Let X be a set and $*$ be a continuous t-norm. Then a fuzzy metric on a set X is a fuzzy set $m : X^2 \times (0,\infty) \to [0,1]$ such that the following conditions are satisfied:

(1) $m(x,y,t) > 0, \forall x, y \in X$ and $t \in (0,\infty)$.
(2) $m(x,y,t) = 1$ if and only if $x = y, \forall\, t \in (0,\infty)$.
(3) $m(x,y,t) = m(y,x,t), \forall x, y \in X$ and $t \in (0,\infty)$.
(4) $m(x,z,t+s) \geq m(x,y,t) * m(y,z,s), \forall x, y, z \in X$ and $t, s \in (0,\infty)$.
(5) $m(x,y,t) : (0,\infty) \to [0,1]$ is continuous, $\forall x, y \in X$ and $t \in (0,\infty)$.

Then the triple $(X, m, *)$ is called a fuzzy metric space.

Definition 2.10. [4] A fuzzy metric $m : X^2 \times (0,\infty) \to [0,1]$ is called strong if in addition to the conditions (1)-(5) of Definition 2.9, the following stronger versions of conditions (4) and (5)(of Definition 2.9) are satisfied:

(1) $m(x,z,t) \geq m(x,y,t) * m(y,z,t), \forall x, y, z \in X$ and $t \in (0,\infty)$.
(2) $m(x,y,t) : (0,\infty) \to [0,1]$ is continuous and non-decreasing, $\forall x, y \in X$ and $t \in (0,\infty)$.

Definition 2.11. [2] Let $(X, m, *)$ be a fuzzy metric space. Then an open ball $B_m(x,r,t)$ with centre x and radius r is given by:

$$\{y \in X | m(x,y,t) > 1 - r\},$$

where $t > 0$.

Remark 1. [2] *In a fuzzy metric space* $(X, m, *)$*, for any* $r \in (0,1)$*, we can find an* $r_1 \in (0,1)$ *such that* $r_1 * r_1 \geq r$*.*

3. $\alpha-$ AND α^*- T_0 FUZZY TOPOLOGICAL SPACES

In this section, we introduce and study $\alpha - T_0$ and $\alpha^* - T_0$ separation axioms in an $I-$fts.

Definition 3.1. Let (X, τ) be an $I-$fts. Then (X, τ) is said to be $\alpha - T_0, \alpha \in [0, 1)$(resp. $\alpha^* - T_0, \alpha \in (0, 1]$) if for $x, y \in X$ such that $x \neq y$, there exists $U \in [\tau]^*_\alpha$(resp., $U \in [\tau]_\alpha$) such that $U(x) \neq U(y)$.

Proposition 3.1. *An $I-$fts (X, τ) is $\alpha - T_0$ iff the $I-$ts $(X, \phi[\tau]^*_\alpha)$ is fuzzy T_0.*

Proof. Let (X, τ) be $\alpha - T_0$. We have to show that $(X, \phi[\tau]^*_\alpha)$ is fuzzy T_0. Let $x, y \in X$ such that $x \neq y$. Then since (X, τ) is $\alpha - T_0$, there exists $U \in [\tau]^*_\alpha$ such that $U(x) \neq U(y)$. This implies that $U \in \phi[\tau]^*_\alpha$ is such that $U(x) \neq U(y)$. So $(X, \phi[\tau]^*_\alpha)$ is fuzzy T_0.
Conversely, assume that $(X, \phi[\tau]^*_\alpha)$ is fuzzy T_0. We have to show that (X, τ) is $\alpha - T_0$. For this, choose $x, y \in X$ such that $x \neq y$. Since $(X, \phi[\tau]^*_\alpha)$ is fuzzy T_0, so there exists $U \in \phi[\tau]^*_\alpha$ such that $U(x) \neq U(y)$. Further, since $[\tau]^*_\alpha$ is a base for $\phi[\tau]^*_\alpha$, so $U = \bigcup_{i \in \Omega} U_i$, where $U_i \in [\tau]^*_\alpha$. We show that there exists $i_1 \in \Omega$ such that $U_{i_1}(x) \neq U_{i_1}(y)$. Assume the contrary that for each $i \in \Omega$,

$$\begin{aligned} & U_i(x) = U_i(y) \\ \Rightarrow \quad & \sup_{i \in \Omega} U_i(x) = \sup_{i \in \Omega} U_i(y) \\ \Rightarrow \quad & (\bigcup_{i \in \Omega} U_i)(x) = (\bigcup_{i \in \Omega} U_i)(y) \\ \Rightarrow \quad & U(x) = U(y), \end{aligned}$$

which is a contradiction. So there exists $i_1 \in \Omega$ such that $U_{i_1} \in [\tau]^*_\alpha$ and $U_{i_1}(x) \neq U_{i_1}(y)$, showing that (X, τ) is $\alpha - T_0$. $\square$

Proposition 3.2. *An $I-$ts (X, τ) is $\alpha^* - T_0$ iff $I-$ts $(X, [\tau]_\alpha)$ is fuzzy T_0.*

Proof is straight forward, hence omitted.

Definition 3.2. Let (X, τ) be an $I-$fts. Then (X, τ) is called T_0 if (X, τ) is $\alpha - T_0$, $\forall \alpha \in [0, 1)$.

Definition 3.3. [18] Let (X,τ) and (X,δ) be two $I-$fts'. Then (X,δ) is said to be finer than (X,τ) if $\delta(A) \geq \tau(A)$, $A \in I^X$.

Proposition 3.3. *If (X,τ) is $\alpha-$(resp. α^*-)T_0 and (X,δ) is finer than (X,τ), then (X,δ) is also $\alpha-$ (resp. α^*-)T_0*

Proof is easy, hence omitted.

Proposition 3.4. *Let (X,τ) be an $I-$fts. Then (X,τ) is $\alpha - T_0$ implies that $RT_0(X,\tau) > \alpha$.*

Proof. Since (X,τ) is $\alpha - T_0$, so for $x, y \in X$ such that $x \neq y$, there exists $U_1 \in [\tau]^*_\alpha$ such that $U_1(x) \neq U_1(y)$. Thus, $RT_0(X,\tau) = \bigvee_{U(x)\neq U(y)} \tau(U) \geq \tau(U_1) > \alpha$. □

Theorem 3.1. *$\alpha - T_0$ and $\alpha^* - T_0$ in an $I-$fts satisfy hereditary property.*

Proof. Let (X,τ) be $\alpha - T_0$ and $(Y,\tau|Y)$ be a subspace of (X,τ). Then we have to show that $(Y,\tau|Y)$ is $\alpha - T_0$. Let $x, y \in Y$ such that $x \neq y$. Then $x, y \in X$. Since (X,τ) is $\alpha - T_0$, so there exists $U \in [\tau]^*_\alpha$ such that $U(x) \neq U(y)$.
Now set $U_1 = U|Y$. Then $(\tau|Y)(U_1) = \bigvee\{\tau(W) : W \in I^X, W|Y = U_1\} \geq \tau(U) > \alpha$. So, there exists $U_1 \in [\tau|Y]^*_\alpha$ such that $U_1(x) \neq U_1(y)$, showing that $(Y,\tau|Y)$ is $\alpha - T_0$. Similarly it can be shown that $\alpha^* - T_0$ is hereditary. □

Theorem 3.2. *Let $\{(X_i,\tau_i) : i \in \Omega\}$ be a family of $I-$fts'. Then their product (X,τ) is $\alpha - T_0$ iff each coordinate space (X_i,τ_i) is $\alpha - T_0$.*

Proof. Let (X,τ) be $\alpha - T_0$. Then by Proposition 3.1, $(X,\phi[\tau]^*_\alpha)$ is fuzzy T_0.

$$\Rightarrow \quad (X, \prod_{i\in\Omega} \phi[\tau_i]^*_\alpha) \text{ is fuzzy } T_0 \quad \text{(Using Theorem 2.3)}$$
$$\Rightarrow \quad (X_i, \phi[\tau_i]^*_\alpha) \text{ is fuzzy } T_0, \forall i \in \Omega \quad \text{(Using Theorem 2.2)}$$
$$\Rightarrow \quad \text{for } x_i, y_i \in X_i \text{ such that } x_i \neq y_i, \text{there exists } U_i \in \phi[\tau_i]^*_\alpha \text{ such that } U_i(x_i) \neq U_i(y_i)$$
$$\Rightarrow \quad \text{for } x_i, y_i \in X_i \text{ such that } x_i \neq y_i, \text{there exists } B_i \in [\tau_i]^*_\alpha \text{ such that } B_i(x_i) \neq B_i(y_i)$$
$$\Rightarrow \quad (X_i,\tau_i) \text{ is } \alpha - T_0, \forall i \in \Omega.$$

Conversely, assume that (X_i, τ_i) is $\alpha - T_0$, $\forall i \in \Omega$. Then by Proposition 3.1, $(X_i, \phi[\tau_i]^*_\alpha)$ is fuzzy T_0, $\forall i \in \Omega$.

$$\Rightarrow \quad (X, \prod_{i\in\Omega} \phi[\tau_i]^*_\alpha) \text{ is fuzzy } T_0 \quad \text{(Using Theorem 2.2)}$$

$$\Rightarrow \quad (X, \phi[\tau]^*_\alpha) \text{ is fuzzy } T_0 \quad \text{(Using Theorem 2.3)}$$

$$\Rightarrow \quad \text{for } x, y \in X \text{ such that } x \neq y, \text{there exists } U \in \phi[\tau]^*_\alpha \text{ such that } U(x) \neq U(y)$$

$$\Rightarrow \quad \text{for } x, y \in X \text{ such that } x \neq y, \text{there exists } B_U \in [\tau]^*_\alpha \text{ such that } B_U(x) \neq B_U(y)$$

$$\Rightarrow \quad (X, \tau) \text{ is } \alpha - T_0.$$

□

Theorem 3.3. *Let (X_i, τ_i) be a family of $I-$fts'. Then their product (X, τ) is $\alpha^* - T_0$ iff each coordinate space (X_i, τ_i) is $\alpha^* - T_0$.*

Proof. Let (X, τ) be $\alpha^* - T_0$.

$$\Leftrightarrow \quad I - \text{ts}\,(X, [\tau]_\alpha) \text{ is fuzzy } T_0$$

$$\Leftrightarrow \quad I - \text{ts}\,(\prod_{i\in\Omega} X_i, \prod_{i\in\Omega} [\tau_i]_\alpha) \text{ is fuzzy } T_0 \quad \text{(cf. Peeters[16], Corollary 8.11)}$$

$$\Leftrightarrow \quad I - \text{ts}\,(X_i, [\tau_i]_\alpha) \text{ is fuzzy } T_0, \forall i \in \Omega \quad \text{(Using Theorem 2.2)}$$

$$\Leftrightarrow \quad I - \text{ts}\,(X_i, \tau_i) \text{ is } \alpha^* - T_0, \forall i \in \Omega \quad \text{(Using Proposition 3.2).}$$

□

4. $\alpha-$ AND $\alpha^* - T_1$ FUZZY TOPOLOGICAL SPACES

In this section, we introduce and study $\alpha - T_1$ and $\alpha^* - T_1$ separation axioms in an $I-$fts.

Definition 4.1. Let (X, τ) be an $I-$fts. Then (X, τ) is said to be $\alpha - T_1, \alpha \in [0, 1)$(resp. $\alpha^* - T_1, \alpha \in (0, 1]$) if for two distinct fuzzy points x_r, y_s in X, there exist $U, V \in [\tau]^*_\alpha$(resp. $U, V \in [\tau]_\alpha$) such that $x_r \in U, x_r \notin V, y_s \notin U, y_s \in V$.

Definition 4.2. Let (X, τ) be an $I-$fts. Then (X, τ) is called T_1 if (X, τ) is $\alpha - T_1$, $\forall \alpha \in [0, 1)$.

Proposition 4.1. *An $I-$fts (X, τ) is $\alpha - T_1$ iff the $I-$ts $(X, \phi[\tau]^*_\alpha)$ is fuzzy T_1.*

Proof. First assume that (X,τ) is $\alpha-T_1$. Then for two distinct fuzzy points x_r and y_s in X, there exist $U,V\in[\tau]^*_\alpha$ such that

$$\begin{aligned} & x_r\in U, y_s\notin U, x_r\notin V, y_s\in V \\ \Rightarrow\quad & U,V\in\phi[\tau]^*_\alpha \text{ such that } x_r\in U, y_s\notin U, x_r\notin V, y_s\in V \\ \Rightarrow\quad & (X,\phi[\tau]^*_\alpha) \text{ is fuzzy } T_1. \end{aligned}$$

Conversely, assume that $(X,\phi[\tau]^*_\alpha)$ is fuzzy T_1. Then to show that (X,τ) is $\alpha-T_1$, let x_r and y_s be two distinct fuzzy points in X. Since $(X,\phi[\tau]^*_\alpha)$ is fuzzy T_1, so there exist $U,V\in\phi[\tau]^*_\alpha$ such that

$$\begin{aligned} & x_r\in U, y_s\notin U, x_r\notin V, y_s\in V \\ \Rightarrow\quad & \exists B_U,B_V\in[\tau]^*_\alpha \text{ such that } x_r\in B_U\subseteq U,\ y_s\in B_V\subseteq V, y_s\notin B_U \text{ and } x_r\notin B_V \\ & (\text{Since } [\tau]^*_\alpha \text{ is the base for } \phi[\tau]^*_\alpha) \\ \Rightarrow\quad & (X,\tau) \text{ is } \alpha-T_1. \end{aligned}$$

□

Proposition 4.2. *An $I-ts$ (X,τ) is α^*-T_1 iff $I-ts$ $(X,[\tau]_\alpha)$ is fuzzy T_1.*

Proof is straight forward, hence omitted.

Proposition 4.3. *If an $I-fts$ is $\alpha-T_1$, then $KT_1(X,\tau)>\alpha$.*

Proof. Let x_λ and y_λ be two distinct fuzzy points in X. Then $x\neq y$ and since (X,τ) is $\alpha-T_1$, so there exist $U,V\in[\tau]^*_\alpha$ such that

$$\begin{aligned} & x_\lambda\in U, x_\lambda\notin V, y_\lambda\notin U, y_\lambda\in V \\ \Rightarrow\quad & \lambda<U(x), \lambda\geq V(x), \lambda\geq U(y), \lambda<V(y) \\ \Rightarrow\quad & U(x)>U(y) \text{ and } V(y)>V(x) \\ \Rightarrow\quad & \bigvee_{U_1(x)>U_1(y)}\tau(U_1)\geq\tau(U)>\alpha \text{ and } \bigvee_{V_1(y)>V_1(x)}\tau(V_1)\geq\tau(V)>\alpha \\ \Rightarrow\quad & KT_1(X,\tau)=\bigvee_{U_1(x)>U_1(y)}\tau(U_1)\wedge\bigvee_{V_1(y)>V_1(x)}\tau(V_1)>\alpha \end{aligned}$$

□

Theorem 4.1. *$\alpha - T_1$ and $\alpha^* - T_1$ in an $I-$fts satisfy hereditary property.*

Proof. Let (X, τ) be an $\alpha - T_1$ $I-$fts and $(Y, \tau|Y)$ be a subspace of (X, τ). Let x_r and y_s be two distinct fuzzy points in Y. Then x_r and y_s are also distinct fuzzy points in X and since (X, τ) is $\alpha - T_1$, so there exist $U, V \in [\tau]^*_\alpha$ such that $x_r \in U, x_r \notin V, y_s \notin U, y_s \in V$. Set $U_1 = U|Y$ and $V_1 = V|Y$. Then $(\tau|Y)(U_1) = \vee\{\tau(W)|W \in I^X, W|Y = U_1\} \geq \tau(U) > \alpha$ and $(\tau|Y)(V_1) = \vee\{\tau(W)|W \in I^X, W|Y = V_1\} \geq \tau(V) > \alpha$. This implies that U_1, $V_1 \in [\tau|Y]^*_\alpha$ such that $x_r \in U_1$, $y_s \notin U_1$, $x_r \notin V_1$, $y_s \in V_1$. Thus, $(Y, \tau|Y)$ is also $\alpha - T_1$.
Similarly it can be shown that $\alpha^* - T_1$ is hereditary. □

Theorem 4.2. *Let $\{(X_i, \tau_i) : i \in \Omega\}$ be a family of $I-$fts'. Then their product (X, τ) is $\alpha - T_1$ iff each coordinate space (X_i, τ_i) is $\alpha - T_1$.*

Proof. Let (X, τ) be $\alpha - T_1$. Then by Proposition 4.1, $(X, \phi[\tau]^*_\alpha)$ is fuzzy T_1.

$\Rightarrow$ $(X, \prod_{i\in\Omega} \phi[\tau_i]^*_\alpha)$ is fuzzy T_1 (Using Theorem 2.3)

$\Rightarrow$ $(X_i, \phi[\tau_i]^*_\alpha)$ is fuzzy $T_1, \forall i \in \Omega$ (Using Theorem 2.1)

$\Rightarrow$ for distinct fuzzy points $(x_i)_r, (y_i)_s$ in X_i, there exist $U_i, V_i \in \phi[\tau_i]^*_\alpha$ such that $(x_i)_r \in U_i, (y_i)_s \notin U_i, (x_i)_r \notin V_i, (y_i)_s \in V_i$

$\Rightarrow$ for distinct fuzzy points $(x_i)_r, (y_i)_s$ in X_i, there exist $U_{ij_1}, V_{ij_2} \in [\tau_i]^*_\alpha$ such that $(x_i)_r \in U_{ij_1}, (y_i)_s \notin U_{ij_1}, (x_i)_r \notin V_{ij_2}, (y_i)_s \in V_{ij_2}$ (Since $[\tau_i]^*_\alpha$ is a base for $\phi[\tau_i]^*_\alpha$)

$\Rightarrow$ (X_i, τ_i) is $\alpha - T_1, \forall i \in \Omega$.

Conversely, assume that (X_i, τ_i) is $\alpha - T_1$, $\forall i \in \Omega$.

$\Rightarrow$ $(X_i, \phi[\tau_i]^*_\alpha)$ is fuzzy $T_1, \forall i \in \Omega$

$\Rightarrow$ $(X, \prod_{i\in\Omega} \phi[\tau_i]^*_\alpha)$ is fuzzy T_1, (Using Theorem 2.1)

$\Rightarrow$ $(X, \phi[\tau]^*_\alpha)$ is fuzzy T_1 (Using Theorem 2.3)

$\Rightarrow$ for x_r, y_s in X, there exist $U, V \in \phi[\tau]^*_\alpha$ such that $x_r \in U, x_r \notin V, y_s \notin U, y_s \in V$

$\Rightarrow$ for x_r, y_s in X, there exist $B_U, B_V \in [\tau]^*_\alpha$ such that $x_r \in B_U, x_r \notin B_V, y_s \notin B_U, y_s \in B_V$

$\Rightarrow$ (X, τ) is $\alpha - T_1$.

□

Theorem 4.3. *Let $(X_i, \tau_i)_{i\in\Omega}$ be a family of $I-$fts'. Then their product (X, τ) is $\alpha^* - T_1$ iff each coordinate space (X_i, τ_i) is $\alpha^* - T_1$.*

This can be proved on the similar lines as in Theorem 3.3, in the case of $\alpha^* - T_0$.

Proposition 4.4. *Let (X, τ) be an $I-$fts. Then $\alpha-$Hausdorff $\Rightarrow \alpha - T_1 \Rightarrow \alpha - T_0$.*

Proof. Let (X, τ) be an $\alpha-$Hausdorff $I-$fts. Then for two distinct fuzzy points x_r and y_s in X, there exist U, $V \in [\tau]^*_\alpha$ such that $x_r \in U$, $y_s \in V$ and $U \cap V = 0_X$. Here $x_r \in U$, $x_r \notin V$, $y_s \in V$, $y_s \notin U$, implying that (X, τ) is $\alpha - T_1$.
Next, assume that (X, τ) is $\alpha - T_1$. Then for two distinct fuzzy points x_r and y_s in X, there exist U, $V \in [\tau]^*_\alpha$ such that $x_r \in U$, $x_r \notin V$, $y_s \in V$, $y_s \notin U$. We show that (X, τ) is $\alpha - T_0$. Let $x, y \in X$ such that $x \neq y$. Then x_λ and y_λ are two distinct fuzzy points in X, so there exist U, $V \in [\tau]^*_\alpha$ such that $x_\lambda \in U$, $x_\lambda \notin V$, $y_\lambda \in V$, $y_\lambda \notin U$.

$$\begin{aligned} &\Rightarrow \quad \lambda < U(x), \lambda < V(y), \lambda \geq V(x), \lambda \geq U(y) \\ &\Rightarrow \quad U(x) \neq U(y) \,\text{and}\, V(x) \neq V(y) \\ &\Rightarrow \quad (X, \tau) \,\text{is}\, \alpha - T_0. \end{aligned}$$

□

The converse of the above Proposition 4.4 does not hold good as can be seen in the following counter examples.

Example 4.1. *Let $X = \{a, b\}$ and $\tau : I^X \to I$ be given by*

$$\tau(A) = \begin{cases} 1, & \textit{if } A = 0_X, 1_X, \chi_{\{a\}} \\ 0, & \textit{otherwise.} \end{cases}$$

Then (X, τ) is an $I-$fts which is $\alpha - T_0$, $\forall \alpha \in [0, 1)$, since for $a, b \in X, a \neq b$, $\exists \chi_{\{a\}} \in [\tau]^*_\alpha$ such that $\chi_{\{a\}}(a) \neq \chi_{\{a\}}(b)$ but it is not $\alpha - T_1$ since for two distinct fuzzy points a_r, b_s in X, there does not exist any $V \in [\tau]^*_\alpha$ such that $b_s \in V$ and $a_r \notin V$.

Example 4.2. *Let X be an infinite set and C_F be a fuzzy set in X given by*

$$C_F(x) = \begin{cases} 1, & \text{if } x \notin F \\ 0, & \text{otherwise.} \end{cases}$$

where F is a finite subset of X. Let $\tau : I^X \to I$ be given by

$$\tau(A) = \begin{cases} 1, & \text{if } A = 0_X \text{ or } C_F \\ 0, & \text{otherwise.} \end{cases}$$

Then (X, τ) is an $I-$fts as shown below:

(1) $\tau(0_X) = 1, \tau(1_X) = \tau(C_\phi) = 1.$

(2) *Let $A, B \in I^X$. Then we have the following cases:*

Case I: If ($A = 0_X$ or C_F) and ($B = 0_X$ or C_G), where F and G are finite subsets of X, then $\tau(A) = 1, \tau(B) = 1$ and $\tau(A \cap B) = 1$.

Case II: If ($A = 0_X$ or C_F) and ($B \neq 0_X$ or C_G), where F and G are finite subsets of X, then $\tau(A) = 1, \tau(B) = 0$ and $\tau(A \cap B) = 0$ or 1.

Case III: If ($A \neq 0_X$ or C_F) and ($B = 0_X$ or C_G), where F and G are finite subsets of X, then $\tau(A) = 0, \tau(B) = 1$ and $\tau(A \cap B) = 0$ or 1..

Case IV: If A and B are different from 0_X and C_F, where F is a finite subset of X then $\tau(A) = 0, \tau(B) = 0$ and $\tau(A \cap B) = 0$.

In all the cases, we have $\tau(A \cap B) \geq \min\{\tau(A), \tau(B)\}$.

(3) *Let $\{A_i\}_{i \in \Omega} \in I^X$. Then similarly proceeding as in (2), we can show that $\tau(\bigcup_{i \in \Omega} A_i) \geq \inf_{i \in \Omega} \tau(A_i)$.*

So, (X, τ) is an $I-$fts which is $\alpha - T_1$, $\alpha \in [0, 1)$, since for two distinct fuzzy points x_r, y_s in X, there exist $C_{\{y\}}$, $C_{\{x\}} \in [\tau]^*_\alpha$ such that $x_r \in C_{\{y\}}, y_s \notin C_{\{y\}}, y_s \in C_{\{x\}}, x_r \notin C_{\{x\}}$, but it is not $\alpha-$Hausdorff since for two distinct fuzzy points x_r, y_s in X, there does not exist any pair of disjoint $U, V \in [\tau]^*_\alpha$ such that $x_r \in U, y_s \in V$.

Proposition 4.5. *Let (X, τ) be an $I-$fts. Then α^*-Hausdorff $\Rightarrow \alpha^* - T_1 \Rightarrow \alpha^* - T_0$.*

This can be proved on the similar lines as in Theorem 4.4, in the case of $\alpha-$Hausdorff.

5. I-fts induced by a strong fuzzy metric

In this section, we show that $I-$fts induced by a strong fuzzy metric is $\alpha-$Hausdorff, $\alpha \in [0,1)$ and α^*-Hausdorff, $\alpha \in (0,1]$, which further implies that this $I-$fts satisfies $\alpha - T_0$, $\alpha^* - T_0$, $\alpha - T_1$ and $\alpha^* - T_1$ separation axioms.
In [4], authors have constructed an $LM-$fuzzy topology $T^m : L^X \to M$ on X by using a strong fuzzy metric space $(X, m, *)$, where L and M are complete sublattices of the unit interval [0,1] containing 0 and 1. In our discussion, we take $L = M = [0,1] = I$, so this $LM-$fuzzy topology reduces to an $I-$fuzzy topology $T^m : I^X \to I$.

Now, we recall some results which are already proved in [14] and [4].

Theorem 5.1. [14] *Let $\alpha \in (0,1)$ and $U \in 2^X$. Then $U \in T^m_\alpha$ iff for each $x \in U$, $\exists \delta \in (0,1)$ such that $B_m(x, \delta, t) \subseteq U$, where $t = \phi^{-1}(\alpha)$ and ϕ is a strictly increasing continuous bijection between $(0,\infty)$ and $(0,1)$. That is, the topology T^m_α is generated by the base $B_\alpha = \{B_m(x, r, t) | x \in X, r \in (0,1)\}$, where $t = \phi^{-1}(\alpha)$.*

Theorem 5.2. [4] *Let $(X, m, *)$ be a strong fuzzy metric space. Then $T^m : I^X \to I$ given by $T^m(A) = sup\{\alpha : A \in \omega(T^m_\alpha)\}$, $\forall A \in I^X$, is an I-fuzzy topology on X, where $\omega(T^m_\alpha)$ is the family of all lower semicontinuous maps from (X, T^m_α) to $[0,1]$ and $[0,1]$ is equipped with the subspace topology of $\mathbb{R}$.*

Here, T^m is called the $I-$fuzzy topology on X induced by the strong fuzzy metric metric m.
Next, we prove the following result by proceeding in the similar way as in the proof of Theorem 3.5 in [2].

Proposition 5.1. *Let $(X, m, *)$ be a fuzzy metric space. Then (X, T^m_α) is Hausdorff, for every $\alpha \in (0,1)$.*

Proof. Let $\alpha \in (0,1)$, $x, y \in X$ such that $x \neq y$. Then $m(x, y, 2t) = r > 0$, where $r \in (0,1)$ and $t = \phi^{-1}(\alpha)$. For each r_0, $r < r_0 < 1$, we can find an $r_1 \in (0,1)$ such that $r_1 * r_1 \geq r_0$, by Remark 1. Now, consider the open balls $B(x, 1 - r_1, t)$ and $B(y, 1 - r_1, t)$ such that $x \in B(x, 1 - r_1, t)$ and $y \in B(y, 1 - r_1, t)$. Further, we prove that $B(x, 1 - r_1, t) \cap B(y, 1 - r_1, t) = \phi$. For if there exists $z \in B(x, 1 - r_1, t) \cap B(y, 1 -$

$r_1, t)$, then

$$\begin{aligned} r &= m(x, y, 2t) \\ &\geq m(x, z, t) * m(z, y, t) \\ &\geq r_1 * r_1 \\ &\geq r_0 > r \end{aligned}$$

which is a contradiction. Therefore, (X, T^m_α) is Hausdorff, for every $\alpha \in (0, 1)$. □

Theorem 5.3. *Let $(X, m, *)$ be a strong fuzzy metric space. Then the I-fts (X, T^m) induced by a strong fuzzy metric m is $\alpha-$Hausdorff, $\alpha \in [0, 1)$ and α^*-Hausdorff, $\alpha \in (0, 1]$.*

Proof. Let x_r and y_s be two fuzzy distinct points in X. To show that (X, T^m) is $\alpha-$Hausdorff, $\alpha \in [0, 1)$(respectively α^*-Hausdorff, $\alpha \in (0, 1]$) we have to find $A, B \in [T^m]^*_\alpha$(resp. $A, B \in [T^m]_\alpha$) such that $x_r \in A, y_s \in B$ and $A \cap B = 0_X$.
Let $\beta \in (0, 1)$. Since $x \neq y$ and by using Proposition 5.1, (X, T^m_β) is Hausdorff, so there exists $B_x = B(x, r_1, t)$ and $B_y = B(y, r_2, t)$ in T^m_β, where $r_1, r_2 \in (0, 1)$ and $t = \phi^{-1}(\beta)$, such that

$$x \in B_x, y \in B_y \text{ and } B_x \cap B_y = \phi.$$

Let $A : X \to [0, 1]$ and $B : X \to [0, 1]$ be two mappings given by

$$A(u) = \begin{cases} 1, & \text{if } u \in B_x \\ 0, & \text{otherwise.} \end{cases}$$

and

$$B(v) = \begin{cases} 1, & \text{if } v \in B_y \\ 0, & \text{otherwise.} \end{cases}$$

Since $A^{-1}(\gamma, 1] = B_x \in T^m_\beta$ and $B^{-1}(\gamma, 1] = B_y \in T^m_\beta$, for $\gamma \in [0, 1]$, so this implies that A and B are left continuous mappings from (X, T^m_β) to [0,1], where [0,1] is equipped with the subspace topology of $\mathbb{R}$.

Since

$$T^m(A) = \sup\{\beta : A \in \omega(T^m_\beta)\} = 1 > \alpha, \forall \alpha \in [0,1),$$

$$T^m(A) = \sup\{\beta : A \in \omega(T^m_\beta)\} = 1 > \alpha, \forall \alpha \in [0,1),$$

$$T^m(A) = \sup\{\beta : A \in \omega(T^m_\beta)\} = 1 \geq \alpha, \forall \alpha \in (0,1],$$

$$T^m(A) = \sup\{\beta : A \in \omega(T^m_\beta)\} = 1 \geq \alpha, \forall \alpha \in (0,1],$$

so this implies that $A, B \in [T^m]^*_\alpha$ and $[T^m]_\alpha$, for $\alpha \in [0,1)$ and $\alpha \in (0,1]$, respectively. Further note that $x_r \in A$ and $y_s \in B$. Finally, we show that $A \cap B = 0_X$ i.e., $\min\{A(u), B(u)\} = 0$, $\forall u \in X$.

Let $u \in X$. Then since $B_x \cap B_y = \phi$, so the following cases are possible:

(1) $u \in B_x, u \notin B_y$.

(2) $u \notin B_x, u \in B_y$.

(3) $u \notin B_x, u \notin B_y$.

In all the cases, it is easy to verify that $\min\{A(u), B(u)\} = 0$. Therefore, (X, T^m) induced by a strong fuzzy metric m is $\alpha-$Hausdorff, $\alpha \in [0,1)$ and α^*-Hausdorff, $\alpha \in (0,1]$. □

6. Conclusion

The notion of $I-$fuzzy topological spaces was introduced by Šostak[18] and Kubiak[11]. Peeters and Šostak[16, 18] introduced and studied subspaces and products of $I-$fuzzy topological spaces. Srivastava et al.[22] introduced and studied $\alpha-$ and α^*-Hausdorff $I-$fuzzy topological space. The improved definition of a fuzzy metric was given by George and Veeramani[2]. In [4], authors have generated an $LM-$fuzzy topology by using a strong fuzzy metric, where L and M are complete sublattices of [0,1] containing 0 and 1. This $LM-$fuzzy topological space reduces to an $I-$ fuzzy topological space if $L = M = I = [0,1]$. In this paper, we have introduced $\alpha - T_0$, $\alpha^* - T_0$, $\alpha - T_1$ and $\alpha^* - T_1$ separation axioms in $I-$fuzzy topological spaces and established several basic desirable results. In particular, it has been proved that these separation axioms satisfy the hereditary, productive and projective properties. Further, we have proved that in an $I-$fuzzy topological space, $\alpha-$Hausdorff$\Rightarrow \alpha - T_1 \Rightarrow \alpha - T_0$ and α^*-Hausdorff $\Rightarrow \alpha^* - T_1 \Rightarrow \alpha^* - T_0$. It has been also shown that an $I-$fuzzy topological space induced by a strong fuzzy metric is α-Hausdorff, for $\alpha \in [0,1)$ and α^*-Hausdorff, for $\alpha \in (0,1]$, which further implies that this $I-$fuzzy topological space satisfies $\alpha - T_0$, $\alpha^* - T_0$, $\alpha - T_1$ and $\alpha^* - T_1$ separation axioms.

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