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ON MAXIMAL IDEAL SPACE OF THE FUNCTIONALLY COUNTABLE SUBRING OF $C(\mathcal{F})$

AMIR VEISI

Abstract. Let X be a Tychonoff space and $\mathcal{F}$, a filter base of dense subsets of X (i.e., it is closed under finite intersection) and let $C(\mathcal{F}) = \lim_{S\in\mathcal{F}} C(S)$, where $C(S)$ is the ring of all real-valued continuous functions on S. It is known that $C(\mathcal{F}) = \bigcup\{C(S) : S \in \mathcal{F}\}$. By $C_c(\mathcal{F})$ ($C^*_c(\mathcal{F})$), we mean a subring of $C(\mathcal{F})$ consisting of (bounded) functions with countable range. In this paper, we study $\mathcal{M}_c$ ($\mathcal{M}^*_c$), the maximal ideal space of $C_c(\mathcal{F})$ ($C^*_c(\mathcal{F})$) with the hull-kernel topology. Equivalent topology for each of them provided. It is shown that both $\mathcal{M}_c$ and $\mathcal{M}^*_c$ are T_4-spaces. More generally, they are homeomorphic. Particularly, we prove that the maximal ideal space of $Q_c(X)$ ($q_c(X)$) and the maximal ideal space of $Q^*_c(X)$ ($q^*_c(X)$) are homeomorphic, where $Q_c(X)$ ($q_c(X)$) is the maximal (classical) ring of quotients of $C_c(X)$, and $Q^*_c(X)$ ($q^*_c(X)$) is the subring consisting of bounded functions.

1. Introduction

Throughout this paper, X denotes a zero-dimensional Hausdorff space, that is, a Hausdorff space with a base of clopen (closed-open) sets. By $C(X)$ (resp. $C^*(X)$), we mean the ring of all real-valued continuous (resp. bounded) functions on X. The subring of $C(X)$ consisting of those functions with countable (resp. finite) image, which is denoted by $C_c(X)$ (resp., $C^F(X)$) is an $\mathbb{R}$-subalgebra of $C(X)$. The subring $C^*_c(X)$ of $C_c(X)$ consists of bounded elements of $C_c(X)$, in fact, we have $C^*_c(X) = C_c(X) \cap C^*(X)$. The rings $C_c(X)$ and $C^F(X)$ are introduced and studied in [6, 7]. We may concentrate on the class of zero-dimensional spaces because it is shown

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in [6, Theorem 4.6] that for any topological space X (not necessarily completely regular), there exists a zero-dimensional space Y which is a continuous image of X with $C_c(X) \cong C_c(Y)$ and $C^F(X) \cong C^F(Y)$. Also, in [6, Remark 7.5], there is a topological space X, such that there is no space Y with $C_c(X) \cong C(Y)$. If $S \subseteq X$, then S is also a zero-dimensional Hausdorff space. For $f \in C_c(S)$, the sets $Z(f) = \{x \in S : f(x) = 0\}$, and its set-theoretic complement $\operatorname{coz}(f) = \{x \in S : f(x) \neq 0\}$, are the zeroset and cozeroset of f, respectively.

From now on, $\mathcal{F}$ is a base for a filter on X consisting of dense subsets of X. Note that $\mathcal{F}$ is closed under finite intersection, i.e., $S_1 \cap S_2 \in \mathcal{F}$ whenever $S_1, S_2 \in \mathcal{F}$. Let

$$C_c(\mathcal{F}) = \lim_{S \in \mathcal{F}} C_c(S), \text{ and } C_c^*(\mathcal{F}) = \lim_{S \in \mathcal{F}} C_c^*(S).$$

In the above notions, the use of the limit is to describe that the rings are direct limits of rings of continuous functions. Here, the direct limit can be described as the union of the rings $C_c(S)$ modulo the equivalence that $f_1 \in C_c(S_1)$; $f_2 \in C_c(S_2)$ are considered equivalent if they agree on $S_1 \cap S_2$. Therefore,

$$C_c(\mathcal{F}) = \bigcup\{C_c(S) : S \in \mathcal{F}\}, \text{ and } C_c^*(\mathcal{F}) = \bigcup\{C_c^*(S) : S \in \mathcal{F}\},$$

see [5, 2.4]. If $\mathcal{F} = \{X\}$, then $C_c(X) = C_c(\mathcal{F})$. In this section, some definitions, preliminaries and concepts are stated. In Section 2, we study the hull-kernel topology on $\mathcal{M}_c$ and $\mathcal{M}_c^*$, the set of maximal ideals of $C_c(\mathcal{F})$ and $C_c^*(\mathcal{F})$ respectively. It is proved that each of them is compact and Hausdorff (and hence a T_4-space). In Section 3, we introduce the real-valued functions $\widehat{f} \in C(\mathcal{M}_c^*)$ (resp. $\widetilde{f} \in C(\mathcal{M}_c)$) for obtaining the equivalent topologies with the hull-kernel topology on $\mathcal{M}_c^*$ (resp. $\mathcal{M}_c$). These are in fact the weak topologies induced by $\widehat{w} := \{\widehat{f} : f \in C_c^*(\mathcal{F})\}$ (resp. $\widetilde{w} := \{\widetilde{f} : f \in C_c^*(\mathcal{F})\}$), where $\widehat{f}(M) = M(f)$, $\|f\| = \|\widehat{f}\|$, also, for $M \in \mathcal{M}_c$, $\widetilde{f}(M)$ is the unique real number such that $|M(f) - \widetilde{f}(M)|$ is infinitely small or zero. In the final section, we prove that the T_4-spaces $\mathcal{M}_c$ and $\mathcal{M}_c^*$ are homeomorphic. In particular, we show that $\max(Q_c(X)) \cong \max(Q_c^*(X))$ and $\max(q_c(X)) \cong \max(q_c^*(X))$, where $Q_c(X)$ (resp. $q_c(X)$) is the maximal (resp. classical) ring of quotients of $C_c(X)$, and $Q_c^*(X)$ (resp. $q_c^*(X)$) is the subring consisting of bounded functions.

Let F be a totally ordered field. Then F is said to be *archimedean* if $\mathbb{Z}$, the set of integers is cofinal, i.e., for every $x \in F$, there exists $n \in \mathbb{Z}$ such that $n \geq x$.

Theorem 1.1. ([8, Theorem 0.21]) An ordered field is archimedean if and only if it is isomorphic to a subfield of the ordered field $\mathbb{R}$.

An element $a \in F$, is called an *infinitely large* element (resp. *infinitely small* element) whenever $a > n$ (resp. $0 < a < \frac{1}{n}$), for each $n \in \mathbb{N}$ (the set of natural numbers). Obviously, a is infinitely large if and only if $\frac{1}{a}$ is infinitely small. Therefore, F is non-archimedean if and only if it contains infinitely large elements. So in this case, F contains infinitely small elements (see also [8, 5.6, and 5.7]). In a partially ordered set A, the symbol $a \vee b$ denotes $\sup\{a, b\}$, i.e., the smallest element c, if one exists, such that $c \geq a$ and $c \geq b$. Likewise, $a \wedge b$ stands for $\inf\{a, b\}$. When both $a \vee b$ and $a \wedge b$ exist, for all $a, b \in A$, then A is called a *lattice*.

In a similar way of [8, 1.12], we obtain the following corollaries.

Corollary 1.1. *$f \in C_c(\mathcal{F})$ is a unit if and only if $Z(f) = \emptyset$.*

Corollary 1.2. *$f \in C_c^*(\mathcal{F})$ is a unit if and only if it is bounded away from zero, i.e., $|f| \geq r$ for some $r > 0$.*

We also need the next results.

Theorem 1.2. ([17, Theorem 17.10]) A compact Hausdorff space X is a T_4-space.

Theorem 1.3. ([17, Theorem 17.14]) A one-to-one continuous map from a compact space X onto a Hausdorff space Y is a homeomorphism.

2. The hull-kernel topology on $\mathcal{M}_c$ and $\mathcal{M}_c^*$

We denote by $\mathcal{M}_c$ and $\mathcal{M}_c^*$, the set of maximal ideals of $C_c(\mathcal{F})$ and $C_c^*(\mathcal{F})$ respectively, i.e., $\mathcal{M}_c = \{M : \text{M is a maximal ideal in } C_c(\mathcal{F})\}$ and $\mathcal{M}_c^* = \{M : \text{M is a maximal ideal in } C_c^*(\mathcal{F})\}$.

Definition 2.1. *For $f, g \in C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$), we call $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in S_f \cap S_g$ whenever S_f and S_g are the domain of f and g. Also, for $M(f), M(g) \in \frac{C_c(\mathcal{F})}{M}$ (resp. $\frac{C_c^*(\mathcal{F})}{M}$), we call $M(f) \leq M(g)$ if and only if $f \leq g$ whenever $f, g \in C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$) and $M \in \mathcal{M}_c$ (resp. $\mathcal{M}_c^*$)*

By the above definition, $C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$) is a partially ordered set. To see the transitivity of $\leq$, let $f \leq g$ and $g \leq h$. Then $f(x) \leq g(x)$ for all $x \in S_f \cap S_g$, and $g(x) \leq h(x)$ for all $x \in S_g \cap S_h$. So $f(x) \leq h(x)$ for all $x \in S_f \cap S_g \cap S_h$. Notice that if $Z \subseteq Y \subseteq X$ and Z is dense in X, then Z is dense in Y. So the density of $S_f \cap S_g \cap S_h$ in X gives its density in $S_f \cap S_h$. Now, for $x \in S_f \cap S_h$, there is a net $(x_\lambda)_{\lambda\in\Lambda} \subseteq S_f \cap S_g \cap S_h$ such that $x_\lambda \to x$, and $f(x_\lambda) \leq h(x_\lambda)$. It gives $f(x) \leq h(x)$, i.e., $f \leq h$ on $S_f \cap S_h$. Moreover, since $C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$) contains $f \vee g = 1/2(f + g + |f - g|)$ and $f \wedge g = -(-f \vee -g)$, it is a lattice-ordered ring, see [8, 0.5, and 0.19].

Corollary 2.1. *The above definition plus letting $M(f) \vee M(g) = M(f \vee g)$ and $M(f) \wedge M(g) = M(f \wedge g)$, turns $\frac{C_c(\mathcal{F})}{M}$ (resp. $\frac{C_c^*(\mathcal{F})}{M}$) into a lattice-ordered ring. In particular, $|M(f)| := M(f) \vee M(-f) = M(f \vee -f) = M(|f|)$.*

Corollary 2.2. *The canonical (ring) homomorphisms*

$$\varphi : C_c(\mathcal{F}) \to \frac{C_c(\mathcal{F})}{M} \quad (M \in \mathcal{M}_c),\ \varphi^* : C_c^*(\mathcal{F}) \to \frac{C_c^*(\mathcal{F})}{M} \quad (M \in \mathcal{M}_c^*),$$

given by $\varphi(f) = M(f)$, and $\varphi^*(f) = M(f)$ are lattice homomorphisms. Moreover, each of the fields $\frac{C_c(\mathcal{F})}{M}$ and $\frac{C_c^*(\mathcal{F})}{M}$ contains a copy of $\mathbb{R}$ (the set of real numbers).

Proof. By Corollary 2.1, φ and φ^* are lattice-homomorphism. For the second part, we note that $\mathbb{R} \cong \{M(r) : r \in \mathbb{R}\} \subseteq \frac{C_c^*(\mathcal{F})}{M}$. Also, $\mathbb{R} \cong \{M(r) : r \in \mathbb{R}\} \subseteq \frac{C_c(\mathcal{F})}{M}$. □

Corollary 2.3. *If M is a maximal ideal in $C_c^*(\mathcal{F})$, then $\frac{C_c^*(\mathcal{F})}{M} \cong \mathbb{R}$.*

Proof. Let $f \in C_c^*(\mathcal{F})$. Then there exists a natural number n and $S \in \mathcal{F}$ such that $\|f\| = \sup_{x\in S}|f(x)| \leq n$. Since $|f| \leq n$, it gives $|M(f)| = M(|f|) \leq M(n) = n$. Hence, $|M(f)|$ is not an infinitely large element. In other words, $\frac{C_c^*(\mathcal{F})}{M}$ is an archimedean field, i.e., $\frac{C_c^*(\mathcal{F})}{M} \subseteq \mathbb{R}$. By Corollary 2.2, the proof is complete. □

Theorem 2.1. *If $f \in C_c^*(\mathcal{F})$, then $\|f\| = \sup_{M\in\mathcal{M}_c^*} |M(f)|$.*

Proof. Take $S \in \mathcal{F}$ such that $f \in C_c^*(S)$. Then $|f(x)| \leq \|f\|$, for all $x \in S$. From Definition 2.1, $|f| \leq \|f\|$ gives $M(|f|) \leq M(\|f\|)$. Hence, $|M(f)| = M(|f|) \leq M(\|f\|) = \|f\|$. Since M is arbitrary, $\sup_{M\in\mathcal{M}_c^*} |M(f)| \leq \|f\|$. Conversely, if $\epsilon > 0$, then $\|f\| - \epsilon$ is not an upper bound for the set $\{|f(x)| : x \in S\}$. So $|f(x)| > \|f\| - \epsilon$, for some $x \in S$. Let $f' = |f| - \|f\| + \epsilon$. Consequently, f' is positive on a neighborhood of x. Hence, $f' \wedge 0$ is not a unit in $C_c^*(\mathcal{F})$. Thus, $f' \wedge 0 \in (f' \wedge 0) \subseteq M$, for some $M \in \mathcal{M}_c^*$, where $(f' \wedge 0)$ is the generated ideal by $f' \wedge 0$. This implies that $M(f') \wedge M = M$. So $M(f') > M$, and hence $|M(f)| = M(|f|) > \|f\| - \epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $|M(f)| \geq \|f\|$. Thus, $\sup_{M \in \mathcal{M}_c^*} |M(f)| \geq \|f\|$. □

Let R be a commutative ring with identity and $\mathcal{M}$, the set of all maximal ideals of R. For $a \in R$, we let

$$\Gamma_a = \{M \in \mathcal{M} : a \in M\} = \{M \in \mathcal{M} : M(a) = 0\}, \text{ and } \Gamma(a) = \mathcal{M} \setminus \Gamma_a.$$

So $\Gamma(a) = \{M \in \mathcal{M} : a \notin M\} = \{M \in \mathcal{M} : M(a) \neq 0\}$.

It is easy to see that the sets Γ_a form a base for the closed sets in $\mathcal{M}$, equivalently, the sets $\Gamma(a)$ form a base for the hull-kernel topology on $\mathcal{M}$.

We state the next lemma without proof since it is straightforward.

Lemma 2.1. *Let I be an ideal in $R, a \in R$ and (a), the generated ideal by a, and let $\Gamma_I = \{M \in \mathcal{M} : I \subseteq M\}$. Then the following statements hold.*

(1) $\Gamma_a = \Gamma_{(a)}$.

(2) $\Gamma_a = \emptyset$ *if and only if a is a unit.*

(3) $\Gamma_I = \emptyset$ *if and only if $I = R$.*

(4) $\bigcap \Gamma_{I_\alpha} = \Gamma_{\bigcup I_\alpha} = \Gamma_{\sum I_\alpha}$, *where I_α is an ideal in R.*

Theorem 2.2. *The space $\mathcal{M}$ is compact.*

Proof. Suppose that A is a subset of R such that $\{\Gamma(a) : a \in A\}$ is an open cover for $\mathcal{M}$. So $\mathcal{M} = \bigcup_{a \in A} \Gamma(a)$ and hence $\bigcap_{a \in A} \Gamma_a = \emptyset$. By Lemma 2.1(4), $\Gamma_{\sum_{a \in A}(a)} = \emptyset$. Hence, $\sum_{a \in A}(a) = R$. Thus, for some finite elements of A, say $a_1, a_2, \ldots, a_n$, we have $\sum_{i=1}^n a_i = 1$. This yields that $\sum_{i=1}^n (a_i) = R$ and thus $\Gamma_{\sum_{i=1}^n (a_i)} = \bigcap_{i=1}^n \Gamma_{a_i} = \emptyset$. Therefore, $\mathcal{M} = \bigcup_{i=1}^n \Gamma(a_i)$, and we are through. □

The following is an immediate consequence of Theorem 2.2.

Corollary 2.4. *The space $\mathcal{M}_c$ (resp. $\mathcal{M}_c^*$) is compact.*

Let I be a proper ideal in $C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$) and let $Z_c[I] = \{Z(f) : f \in I\}$. We observe that $Z_c[I]$ is a z_c-*filter*, i.e., $\emptyset \notin Z_c[I]$, if $Z(f), Z(g) \in Z_c[I]$ (we can suppose that $f, g \in I$), then $Z(f) \cap Z(g) = Z(f^2 + g^2) \in Z_c[I]$, and further if $Z(f) \subseteq Z(h)$ where $f \in I$ and $h \in C_c(\mathcal{F})$ (resp. $h \in C_c^*(\mathcal{F})$, then $fh \in I$ and $Z(h) = Z(f) \cup Z(h) = Z(fh) \in Z_c[I]$. We observe that $Z_c^{-1}[Z_c[I]] := \{f : Z(f) \in Z_c[I]\} \supseteq I$ (note, $Z_c^{-1}[Z_c[I]]$ is a proper ideal). If the equality holds we call I a *z_c-ideal*. This is equivalent to say that if $Z(f) \subseteq Z(g)$ and $f \in I$, then $g \in I$. Therefore, each maximal ideal in $C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$) as same as an arbitrary intersection of them is a z_c-ideal.

In the same way of [8, Theorem 2.9], we obtain the next theorem.

Theorem 2.3. *Let I be a z_c-ideal in $C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$). Then the following statements are equivalent.*

(1) *I is prime.*
(2) *I contains a prime ideal.*
(3) *For all $g, h \in C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$), if $gh = 0$, then $g \in I$ or $h \in I$.*
(4) *For every $f \in C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$), there is a zero set in $Z_c[I]$ on which f does not change sign.*

Theorem 2.4. *$C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$) is a Gelfand ring (a pm-ring, i.e., every prime ideal is contained in a unique maximal ideal).*

Proof. Suppose that P is a prime ideal, and, M_1 and M_2 are distinct maximal ideals of $C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$) which contains P. Hence, $M_1 \cap M_2$ is a z_c-ideal containing P. By Theorem 2.3, $M_1 \cap M_2$ is prime, which is a contradiction. □

Remark 1. *It is known that the structure space (maximal ideals with the hull-kernel topology) of a pm-ring is Hausdorff. For more details, see* [3, Theorem 1.2].

By combining Corollary 2.4, Remark 1 and Theorem 1.2, we get the following.

Corollary 2.5. *The spaces $\mathcal{M}_c$ and $\mathcal{M}_c^*$ are T_4-spaces.*

An ideal I in a partially ordered ring is called *convex* if $0 \leq x \leq y$ and $y \in I$, then $x \in I$, or equivalently, if $a \leq b \leq c$ and $a, c \in I$, then $b \in I$. By definition, an ideal I in a lattice-ordered ring is said to be *absolutely convex*, whenever $|x| \leq |y|$ and $y \in I$ imply that $x \in I$. So every absolutely convex ideal is convex, see [8, 5.1].

Lemma 2.2. *If I is a z_c-ideal in $C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$), then I is absolutely convex.*

Proof. Suppose that $f, g \in C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$) and $|f| \leq |g|$, where $g \in I$. So there are S_f, $S_g \in \mathcal{F}$ such that $f \in C_c(S_f)$ and $g \in C_c(S_g)$. Let $S = S_f \cap S_g$. Then $Z(g) \subseteq Z(f)$, on S. Now, since $g \in I$ and I is a z_c-ideal, $f \in I$, and we are done. □

Corollary 2.6. *Every maximal ideal of $C_c(\mathcal{F})$ (resp. $C_c^*(\mathcal{F})$) is absolutely convex.*

Proposition 2.1. *Let $M \in \mathcal{M}_c$ (resp. $\mathcal{M}_c^*$) and the topology on $\mathcal{M}_c$ (resp. $\mathcal{M}_c^*$) be the hull-kernel topology. Then each of the sets $\{M : M(f) \neq r\}$, $\{M : M(f) > r\}$ and $\{M : M(f) < r\}$ is open, where $r \in \mathbb{R}$ and $f \in C_c(\mathcal{F})$ (resp. $f \in C_c^*(\mathcal{F})$). So each of the sets $\{M : M(f) = r\}$, $\{M : M(f) \leq r\}$, and $\{M : M(f) \geq r\}$ is closed.*

Proof. The results are obtained by the facts that $\{M : M(f) \neq r\} = \Gamma(f - r), \{M : M(f) > r\} = \{M : M(f \vee r) \neq r\}$, and $\{M : M(f) < r\} = \{M : M(-f) > -r\}$. □

Corollary 2.7. *Let $M \in \mathcal{M}_c$ (resp. $\mathcal{M}_c^*$). Then each of the sets $\{M : |M(f)| \neq r\}$, $\{M : |M(f)| > r\}$ and $\{M : |M(f)| < r\}$ is open, where $r \in \mathbb{R}$ and $f \in C_c(\mathcal{F})$ (resp. $f \in C_c^*(\mathcal{F})$). Hence, each of the sets $\{M : |M(f)| = r\}$, $\{M : |M(f)| \leq r\}$ and $\{M : |M(f)| \geq r\}$ is closed.*

Proof. Using the following facts and Proposition 2.1, we get the results.

$$\{M : |M(f)| < r\} = \{M : M(f) < r\} \cap \{M : -r < M(f)\},$$

$$\{M : |M(f)| > r\} = \{M : M(f) > r\} \cup \{M : M(f) < -r\}, \text{ and}$$

$$\{M : |M(f)| \neq r\} = \{M : |M(f)| > r\} \cup \{M : |M(f)| < r\}.$$

□

3. The functions $\widehat{f}$ on $\mathcal{M}_c^*$ (resp. $\widetilde{f}$ on $\mathcal{M}_c$) and their applications

By Corollary 2.3, $\frac{C_c^*(\mathcal{F})}{M} \cong \mathbb{R}$ for each $M \in \mathcal{M}_c^*$. So this permits us to define

$$\widehat{f} : \mathcal{M}_c^* \to \mathbb{R} \text{ with } \widehat{f}(M) = M(f) \text{ for each } f \in C_c^*(\mathcal{F}).$$

Proposition 3.1. *Let $f, g \in C_c^*(\mathcal{F})$, $r \in \mathbb{R}$ and $M \in \mathcal{M}_c^*$. Then the following statements hold.*

(1) $\widehat{f+g} = \widehat{f} + \widehat{g}$, $\widehat{fg} = \widehat{f}\widehat{g}$, *and* $\widehat{rf} = r\widehat{f}$.
(2) $\|f\| = \|\widehat{f}\|$.
(3) $\widehat{f}$ is continuous.
(4) Let $\widehat{w} := \{\widehat{f} : f \in C_c^*(\mathcal{F})\}$ and $\tau_{\widehat{w}}$ be the weak topology on $\mathcal{M}_c^*$ induced by $\widehat{w}$. Then $\{\widehat{f}^{-1}((a,b)) : a,b \in \mathbb{R}\}$ is a base for $\tau_{\widehat{w}}$.

Proof. (1). Evident. (2). Note that if n is an upper bound for f, then $|\widehat{f}(M)| = |M(f)| = M(|f|) \leq M(n) = n$. So $\widehat{f}$ is bounded, and thus $\|\widehat{f}\| = \sup_{M \in \mathcal{M}_c^*} |M(f)|$. By Theorem 2.1, $\|f\| = \|\widehat{f}\|$. (3). Observe that

$$\begin{aligned}\widehat{f}^{-1}((a,b)) &= \{M \in \mathcal{M}_c^* : a < M(f) < b\} \\ &= \{M \in \mathcal{M}_c^* : a < M(f)\} \cap \{M \in \mathcal{M}_c^* : M(f) < b\}.\end{aligned}$$

By Proposition 2.1, the result holds. (4). It follows from the definition of the weak topology. □

Proposition 3.2. *The family* $\{\widehat{g}^{\ -1}((-r,r)) : g \in C_c^*(\mathcal{F}),\ r > 0\}$ *is a base for* $\tau_{\widehat{w}}$ *on* $\mathcal{M}_c^*$.

Proof. It is sufficient to show the following equality,

$\{\widehat{f}^{\ -1}((a,b)) : f \in C_c^*(\mathcal{F})$, and $a,b \in \mathbb{R}\} = \{\widehat{g}^{\ -1}((-r,r)) : g \in C_c^*(\mathcal{F}), r > 0\}$. To see this, let $M \in \widehat{f}^{\ -1}((a,b))$. Then $a < \widehat{f}(M) = M(f) < b$. Hence,

$$a - \tfrac{a+b}{2} < M(f) - \tfrac{a+b}{2} < b - \tfrac{a+b}{2}.$$

Now, if we put $s = \frac{a+b}{2}, r = \frac{b-a}{2}$, and $g = f - s$, then $-r < M(g) < r$. So $M \in \widehat{g}^{\ -1}((-r,r))$. The reverse inclusion is obvious. □

Lemma 3.1. *The hull-kernel topology on* $\mathcal{M}_c^*$ *(denoted by* $\tau_{\widehat{S}}$*) coincides with* $\tau_{\widehat{w}}$.

Proof. Consider $V = \widehat{f}^{\ -1}((-r,r))$, where $r > 0$ and $f \in C_c^*(\mathcal{F})$, as a basic open set in $\tau_{\widehat{w}}$. Hence, $V = \{M \in \mathcal{M}_c^* : |\widehat{f}(M)| = |M(f)| < r\}$. By Corollary 2.7, V is an open set in $\mathcal{M}_c^*$, i.e., $\tau_{\widehat{w}} \subseteq \tau_{\widehat{S}}$. Also, the equality

$$\Gamma(f) = \{M \in \mathcal{M}_c^* : M(f) \neq 0\} = \widehat{f}^{\ -1}(\mathbb{R} \setminus \{0\}),$$

gives $\Gamma(f) \in \tau_{\widehat{w}}$, i.e., $\tau_{\widehat{S}} \subseteq \tau_{\widehat{w}}$. □

Remind that a subset S of a ring R is called dense in R if $\mathrm{Ann}_R S = \{r \in R : rs = 0, \text{ for all } s \in S\} = 0$.

Theorem 3.1. *$C_c^*(\mathcal{F})$ is a dense $\mathbb{R}$-subalgebra of $C(\mathcal{M}_c^*)$.*

Proof. Let $\psi : C_c^*(\mathcal{F}) \to C(\mathcal{M}_c^*)$ be defined by $\psi(f) = \widehat{f}$. By Proposition 3.1, ψ is an algebra homomorphism. Also, if we let $0 \neq f \in C_c^*(\mathcal{F})$, then $0 \neq \|f\| = \|\widehat{f}\|$. This yields that $\widehat{f} \neq 0$ and hence ψ is a one-to-one mapping. Thus, $C_c^*(\mathcal{F})$ is a subring (in fact, an $\mathbb{R}$-subalgebra) of $C(\mathcal{M}_c^*)$. Briefly, we can state

$$C_c^*(\mathcal{F}) \cong \psi(C_c^*(\mathcal{F})) = \{\widehat{f} : f \in C_c^*(\mathcal{F})\} \subseteq C(\mathcal{M}_c^*).$$

Now, suppose that $g \in C(\mathcal{M}_c^*)$ such that $g.C_c^*(\mathcal{F}) = 0$. So the function $g\widehat{1}$ is zero, i.e., $g(M)\widehat{1}(M) = g(M)M(1) = g(M) = 0$, for each $M \in \mathcal{M}_c^*$. Hence, $g = 0$ which gives $C_c^*(\mathcal{F})$ is dense in $C(\mathcal{M}_c^*)$, and we are done. □

Recall that $\frac{C_c^*(\mathcal{F})}{M} \cong \mathbb{R}$, for each $M \in \mathcal{M}_c^*$ (see Corollary 2.3). But whenever $M \in \mathcal{M}_c$, the field $\frac{C_c(\mathcal{F})}{M}$ may contain $\mathbb{R}$ properly, in other words, it contains infinitely large elements. In the rest of this section, our discussion is on the space $\mathcal{M}_c$.

Lemma 3.2. *Let $M \in \mathcal{M}_c$ and $M(f) \in \frac{C_c(\mathcal{F})}{M}$. If $|M(f)|$ is not infinitely large, then there exists a unique real number s such that $|M(f) - s|$ is infinitely small or zero.*

Proof. Recall that $|M(f)| = \sup\{M(f), M(-f)\} = M(f) \vee M(-f) = M(|f|)$. Since $|M(f)|$ is not infinitely large, there exists a natural number n_1 such that $|M(f)| < n_1$. Let $A = \{r \in \mathbb{R} : r < M(f)\}$. Observe that $-n_1 \in A$ and n_1 is an upper bound for A. So A has a supremum in $\mathbb{R}$. Let $\sup A = s$. If $M(f) \in \mathbb{R}$, then $M(f) = s$ and hence $|M(f) - s| = 0$ clearly. In the case $M(f) \notin \mathbb{R}$, we see that $s - \frac{1}{n} < s < M(f) < s + \frac{1}{n}$. So $|M(f) - s| < \frac{1}{n}$, for all n. Hence, $|M(f) - s|$ is infinitely small. For the uniqueness of s, suppose that $s' \in \mathbb{R}$ also satisfies the conditions of the lemma. Therefore,

$$|s - s'| \leq |s - M(f)| + |M(f) - s'| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} \text{ for all } n.$$

So $s = s'$, and we are done. □

Corollary 3.1. *If $M \in \mathcal{M}_c$ and $f \in C_c^*(\mathcal{F})$, then there exists a unique real number s such that $|M(f) - s|$ is infinitely small or zero.*

Proof. Note that $M(f) \in \frac{C_c(\mathcal{F})}{M}$ and $\|f\| \leq n$, for some $n \in \mathbb{N}$. So $|M(f)| = M(|f|) \leq \|f\| \leq n$, and thus $|M(f)|$ is not infinitely large. Lemma 3.2 now gives the result. □

The uniqueness of s in the previous results will enable us to define the real-valued functions $\widetilde{f}$ on $\mathcal{M}_c$ with several interesting properties.

Definition 3.1. *For $f \in C_c^*(\mathcal{F})$ and $M \in \mathcal{M}_c$, we define $\widetilde{f} : \mathcal{M}_c \to \mathbb{R}$ with $\widetilde{f}(M) = s$.*

Lemma 3.3. *Let $M \in \mathcal{M}_c$, $M(f), M(g), |M(f)|$ and $|M(g)|$ are infinitely small elements of $\frac{C_c(\mathcal{F})}{M}$. Then each of the elements $|M(f) - M(g)|$, $|M(f) + M(g)|$, and $|M(f)| + |M(g)|$ is also infinitely small. Furthermore, if $|M(h)|$ is not an infinitely large element in $\frac{C_c(\mathcal{F})}{M}$, then $|M(fh)|$ is an infinitely small element.*

Proof. First, we note that $0 < |M(f)| + |M(g)| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$ for each $n \in \mathbb{N}$. Therefore, $0 < |M(f) - M(g)| < \frac{1}{n}$. Also, we have $0 < |M(f) + M(g)| < \frac{1}{n}$. Since $|M(h)|$ is not infinitely large, there exists $n_0 \in \mathbb{N}$ such that $|M(h)| < n_0$, and further $|M(f)| < \frac{1}{nn_0}$ for all n. So $0 < |M(f)M(h)| < \frac{1}{n}$, i.e., $|M(fh)|$ is infinitely small. □

Proposition 3.3. *If $f, g \in C_c^*(\mathcal{F})$ and $r \in \mathbb{R}$, then the following statements hold.*

(1) $\widetilde{(f+g)} = \widetilde{f} + \widetilde{g}$.
(2) $\widetilde{fg} = \widetilde{f}\widetilde{g}$.
(3) $\widetilde{rf} = \widetilde{r}\widetilde{f} = r\widetilde{f}$.

Proof. (1). Let $M \in \mathcal{M}_c$. By Corollary 3.1 and Definition 3.1, $|M(f) - \widetilde{f}(M)|$ and $|M(g) - \widetilde{g}(M)|$ are infinitely small or zero. According to Lemma 3.3, $|M(f+g) - (\widetilde{f} + \widetilde{g})(M)|$ is infinitely small or zero. Also, since $f + g$ is bounded, $|M(f+g) - \widetilde{(f+g)}(M)|$ is infinitely small or zero. By the uniqueness of s in Lemma 3.2, the result holds. (2). Note that $|M(fg) - \widetilde{fg}(M)|$ is infinitely small or zero. Now, let $A = |M(fg) - \widetilde{f}(M)\widetilde{g}(M)|$. Then

$$A = |M(fg) - \widetilde{f}(M)\widetilde{g}(M) + \widetilde{f}(M)M(g) - \widetilde{f}(M)M(g)| \tag{3.1}$$

$$= |\widetilde{f}(M)(M(g) - \widetilde{g}(M)) + M(g)(M(f) - \widetilde{f}(M))| \tag{3.2}$$

$$\leq |\widetilde{f}(M)||M(g) - \widetilde{g}(M)| + |M(g)||M(f) - \widetilde{f}(M)|. \tag{3.3}$$

Since $|M(g)|$ is bounded and $|\widetilde{f}(M)|$ is real, the phrase in relation (3.3) is infinitely small or zero. By Lemmas 3.2 and 3.3, the result is obtained. (3). It follows from (2), by replacing g with r. □

The next result is immediate. Compare it with Theorem 3.1.

Corollary 3.2. *The mapping $\varphi : C_c^*(\mathcal{F}) \to C(\mathcal{M}_c)$ given by $\varphi(f) = \widetilde{f}$ is an algebra homomorphism.*

The weak topology induced by the family $\widetilde{w} := \{\widetilde{f} : f \in C_c^*(\mathcal{F})\}$ on $\mathcal{M}_c$ is denoted by $\tau_{\widetilde{w}}$. In the same way of Proposition 3.2, we have the next result.

Proposition 3.4. *The family $\{\widetilde{g}^{\,-1}((-r, r)) : g \in C_c^*(\mathcal{F}),\ r > 0\}$ is a base for $\tau_{\widetilde{w}}$ on $\mathcal{M}_c$.*

Recall that the two ideals of a commutative ring are called coprime, whenever the sum of them is the whole of the ring. So every two distinct maximal ideals of a ring, are coprime.

Lemma 3.4. *If M_1 and M_2 are distinct maximal ideals of $C_c(\mathcal{F})$. Then the ideals $M_1' := M_1 \cap C_c^*(\mathcal{F})$ and $M_2' := M_2 \cap C_c^*(\mathcal{F})$ are coprime.*

Proof. Since $M_1 + M_2 = C_c(\mathcal{F})$, there exist $f \in M_1$ and $g \in M_2$ such that $1 = f + g$. We note that $\frac{f}{1+|f|} \in M_1' \backslash M_2'$ and $\frac{g}{1+|g|} \in M_2' \backslash M_1'$. Since maximal ideals are absolutely convex (Corollary 2.6), we have $|f| \in M_1$ and $|g| \in M_2$. Notice that $f + g = 1$ implies that $|f| + |g| \neq 0$. Now, let

$$f' = \frac{|f|}{|f| + |g|}, \text{ and } g' = \frac{|g|}{|f| + |g|}.$$

Then $f' \in M_1'$, $g' \in M_2'$, and, $f' + g' = 1$. Thus $M_1' + M_2' = C_c^*(\mathcal{F})$. □

Theorem 3.2. *Let $\widetilde{w}$ and $\tau_{\widetilde{w}}$ be as defined previously. Then $(\mathcal{M}_c, \tau_{\widetilde{w}})$ is Hausdorff.*

Proof. Let M_1 and M_2 be distinct maximal ideals of $C_c(\mathcal{F})$, and let M_1' and M_2' be their intersections with $C_c^*(\mathcal{F})$, respectively. By Lemma 3.4, $f + g = 1$, for some $f \in M_1'$ and $g \in M_2'$. Notice that $|M_1(f) - \widetilde{f}(M_1)| = |\widetilde{f}(M_1)|$ and $|M_2(g) - \widetilde{g}(M_2)| = |\widetilde{g}(M_2)|$ are infinitely small or zero. Since the real numbers $|\widetilde{f}(M_1)|$ and $|\widetilde{g}(M_2)|$

cannot be infinitely small, they are zero. Hence, $\widetilde{f}(M_2) = \widetilde{(1-g)}(M_2) = \widetilde{1}(M_2) - \widetilde{g}(M_2) = 1$. Now, if we put $U_1 = \widetilde{f}^{\,-1}((-\infty, \frac{1}{2}))$ and $U_2 = \widetilde{f}^{\,-1}((\frac{1}{2}, \infty))$, then U_1 and U_2 are disjoint open sets containing M_1 and M_2, respectively. So $\mathcal{M}_c$ is Hausdorff. □

Corollary 3.3. *Let $\tau_{\widetilde{S}}$ be the hull-kernel topology on $\mathcal{M}_c$. Then $\tau_{\widetilde{S}} = \tau_{\widetilde{w}}$.*

Proof. Let $X = (\mathcal{M}_c, \tau_{\widetilde{S}})$, $Y = (\mathcal{M}_c, \tau_{\widetilde{w}})$ and let $i : X \to Y$ be the identity function. Recall that X is compact (Corollary 2.4) and Y is Hausdorff (Theorem 3.2). Moreover, notice that $\widetilde{g}^{\,-1}((-r, r)) = \bigcup_{0<r'<r}\{M : |M(g)| < r'\}$, see Proposition 3.4 and Corollary 2.7. So i is continuous. Now, Theorem 1.3 completes the proof. □

4. A homeomorphism between T_4-spaces $\mathcal{M}_c$ and $\mathcal{M}_c^*$

In this section, we show that the topological spaces $\mathcal{M}_c$ and $\mathcal{M}_c^*$ are homeomorphic.

Lemma 4.1. *Let $M \in \mathcal{M}_c$ and $\tau^*(M) = \{f \in C_c^*(\mathcal{F}) : \widetilde{f}(M) = 0\}$. Then $\tau^*(M)$ is a maximal ideal in $C_c^*(\mathcal{F})$.*

Proof. By Proposition 3.3, $\tau^*(M)$ is an ideal of $C_c^*(\mathcal{F})$. We claim that $1 \notin \tau^*(M)$. Otherwise, $\widetilde{1}(M) = 0$. Consequently, $1 = |M(1)| = |M(1) - \widetilde{1}(M)|$ is infinitely small or zero, a contradiction. Hence, $\tau^*(M)$ is a proper ideal. Recall that if $f \in M \cap C_c^*(\mathcal{F})$, then $M(f) = 0$ and thus $\widetilde{f}(M) = 0$. So $\tau^*(M) \supseteq M \cap C_c^*(\mathcal{F})$. Suppose that $f \in C_c^*(\mathcal{F}) \setminus \tau^*(M)$. Thus, $f \notin M$ and therefore $C_c(\mathcal{F}) = (M, f)$ (note, $M \in \mathcal{M}_c$). So $1 = m + fg$, for some $m \in M$ and some $g \in C_c(\mathcal{F})$. Therefore, $1 = M(1) = M(m) + M(f)(M(g) = M(f)M(g)$. Since $f \notin \tau^*(M)$, we have $0 \neq \widetilde{f}(M)$ and thus $|M(f)|$ is not infinitely small. Otherwise, by Lemma 3.3, $|\widetilde{f}(M)| = |M(f) - \widetilde{f}(M) - M(f)|$ is infinitely small, a contradiction. Hence, $|M(g)|$ is not infinitely large. There is a unique real number s such that $|M(g) - s|$ is infinitely small or zero (Lemma 3.2). Evidently, $s - 1 < M(g) < s + 1$. Let

$$g' = ((s-1) \vee g) \wedge (s+1).$$

Clearly $g' \in C_c^*(\mathcal{F})$ and $M(g) = M(g')$. Therefore, $M(f)M(g') = 1$. So $fg' - 1 \in M \cap C_c^*(\mathcal{F}) \subseteq \tau^*(M)$. Since $1 = (1 - fg') + fg'$, we have $(\tau^*(M), f) = C_c^*(\mathcal{F})$. This gives $\tau^*(M)$ is a maximal ideal in $C_c^*(\mathcal{F})$, and we are trough. □

Remark 2. *If* $M \in \mathcal{M}_c$, $0 \neq s \in \mathbb{R}$ *and* $\tau_s^*(M) = \{f \in C_c^*(\mathcal{F}) : \widetilde{f}(M) = s\}$, *then* $\tau_s^*(M)$ *contains the unit* s. *So it is not a proper ideal, i.e.,* $\tau_s^*(M) = C_c^*(\mathcal{F})$. *Hence, the assumption* $\widetilde{f}(M) = 0$ *is necessary in the definition of* $\tau^*(M)$ *in Lemma 4.1.*

Lemma 4.2. *Let* $M^* \in \mathcal{M}_c^*$ *and* $F = \{(|f| - r) \vee 0 : f \in M^*,\ \text{and}\ r > 0\}$. *Then* I, *the generated ideal by* F *in* $C_c(\mathcal{F})$, *is a proper ideal (and hence it is contained in a maximal ideal of* $C_c(\mathcal{F})$*).*

Proof. Suppose $f_i \in M^*, 1 \leq i \leq n$, and $r_i > 0$. Since $f = \sum_{i=1}^n f_i^2 \in M^*$, neither f nor f_i is a unit, hence it is not bounded away from zero (Corollary 1.2). Therefore, $\mathrm{Im} f_i \cap (-r_i, r_i) \neq \emptyset$, also, $\mathrm{Im} f \cap (-r^2, r^2) \neq \emptyset$, where $r = \min\{r_i : 1 \leq i \leq n\}$. Let $A = \{x : f(x) < r^2\}$. Then $A \neq \emptyset$ and $f_i^2 \leq f < r^2$, on A. Thus, $|f_i| - r < 0$, and $A \subseteq \bigcap_{i=1}^n Z((|f_i| - r) \vee 0)$. Now, we claim that $1 \notin I = (F)$, otherwise,

$$1 = \sum_{i=1}^n g_i((|f_i| - r) \vee 0),$$

where $g_i \in C_c(\mathcal{F})$. Therefore, $\emptyset = Z(1) \supseteq \bigcap_{i=1}^n Z((|f_i| - r) \vee 0) \supseteq A$, a contradiction. So I is a proper ideal of $C_c(\mathcal{F})$. □

Theorem 4.1. *Let* $\tau^*(M)$ *be as defined in Lemma 4.1. Then the following statements hold.*

(1) *The mapping* $\tau^* : \mathcal{M}_c \to \mathcal{M}_c^*$, *which* $M \mapsto \tau^*(M)$ *is a one-to-one correspondence.*
(2) $\widehat{f} \circ \tau^* = \widetilde{f}$, *for every* $f \in C_c^*(\mathcal{F})$.
(3) τ^* *is continuous.*

Proof. (1). Let $M_1, M_2 \in \mathcal{M}_c$ be distinct, and let f be the function in the proof of Theorem 3.2. We saw there that $\widetilde{f}(M_1) = 0 \neq \widetilde{f}(M_2) = 1$. So $\widetilde{f} \in \tau^*(M_1) \setminus \tau^*(M_2)$. Hence, τ^* is one-to-one. Next, suppose that $M^* \in \mathcal{M}_c^*$. In Lemma 4.2, the existence of a maximal ideal M of $C_c(\mathcal{F})$ corresponding to M^* has been proved. We now claim that $\tau^*(M) = M^*$. Let $f \in M^*$ and $r > 0$. Using Lemma 4.2, we get $(|f| - r) \vee 0 \in M$. So $|M(f)| = M(|f|) < r$. Since r is an arbitrary positive number, $|0 - M(f)| = M(|f|)$ is infinitely small or zero. Remind that $|\widetilde{f}(M) - M(f)|$ is infinitely small or zero. So $\widetilde{f}(M) = 0$ and hence $M^* \subseteq \tau^*(M)$. Since M^* is maximal,

$M^* = \tau^*(M)$, which gives τ^* is onto. The proof of (1) is now complete. (2). Assume that $M \in \mathcal{M}_c$ and $\widehat{f} \circ \tau^*(M) = r$, for some $r \in \mathbb{R}$. So $\tau^*(M)(f) = r = \tau^*(M)(r)$ and thus $f - r \in \tau^*(M)$. Consequently, $\widetilde{(f-r)}(M) = 0 = (\widetilde{f} - \widetilde{r})(M)$. Therefore, $\widetilde{f}(M) = \widetilde{r}(M) = r$. (3). Let $f \in C_c^*(\mathcal{F})$. Consider $\Gamma_f = \{M \in \mathcal{M}_c^* : M(f) = 0\}$ and $\Gamma_{f'} = \{M \in \mathcal{M}_c : M(f) = 0\}$ as basic closed sets in $\mathcal{M}_c^*$ and $\mathcal{M}_c$ respectively. Then $\widehat{f}(M) = 0$, for each $M \in \Gamma_f$. Hence,

$$\tau^{*-1}(\Gamma_f) = \{M \in \mathcal{M}_c : \tau^*(M)(f) = 0 = \widehat{f}(\tau^*(M))\}.$$

By (2), $\widetilde{f}(M) = 0$. Recall that $|M(f) - \widetilde{f}(M)| = |M(f)|$ is infinitely small or zero. Since f is bounded, $|M(f)|$ is not infinitely small. So it is zero, i.e., $f \in M$ and thus $M \in \Gamma_{f'}$. Consequently, $\tau^{*-1}(\Gamma_f) = \Gamma_{f'}$, i.e., τ^* is continuous. □

We are now in a position to say that the T_4-spaces $\mathcal{M}_c$ and $\mathcal{M}_c^*$ are homeomorphic.

Theorem 4.2. *The T_4-spaces $\mathcal{M}_c$ and $\mathcal{M}_c^*$ are homeomorphic.*

Proof. By Theorem 4.1, τ^* is one-to-one, onto, and continuous. Since $\mathcal{M}_c$ is compact and $\mathcal{M}_c^*$ is Hausdorff, the result holds by Theorems 1.3. □

We conclude the article with the following result. But before that, a summary of rings $Q_c(X)$ and $q_c(X)$ is mentioned. The maximal (resp. classical) ring of quotients of $C_c(X)$ will be denoted by $Q_c(X)$ (resp. $q_c(X)$). These rings have been characterized in [12] and [2], respectively. By $Q_c^*(X)$ (resp. $q_c^*(X)$), we mean the subring of $Q_c(X)$ (resp. $q_c(X)$) consisting of bounded functions. Also, $\max(Q_c(X))$ (resp. $\max(q_c(X))$) denotes the maximal ideal space of $Q_c(X)$ (resp. $q_c(X)$).

Corollary 4.1. *Let X be a zero-dimensional space. Then*

$$\max(Q_c(X)) \cong \max(Q_c^*(X)), \textit{ and } \max(q_c(X)) \cong \max(q_c^*(X))$$

Proof. Recall that a subset of X is a σ-clopen set in X if it is a countable union of clopen (closed-open) sets in X. Let

$$\mathcal{F}_1 = \{V \subseteq X : V \text{ is a dense open subset of } X\}, \text{ and}$$

$$\mathcal{F}_2 = \{V \subseteq X : V \text{ is a dense } \sigma\text{-clopen subset of } X\}.$$

Then both $\mathcal{F}_1$ and $\mathcal{F}_2$ are filter base on X. In view of [12, Theorem 2.12] and [2, Theorem 2.2], we have $Q_c(X) = \lim_{V\in\mathcal{F}_1} C_c(V)$ and $q_c(X) = \lim_{V\in\mathcal{F}_2} C_c(V)$. Hence, $Q_c(X) = C_c(\mathcal{F}_1)$ and $q_c(X) = C_c(\mathcal{F}_2)$. Theorem 4.2 now yields the result. □

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References

[1] A. R. Aliabad, F. Azarpanah, and M. Namdari, Rings of continuous functions vanishing at infinity, *Comment. Math. Univ. Carolin.*, **3** (2004), 519–533.

[2] P. Bhattacharjee, M. L. Knox, and W. Wm McGovern, The classical ring of quotients of $C_c(X)$, *Appl. Gen. Topol.*, **15(2)** (2014), 147–154.

[3] G. De Marco, and A. Orsatti, Commutative rings in which every prime ideal is contained in a unique maximal ideal, *Proc. Amer. Math. Soc.*, **30(3)** (1971), 459–466.

[4] R. Engelking, *General Topology*, Heldermann Verlag Berlin, (1989).

[5] N. J. Fine, L. Gillman, and J. Lambek, *Rings of quotients of rings of functions*, Lecture Notes Series Mc-Gill University Press, Montreal, (1966).

[6] M. Ghadermazi, O. A. S. Karamzadeh, and M. Namdari, On the functionally countable subalgebra of $C(X)$, *Rend. Sem. Mat. Univ. Padova*, **129** (2013), 47–69.

[7] M. Ghadermazi, O. A. S. Karamzadeh, and M. Namdari, $C(X)$ versus its functionally countable subalgebra, *Bull. Iran. Math. Soc.*, **45(1)** (2019), 173–187.

[8] L. Gillman, and M. Jerison, *Rings of continuous functions*, Springer-Verlag, (1976).

[9] O. A. S. Karamzadeh, and Z. Keshtkar, On c-realcompact spaces, *Quest. Math.*, **41(8)** (2018), 1135–1167.

[10] O. A. S. Karamzadeh, M. Namdari, and S. Soltanpour, On the locally functionally countable subalgebra of $C(X)$, *Appl. Gen. Topol.*, **16(2)** (2015), 183–207.

[11] M. A. Mulero, Algebraic properties of rings of continuous functions, *Fund. Math.*, **149** (1996), 55–66.

[12] M. Namdari, and A. Veisi, Rings of quotients of the subalgebra of $C(X)$ consisting of functions with countable image, *Inter. Math. Forum*, **7** (2012), 561–571.

[13] A. Veisi, e_c-Filters and e_c-ideals in the functionally countable subalgebra of $C^*(X)$, *Appl. Gen. Topol.*, **20(2)** (2019), 395–405.

[14] A. Veisi, On the m_c-topology on the functionally countable subalgebra of $C(X)$, *Journal of Algebraic Systems*, **9(2)** (2022), 335-345.

[15] A. Veisi, Closed ideals in the functionally countable subalgebra of $C(X)$, *Appl. Gen. Topol.*, (2022), doi:10.4995/agt.2022.15844.

[16] A. Veisi, and A. Delbaznasab, Metric spaces related to Abelian groups, *Appl. Gen. Topol.*, **22(1)** (2021), 169-181.

[17] S. Willard, *General Topology*, Addison-Wesley, (1970).

Faculty of Petroleum and Gas, Yasouj University, Gachsaran, Iran

Email address: aveisi@yu.ac.ir