Jordan Journal of Mathematics and Statistics (JJMS), 15(2), 2022, pp 325 - 338
DOI: https://doi.org/10.47013/15.2.11

HOMODERIVATIONS ON A LATTICE

MOURAD YETTOU (1) AND ABDELAZIZ AMROUNE (2)

Abstract. In this paper, the concept of homoderivation on a lattice as a combination of two concepts of *meet*-homomorphisms and derivations is introduced. Some characterizations and properties of homoderivations are provided. The relationship between derivations and homoderivations on a lattice is established. Also, an interesting class of homoderivations namely isotone homoderivations is studied. A characterization of the isotone homoderivations in terms of the *meet*-homomorphisms is given. Furthermore, a sufficient condition for a homoderivation to become isotonic is established.

1. Introduction

The notion of derivation appeared on the ring structures and it has many applications (see, e.g. [6, 7]). Szász [13] has extended this notion of derivation to the setting of lattice structures. He has defined a derivation on a given lattice L as a function d satisfying the following two conditions:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) \text{ and } d(x \vee y) = d(x) \vee d(y), \text{ for any } x, y \in L.$$

Xin et al. [17] have ameliorated this notion of derivation on a lattice by considering only the first condition, and they have shown that the second condition obviously holds for the isotone derivations on a distributive lattice. In the same paper, they have characterized distributive and modular lattices in terms of their isotone derivations. Later on, Xin [16] has focused his attention on the structure of the set of the fixed points of a derivation on a lattice, and he has shown some relationships between this set and the lattice ideals.

2000 *Mathematics Subject Classification.* 03G10, 06B05, 06B10, 06B99.

Key words and phrases. Lattice, derivation, homoderivation, isotone homoderivation.

Received: Nov. 22, 2020 Accepted: Feb. 25, 2021 .

This notion of derivation on a lattice is witnessing increased attention. It was applied in partially ordered sets [2, 20], in distributive lattices [1, 19], in semilattices [18], in bounded hyperlattices [14], in quantales and residuated lattices [10, 15].

In the ring structures, El Sofy [9] has introduced the notion of homoderivation on a ring $(R, +, \cdot)$ as an additive mapping $h : R \to R$ satisfies

$$h(x \cdot y) = (h(x) \cdot h(y)) + (h(x) \cdot y) + (x \cdot h(y)), \text{ for any } x, y \in R.$$

Recently, this notion of homoderivation is also witnessing increased attention. It was studied by many authors [3, 4, 5, 12].

Inspired by the notion of homoderivation on ring structures, we introduce the notion of homoderivation on a lattice structure by combining the two concepts of *meet*-homomorphisms and derivations introduced by Xin et al. [17]. More specifically, we investigate some characterizations and properties of homoderivations on a lattice. We establish the relationship between derivations and homoderivations on a lattice. Moreover, we study the isotone homoderivations on a lattice as an interesting class of homoderivations. We provide a characterization of the isotone homoderivations on a lattice in terms of the *meet*-homomorphisms. Further, we give a sufficient condition for a homoderivation to be isotone.

The rest of the paper is organized as follows. In Section 2, we recall some necessary concepts and properties of lattices and derivations on lattices. In Section 3, we introduce the notion of homoderivation on a lattice and investigate some of its characterizations and properties. In Section 4, we study the isotone homoderivations on a lattice as a particular class of homoderivations and we provide a characterization of them. Finally, we present some conclusions and discuss future research in Section 5.

2. Basic concepts

In this section, we recall some basic concepts and properties of lattices and derivations on lattices that will be needed in this paper.

2.1. **Lattices.** An *order relation* $\leqslant$ on a set X is a binary relation on X that is *reflexive*, *antisymmetric* and *transitive*. A set X equipped with an order relation $\leqslant$ is called a *partially ordered set* (*a poset*, for short), denoted $(X, \leqslant)$.

Let $(X, \leqslant)$ be a poset and A be a non-empty subset of X. An element $x_0 \in X$ is called a *lower bound* of A if $x_0 \leqslant x$, for any $x \in A$. The element x_0 is called the *greatest lower bound* (or the *infimum*) of A, if x_0 is a lower bound of A and $m \leqslant x_0$, for any lower bound m of A. Dually, An element $x_1 \in X$ is called an *upper bound* of A if $x \leqslant x_1$, for any $x \in A$. The element x_1 is called the *smallest upper bound* (or the *supremum*) of A, if x_1 is an upper bound of A and $x_1 \leqslant t$, for any upper bound t of A.

A poset $(L, \leqslant)$ is called a *meet*-semilattice if any two elements x and y of L have a greatest lower bound, denoted by $x \wedge y$ and called the *meet* (*infimum*) of x and y. Analogously, it is called a *join*-semilattice if any two elements x and y of L have a smallest upper bound, denoted by $x \vee y$ and called the *join* (*supremum*) of x and y. A poset $(L, \leqslant)$ is called a lattice if it is both a *meet*- and *join*-semilattice. Usually, the notation $(L, \leqslant, \wedge, \vee)$ is used to describe a lattice.

Let $(L, \leqslant, \wedge, \vee)$ be a lattice and x_0, x_1 be two elements of L. The element x_0 is called the *smallest element* of L if $x_0 \leqslant x$, for any $x \in L$. Daully, The element x_1 is called the *greatest element* of L if $x \leqslant x_1$, for any $x \in L$.

A lattice $(L, \leqslant, \wedge, \vee)$ is called *bounded* if it has a smallest and a greatest element, respectively denoted by 0 and 1. Usually, the notation $(L, \leqslant, \wedge, \vee, 0, 1)$ is used to describe a bounded lattice. For example, let $(L_1 = [0, 1], min, max)$ and $(L_2 =]0, 1[, min, max)$ be two lattices ordered by the usual order of the real numbers. Obvious that 0 is the smallest element and 1 is the greatest element of L_1. Then, $(L_1 = [0, 1], min, max, 0, 1)$ is a bounded lattice. But, L_2 has not a smallest and a greatest element. Thus, $(L_2 =]0, 1[, min, max)$ is not a bounded lattice.

A lattice $(L, \leqslant, \wedge, \vee)$ is called *distributive* if it satisfies one of the following two equivalent conditions:

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, for any $x, y, z \in L$;

(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, for any $x, y, z \in L$.

Let $(L_1, \leqslant_1, \wedge_1, \vee_1)$ and $(L_2, \leqslant_2, \wedge_2, \vee_2)$ be two lattices. A mapping $\psi : L_1 \to L_2$ is called a *meet*-homomorphism, if $\psi(x \wedge_1 y) = \psi(x) \wedge_2 \psi(y)$, for any $x, y \in L_1$.

Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $x, y \in L$. The element y covers the element x if $x < y$ (i.e., $x \leqslant y$ and $x \neq y$) and there is not an element $z \in L$ such that $x < z < y$.

The *Hasse diagram* of a finite lattice $(L, \leqslant, \wedge, \vee)$ is a digraph whose vertexes are the elements of L and which has line segments between some their vertexes. If an element $y \in L$ covers an element $x \in L$, we get the vertex y is higher up than the vertex x and they are connected with a line segment. For example, let $(D(12), |, gcd, lcm)$ be the finite lattice of the positive divisors of 12 ordered by the divisibility order $|$. The Hasse diagram of $(D(12), |, gcd, lcm)$ is given in the Figure 1.

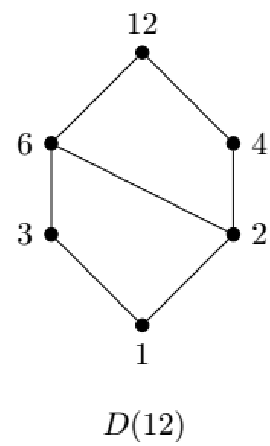

FIGURE 1. The Hasse diagram of the finite lattice $(D(12), |, gcd, lcm)$.

Further information on lattices can be found in [8, 11].

2.2. **Derivations on lattices.** In this subsection, we recall some basic concepts and properties of derivations on lattices that will be needed throughout this paper.

Definition 2.1. [17] *Let $(L, \leqslant, \wedge, \vee)$ be a lattice. A function $d : L \to L$ is called a derivation on L if it satisfies the following condition:*

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)), \text{ for any } x, y \in L\,.$$

In the rest of the paper, we shortly write dx instead of $d(x)$.

Definition 2.2. [17] *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d a derivation on L. The derivation d is called isotone if it satisfies the following condition:*

$$x \leqslant y \text{ implies } dx \leqslant dy, \text{ for any } x, y \in L\,.$$

Definition 2.3. [17] *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and α an element of L. A principal derivation on L is a function $d_\alpha : L \to L$ defined as*

$$d_\alpha(x) = \alpha \wedge x, \text{ for any } x \in L.$$

The following proposition gives some properties of derivations on a lattice.

Proposition 2.1. [17] *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d a derivation on L. Then the following holds:*

(i) $dx \leqslant x$, for any $x \in L$;

(ii) $dx \wedge dy \leqslant d(x \wedge y)$, for any $x, y \in L$

For more details concerning derivations on lattices, we refer to [16, 17].

3. Homoderivations on a lattice

In this section, we introduce the notion of homoderivation on a lattice and investigate some of its characterizations and properties. Further, we show the relationship between derivations and homoderivations on a lattice.

Definition 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice. A homoderivation on L is a function $h : L \to L$ satisfies the following condition:*

$$h(x \wedge y) = (h(x) \wedge h(y)) \vee (h(x) \wedge y) \vee (x \wedge h(y)), \text{ for any } x, y \in L\,.$$

In the rest of the paper, we shortly write hx instead of $h(x)$.

Example 3.1. *(i) Let $(L, \leqslant, \wedge, \vee)$ be a lattice. The identity function of L (i.e., $h(x) = x$, for any $x \in L$) is a homoderivation on L;*

(ii) Let $(L, \leqslant, \wedge, \vee, 0)$ be a lattice with the smallest element $0 \in L$. The null function of L (i.e., $hx = 0$, for any $x \in L$) is a homoderivation on L.

Example 3.2. *Let $(L = \{0, a, b, 1\}, \leqslant, \wedge, \vee)$ be the bounded lattice given by the Hasse diagram in Figure 2 and h_1, h_2, h_3, h_4 be four functions on L defined in the following table:*

x	0	a	b	1
$h_1(x)$	1	b	a	0
$h_2(x)$	0	a	b	0
$h_3(x)$	a	a	1	1
$h_4(x)$	b	1	b	1

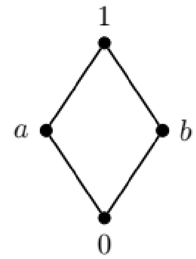

FIGURE 2. The Hasse diagram of $(L = \{0, a, b, 1\}, \leqslant, \wedge, \vee)$.

The functions h_1, h_2, h_3 and h_4 are homoderivations on L. Indeed, the function h_1 satisfies

(1) $h_1(0 \wedge a) = h_1(0) = 1$ *and* $(h_1(0) \wedge h_1(a)) \vee (h_1(0) \wedge a) \vee (0 \wedge h_1(a)) = (1 \wedge b) \vee (1 \wedge a) \vee (0 \wedge b) = b \vee a \vee 0 = 1$;

(2) $h_1(0 \wedge b) = h_1(0) = 1$ *and* $(h_1(0) \wedge h_1(b)) \vee (h_1(0) \wedge b) \vee (0 \wedge h_1(b)) = (1 \wedge a) \vee (1 \wedge b) \vee (0 \wedge a) = a \vee b \vee 0 = 1$;

(3) $h_1(0 \wedge 1) = h_1(0) = 1$ *and* $(h_1(0) \wedge h_1(1)) \vee (h_1(0) \wedge 1) \vee (0 \wedge h_1(1)) = (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 0) = 0 \vee 1 \vee 0 = 1$;

(4) $h_1(a \wedge b) = h_1(0) = 1$ *and* $(h_1(a) \wedge h_1(b)) \vee (h_1(a) \wedge b) \vee (a \wedge h_1(b)) = (b \wedge a) \vee (b \wedge b) \vee (a \wedge a) = 0 \vee b \vee a = 1$;

(5) $h_1(a \wedge 1) = h_1(a) = b$ *and* $(h_1(a) \wedge h_1(1)) \vee (h_1(a) \wedge 1) \vee (a \wedge h_1(1)) = (b \wedge 0) \vee (b \wedge 1) \vee (a \wedge 0) = 0 \vee b \vee 0 = b$;

(6) $h_1(b \wedge 1) = h_1(b) = a$ *and* $(h_1(b) \wedge h_1(1)) \vee (h_1(b) \wedge 1) \vee (b \wedge h_1(1)) = (a \wedge 0) \vee (a \wedge 1) \vee (b \wedge 0) = 0 \vee a \vee 0 = a$;

(7) $h_1(x \wedge x) = h_1(x)$ *and* $(h_1(x) \wedge h_1(x)) \vee (h_1(x) \wedge x) \vee (x \wedge h_1(x)) = h_1(x) \vee (h_1(x) \wedge x) = h_1(x)$, *for any* $x \in L = \{0, a, b, 1\}$.

Therefore, h_1 is a homoderivation on L. The other homoderivations h_2, h_3 and h_4 can be proved similarly.

Remark 3.1. *We note that Xin et al. [17] have shown that the only bijective derivation on a given lattice L is the identity function of L. But here, we can see that the identity function of L is not the only bijective homoderivation on L. Indeed, as can be seen from Example 3.2 that the homoderivation h_1 is bijective and it is not the identity function of L.*

In the following proposition, we show that any derivation on a given lattice is a homoderivation.

Proposition 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $d : L \to L$ be a function. If d is a derivation on L, then d is a homoderivation.*

Proof. From Proposition 2.1 (ii) we have $dx \wedge dy \leqslant d(x \wedge y)$, for any $x, y \in L$. Hence, $d(x \wedge y) = (dx \wedge dy) \vee d(x \wedge y)$, for any $x, y \in L$. Since d is a derivation on L, it follows that $d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$, for any $x, y \in L$. Thus, $d(x \wedge y) = (dx \wedge dy) \vee (dx \wedge y) \vee (x \wedge dy)$, for any $x, y \in L$. Consequently, d is a homoderivation on L. □

Remark 3.2. *We note that the converse implication of Proposition 3.1 does not necessarily hold. As can be seen that the homoderivation h_1 given in Example 3.2 is not a derivation. Indeed,*

$$h_1(0 \wedge a) = 1 \neq (h_1(0) \wedge a) \vee (0 \wedge h_1(a)) = a\,.$$

In view of Proposition 3.1 and Remark 3.2, we conclude that the set of all derivations on L (denoted $D(L)$) is a proper subset of the set of all homoderivations on L (denoted $H(L)$), i.e., $D(L) \subsetneq H(L)$.
The following theorem determines the necessary and sufficient condition under which a homoderivation on a lattice is a derivation.

Theorem 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and h a homoderivation on L. Then h is a derivation if and only if $hx \leqslant x$, for any $x \in L$.*

Proof. The proof of the direct implication follows from proposition 2.1 (i). Conversely, let $x, y \in L$. The fact that $hx \leqslant x$ and $hy \leqslant y$ implies that $hx \wedge hy \leqslant hx \wedge y$ and $hx \wedge hy \leqslant x \wedge hy$. Then $hx \wedge hy \leqslant (hx \wedge y) \vee (x \wedge hy)$. Since h is a homoderivation on L, it follows that

$$h(x \wedge y) = (hx \wedge hy) \vee (hx \wedge y) \vee (x \wedge hy) = (hx \wedge y) \vee (x \wedge hy)\,.$$

Therefore, h is a derivation on L. □

Proposition 3.2. *Let $(L, \leqslant, \wedge, \vee, 1)$ be a lattice with the greatest element $1 \in L$ and h be a homoderivation on L. Then $x \wedge h1 \leqslant hx$, for any $x \in L$.*

Proof. Let $x \in L$, since h is a homoderivation on L, it follows that

$$\begin{aligned} hx = h(x \wedge 1) &= (hx \wedge h1) \vee (hx \wedge 1) \vee (x \wedge h1) \\ &= (hx \wedge h1) \vee hx \vee (x \wedge h1) \\ &= hx \vee (x \wedge h1)\,. \end{aligned}$$

Thus, $x \wedge h1 \leqslant hx$. □

The above proposition leads to the following characterization.

Corollary 3.1. *Let $(L, \leqslant, \wedge, \vee, 1)$ be a lattice with the greatest element $1 \in L$ and h be a homoderivation on L. Then $x \leqslant hx$, for any $x \in L$ if and only if 1 is a fixed point of h (i.e., $h(1) = 1$).*

Proposition 3.3. *Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice and h be a homoderivation on L. Then $h0 = 1$ if and only if $hx \vee x = 1$, for any $x \in L$.*

Proof. We suppose that $h0 = 1$ and let $x \in L$. Since h is a homoderivation on L, it follows that $h0 = h(0 \wedge x) = (h0 \wedge hx) \vee (h0 \wedge x) \vee (0 \wedge hx) = hx \vee x$. Therefore, $hx \vee x = 1$. To prove the converse implication, we suppose that $x \vee hx = 1$, for any $x \in L$. Then $h0 = 0 \vee h0 = 1$, hence $h0 = 1$. □

Proposition 3.4. *Let $(L, \leqslant, \wedge, \vee, 0)$ be a distributive lattice and h be a homoderivation on L. Then $h0 \leqslant hx \vee x$, for any $x \in L$.*

Proof. Since h is a homoderivation on L and L is distributive, it follows that $h0 = h(0 \wedge x) = (h0 \wedge hx) \vee (h0 \wedge x) \vee (0 \wedge hx) = (h0 \wedge hx) \vee (h0 \wedge x) = h0 \wedge (hx \vee x)$, for any $x \in L$. Thus, $h0 \leqslant hx \vee x$, for any $x \in L$. □

4. Isotone homoderivations on a lattice

In this section, we study the isotone homoderivations on a lattice as a particular class of homoderivations. More precisely, we provide a characterization of the isotone homoderivations in terms of the *meet*-homomorphisms and we show a sufficient condition for a homoderivation on a lattice becomes isotone.

Definition 4.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice. A homoderivation h on L is called isotone if it satisfies the following condition:*

$$x \leqslant y \text{ implies } hx \leqslant hy, \text{ for any } x, y \in L.$$

Example 4.1. *Let $(L = \{0, a, b, c, d, 1\}, \leqslant, \wedge, \vee)$ be the bounded lattice given by the Hasse diagram in Figure 3. Let h_1 and h_2 be two functions on L defined in the following table:*

x	0	a	b	c	d	1
$h_1(x)$	c	c	c	c	d	1
$h_2(x)$	c	c	c	d	d	1

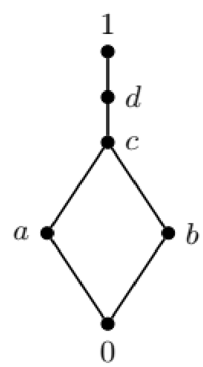

FIGURE 3. The Hasse diagram of $(L = \{0, a, b, c, d, 1\}, \leqslant, \wedge, \vee)$.

The functions h_1 and h_2 are isotone homoderivations on L. Indeed,

(i) the function h_1 is a homoderivation:

(1) $h_1(0 \wedge a) = h_1(0) = c$ *and* $(h_1(0) \wedge h_1(a)) \vee (h_1(0) \wedge a) \vee (0 \wedge h_1(a)) = (c \wedge c) \vee (c \wedge a) \vee (0 \wedge c) = c \vee a \vee 0 = c$;

(2) $h_1(0 \wedge b) = h_1(0) = c$ *and* $(h_1(0) \wedge h_1(b)) \vee (h_1(0) \wedge b) \vee (0 \wedge h_1(b)) = (c \wedge c) \vee (c \wedge b) \vee (0 \wedge c) = c \vee b \vee 0 = c$;

(3) $h_1(0 \wedge c) = h_1(0) = c$ *and* $(h_1(0) \wedge h_1(c)) \vee (h_1(0) \wedge c) \vee (0 \wedge h_1(c)) = (c \wedge c) \vee (c \wedge c) \vee (0 \wedge c) = c \vee c \vee 0 = c$;

(4) $h_1(0 \wedge d) = h_1(0) = c$ *and* $(h_1(0) \wedge h_1(d)) \vee (h_1(0) \wedge d) \vee (0 \wedge h_1(d)) = (c \wedge d) \vee (c \wedge d) \vee (0 \wedge d) = c \vee c \vee 0 = c$;

(5) $h_1(0 \wedge 1) = h_1(0) = c$ *and* $(h_1(0) \wedge h_1(1)) \vee (h_1(0) \wedge 1) \vee (0 \wedge h_1(1)) = (c \wedge 1) \vee (c \wedge 1) \vee (0 \wedge 1) = c \vee c \vee 0 = c$;

(6) $h_1(a \wedge b) = h_1(0) = c$ *and* $(h_1(a) \wedge h_1(b)) \vee (h_1(a) \wedge b) \vee (a \wedge h_1(b)) = (c \wedge c) \vee (c \wedge b) \vee (a \wedge c) = c \vee b \vee a = c$;

(7) $h_1(a \wedge c) = h_1(a) = c$ *and* $(h_1(a) \wedge h_1(c)) \vee (h_1(a) \wedge c) \vee (a \wedge h_1(c)) = (c \wedge c) \vee (c \wedge c) \vee (a \wedge c) = c \vee c \vee a = c$;

(8) $h_1(a \wedge d) = h_1(a) = c$ *and* $(h_1(a) \wedge h_1(d)) \vee (h_1(a) \wedge d) \vee (a \wedge h_1(d)) = (c \wedge d) \vee (c \wedge d) \vee (a \wedge d) = c \vee c \vee a = c$;

(9) $h_1(a \wedge 1) = h_1(a) = c$ *and* $(h_1(a) \wedge h_1(1)) \vee (h_1(a) \wedge 1) \vee (a \wedge h_1(1)) = (c \wedge 1) \vee (c \wedge 1) \vee (a \wedge 1) = c \vee c \vee a = c$;

(10) $h_1(b \wedge c) = h_1(b) = c$ *and* $(h_1(b) \wedge h_1(c)) \vee (h_1(b) \wedge c) \vee (b \wedge h_1(c)) = (c \wedge c) \vee (c \wedge c) \vee (b \wedge c) = c \vee c \vee b = c$;

(11) $h_1(b \wedge d) = h_1(b) = c$ *and* $(h_1(b) \wedge h_1(d)) \vee (h_1(b) \wedge d) \vee (b \wedge h_1(d)) = (c \wedge d) \vee (c \wedge d) \vee (b \wedge d) = c \vee c \vee b = c$;

(12) $h_1(b \wedge 1) = h_1(b) = c$ *and* $(h_1(b) \wedge h_1(1)) \vee (h_1(b) \wedge 1) \vee (b \wedge h_1(1)) = (c \wedge 1) \vee (c \wedge 1) \vee (b \wedge 1) = c \vee c \vee b = c$;

(13) $h_1(c \wedge d) = h_1(c) = c$ *and* $(h_1(c) \wedge h_1(d)) \vee (h_1(c) \wedge d) \vee (c \wedge h_1(d)) = (c \wedge d) \vee (c \wedge d) \vee (c \wedge d) = c \wedge d = c$;

(14) $h_1(c \wedge 1) = h_1(c) = c$ *and* $(h_1(c) \wedge h_1(1)) \vee (h_1(c) \wedge 1) \vee (c \wedge h_1(1)) = (c \wedge 1) \vee (c \wedge 1) \vee (c \wedge 1) = c \wedge 1 = c$;

(15) $h_1(d \wedge 1) = h_1(d) = d$ *and* $(h_1(d) \wedge h_1(1)) \vee (h_1(d) \wedge 1) \vee (d \wedge h_1(1)) = (d \wedge 1) \vee (d \wedge 1) \vee (d \wedge 1) = d \wedge 1 = d$;

(16) $h_1(x \wedge x) = h_1(x)$ *and* $(h_1(x) \wedge h_1(x)) \vee (h_1(x) \wedge x) \vee (x \wedge h_1(x)) = h_1(x) \vee (h_1(x) \wedge x) = h_1(x)$, *for any* $x \in L = \{0, a, b, c, d, 1\}$.

(ii) the function h_1 *satisfies if* $x \leqslant y$, *then* $h_1(x) \leqslant h_1(y)$, *for any* $x, y \in L = \{0, a, b, c, d, 1\}$. *Thus* h_1 *is isotone.*

The isotone homoderivation h_2 *can be proved similarly.*

In the following theorem, we provide a characterization of the isotone homoderivations on a lattice in terms of the *meet*-homomorphisms.

Theorem 4.1. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* h *a homoderivation on* L. *Then* d *is isotone if and only if* h *is a meet-homomorphism.*

Proof. On the one hand, the fact that h is a homoderivation on L guarantees that $hx \wedge hy \leqslant h(x \wedge y)$, for any $x, y \in L$. On the other hand, since h is isotone, it holds that $h(x \wedge y) \leqslant hx$ and $h(x \wedge y) \leqslant hy$, for any $x, y \in L$. Then $h(x \wedge y) \leqslant hx \wedge hy$, for any $x, y \in L$. Thus, $h(x \wedge y) = hx \wedge hy$, for any $x, y \in L$. Consequently, h is a *meet*-homomorphism on L. Conversely, we suppose that h is a *meet*-homomorphism and let $x, y \in L$ such that $x \leqslant y$. Then $hx = h(x \wedge y) = hx \wedge hy$. Thus, $hx \leqslant hy$. Therefore, h is isotone. □

The following theorem gives a sufficient condition for a homoderivation on a lattice to be isotone.

Theorem 4.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and h a homoderivation on L. If $x \leqslant hx$ for any $x \in L$, then h is isotone.*

Proof. Let $x, y \in L$ such that $x \leqslant y$, then $hx = h(x \wedge y)$. The fact that $x \leqslant hx$ and $y \leqslant hy$ we obtain that $x \leqslant hy$, $x \leqslant (hx \wedge hy)$ and $(hx \wedge y) \leqslant (hx \wedge hy)$. Since d is a homoderivation on L and $x \leqslant y$, it follows that

$$\begin{aligned} hx = h(x \wedge y) &= (hx \wedge hy) \vee (hx \wedge y) \vee (x \wedge hy) \\ &= (hx \wedge hy) \vee (hx \wedge y) \vee x \\ &= (hx \wedge hy) \vee x \vee (hx \wedge y) \\ &= (hx \wedge hy) \vee (hx \wedge y) \\ &= hx \wedge hy\,. \end{aligned}$$

Hence, $hx \leqslant hy$. Thus, h is isotone. □

In the following, we give a counter-example to show that the sufficient condition in Theorem 4.2 does not a necessary condition.

Example 4.2. *Let $(D(12), |, gcd, lcm)$ be the lattice of the positive divisors of 12 given by the Hasse diagram in Figure 4. Let h be the principal derivation on $D(12)$ defined as*

$$h(x) = gcd(3, x), \textit{ for any } x \in D(12).$$

Since any principal derivations on a lattice is isotone, Proposition 3.1 guarantees that h is an isotone homoderivation on $D(12)$. But, the fact that $4 \nmid h(4)$ guarantees that the converse implication of Theorem 4.2 does not necessarily hold.

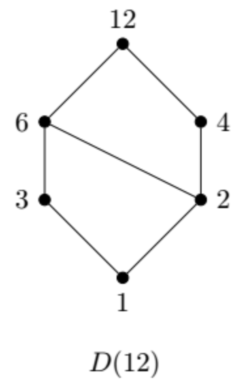

FIGURE 4. The Hasse diagram of the lattice $(D(12), |, gcd, lcm)$.

5. Conclusion and future research

In this paper, we have introduced the notion of homoderivation on a lattice as a combination of the two concepts of *meet*-homomorphisms and derivations. We have investigated some characterizations and properties of homoderivations on a lattice. We have established the relationship between derivations and homoderivations on a lattice. Moreover, we have studied the isotone homoderivations on a lattice as an interesting class of homoderivations. We have provided a characterization of the isotone homoderivations on a lattice in terms of the *meet*-homomorphisms. Further, we have given a sufficient condition for a homoderivation to be isotone.
Finally, we intend to extend this notion of homoderivation to other interesting algebraic structures.

Acknowledgement

This research work was supported by the General Direction of Scientific Research and Technological Development (DGRSDT)- Algeria.

References

[1] A. Amroune, L. Zedam and M. Yettou, (F, G)-derivations on a lattice, *Kragujevac Journal of Mathematics* **46** (2022), 773-778.

[2] A.Y. Abdelwanis and A. Boua, On generalized derivations of partially ordered sets, *Communications in Mathematics* **27** (2019), 69-78.

[3] E.F. Alharfie and N.M. Muthana, Homoderivation of prime rings with involution, *Bulleting of the International Mathematical Virtual Institute* **9** (2019), 301-304.

[4] E.F. Alharfie and N.M. Muthana, The commutativity of prime rings with homoderivations, *International Journal of Advanced and Applied Sciences* **5** (2018), 79-81.

[5] A. Al-Kenani, A. Melaibari and N. Muthana, Homoderivations and commutativity of *-prime rings, *East-West Journal of Mathematics* **17** (2015), 117-126.

[6] M. Ashraf, S. Ali and C. Haetinger, On derivations in rings and their applications, *The Aligarh Bulletin of Mathematics* **25** (2006), 79–107.

[7] I. Banič, Integrations on rings. *Open Mathematics* **15** (2017), 365–373.

[8] B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order*, 2nd edition, Cambridge University Press, Cambridge, 2002.

[9] M. M. El Sofy Aly, *Rings with some kinds of mappings*, M.Sc. Thesis, Cairo University, Branch of Fayoum, 2000.

[10] P. He, X.L. Xin and J. Zhan, On derivations and their fixed point sets in residuated lattices, *Fuzzy Sets and Systems* **303** (2016), 97–113.

[11] B. Kolman, R.C. Busby and S.C. Ross, *Discrete Mathematical Structures*. 4th edition, Prentice Hall PTR, 2000.

[12] A. Melaibari, N. Muthana and A. Al-Kenani, Homoderivations on rings. *General Mathematics Notes* **35** (2016), 1-8.

[13] G. Szász, Derivations of lattices, *Acta Scientiarum Mathematicarum* **37** (1975), 149–154.

[14] J. Wang, Y. Jun, X.L. Xin, T.Y. Li and Y. Zou, On derivations of bounded hyperlattices, *Journal of Mathematical Research with Applications* **36** (2016), 151–161.

[15] Q. Xiao and W. Liu, On derivations of quantales, *Open Mathematics* **14** (2016), 338–346.

[16] X.L. Xin, The fixed set of a derivation in lattices, *Fixed Point Theory and Application* **218** (2012), 1–12

[17] X.L. Xin, T.Y. Li and J.H. Lu, On derivations of lattices, *Information Sciences* **178** (2008), 307–316.

[18] Y.H. Yon and K.H. Kim, On f-Derivations from semilattices to lattices, *Commun. Korean Math. Soc.* **29** (2014), 27–36.

[19] L. Zedam, M. Yettou and A. Amroune, f-fixed points of isotone f-derivations on a lattice, *Discussiones Mathematicae-General Algebra and Applications* **39** (2019); 69–89.

[20] H. Zhang and Q. Li, On derivations of partially ordered sets, *Mathematica Slovaca* **67** (2017), 17-22.

(1) Department of the preparatory formation, National higher school of hydraulics, Blida, Algeria.

Email address: m.yettou@ensh.dz, mourad.yettou@univ-msila.dz

(2) Laboratory of Pure and Applied Mathematics, Department of Mathematics, University of M'sila, P.O. Box 166 Ichbilia, Msila 28000, Algeria.

Email address: abdelaziz.amroune@univ-msila.dz