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ON STRONGLY $g(x)$-INVO CLEAN RINGS

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Abstract. In this paper we characterize strongly $g(x)$-invo clean rings and specify the relation between strongly $g(x)$-invo clean rings and strongly clean rings. Further, some general properties and relevant examples of strongly $g(x)$-invo clean rings are presented.

1. Introduction

In this paper the ring R is assumed to be associative with identity element 1. Let $U(R)$ denote the set of units and $Inv(R)$ denote the subset of $U(R)$ consisting of all involutions $Inv(R)$ (square roots of 1), $Id(R)$ the set of all idempotents and $J(R)$ denote the Jacobson radical. We denote the rings of all $n \times n$ matrices by $M_n(R)$. The center of a ring R is denoted by $C(R)$ and $g(x)$ is a polynomial in $C(R)[x]$, the notation of clean rings was introduced in 1977 by Nicholson [7], a ring R is called clean if for every element $r \in R$ there exits a unit $u \in U(R)$ and an idempotent $e \in Id(R)$ such that $r = u + e$. If $ue = eu$, then R is called strongly clean. In 2002 [2], Camillo and Simón defined $g(x)$-clean rings. A ring R is called $g(x)$-clean if for every $r \in R$, $r = u + s$ where $u \in U(R)$ and $g(s) = 0$ for some $g(x)$ in $C(R)[x]$. Wang and Chen in [8] proved that R is $g_1(x)$-clean if and only if R is clean and $b - a$ is invertible. Moreover, for $g_2(x) \in (x - a)(x - b)C[x]$, where $a - b \in C$ and $b - a$ is a unit in R, if R is clean, then it is $g_2(x)$-clean, and if R is $g_2(x)$-clean for any $g_2(x) \in (x - a)(x - b)C[x]$, then it is clean. Strongly clean rings were studied

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by Yang in [9], he dealt with the question of when a matrix ring is strongly clean. Among many results, he defined the conditions for the matrix ring $M_n(R)$ over a commutative ring R to be strongly clean, and that the 2×2 matrix ring $M_2(\mathbb{Z}_{(2)})$ is not strongly clean where $\mathbb{Z}_2$ is the localization of $\mathbb{Z}$ at the prime ideal (2). Strongly $g(x)$-clean rings were studied in [6], R is called strongly $g(x)$-clean if every element $r \in R$ can be written as $r = s + u$ with $g(s) = 0$, u a unit of R, and $su = us$, strongly clean rings are strongly $(x^2 - x)$-clean rings; though, there are strongly $g(x)$-clean rings which are not strongly clean and vice versa.
Similarly, in 2020 $g(x)$-invo clean rings were defined by Abed Alhaleem and Handam in [1], an element $r \in R$ is called $g(x)$-invo clean if $r = v + s$ where $v \in Inv(R)$ and $g(s) = 0$, R is $g(x)$-invo clean ring if every element is $g(x)$-invo clean. Clearly, every $(x^2 - x)$-invo clean ring is invo clean. Moreover, some examples of $g(x)$-invo clean are these: ring $M_2(\mathbb{Z}_2)$ is $(x^3 - x)$-invo clean ring and $\mathbb{Z}_7$ is $(x^6 - 1)$-invo clean ring which is not invo-clean ring. In [3], Danchev introduced and studied invo-clean unital rings and strongly invo-clean unital rings, a ring R is said to be invo-clean if for every $r \in R$ can be written as $r = v + e$ where $v \in Inv(R)$ and $e \in Id(R)$. An invo-clean ring with $ve = ev$ is called strongly invo-clean. Also, simple examples of invo-clean rings are these: $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_8$. Oppositely, both $\mathbb{Z}_5$, $\mathbb{Z}_7$ are not invo-clean rings.
In this paper, we introduce the notion of strongly $g(x)$-invo clean ring and determine the relation between strongly $g(x)$-invo clean rings and strongly invo-clean rings. Some general properties of strongly $g(x)$-invo clean rings are discussed and several examples will be given.

2. Strongly $g(x)$-invo clean ring

In this section, we define the strongly $g(x)$-invo clean rings and present some related examples.

Definition 2.1. Let R be a ring and $Inv(R) = \{x \in R |\ x^2 = 1\}$, $C(R)$ be the center of R and $g(x) \in C(R)[x]$. An element $r \in R$ is called strongly $g(x)$-invo clean if there exits $v \in Inv(R)$ and s a root of $g(x)$ such that $r = v + s$ and $vs = vs$. A ring R is strongly $g(x)$-invo clean ring if every element in R is strongly $g(x)$-invo clean.

Strongly invo-clean rings are strongly $(x^2 - x)$-invo clean rings. However, there are strongly $g(x)$-invo clean rings which are not strongly invo-clean and vice versa.

Example 2.1. *Let R be a Boolean ring with more than two elements and let $c \in R$ with $0 \neq c \neq 1$. Define $g(x) = (x+1)(x+c)$. Then R is strongly invo-clean but R is not strongly $g(x)$-invo clean.*
Since $e = (2e-1) + (1-e)$ with $(2e-1)^2 = 1$ and $(1-e)^2 = 1-e$, then every idempotent is an invo-clean element. Thus, R is strongly invo-clean. But, if $c = v+s$ where $v \in Inv(R)$, $g(s) = 0$ and $vs = sv$, then it must be that $v = 1$ and $s = c - v$. But, $g(c-1) \neq 0$. Hence, R is not strongly $g(x)$-invo clean.

Example 2.2. *Let $R = \mathbb{Z}_{(p)} = \{\frac{m}{n} \in \mathbb{Q} : gcd(p,n) = 1$ and p prime$\}$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ and $g(x) = (x-a)(x^2+1) \in C(R)[x]$. Then R is strongly invo-clean but its not strongly $g(x)$-invo clean. Suppose R is g(x)-invo clean, then there exist an involution v and a root s of $g(x)$ such that $a = v + s$. Since $g(s) = (s-a)(s^2+1) = 0$ and $s - a \in Inv(R)$, so we get $(s^2+1) = 0$, which cannot be true in $\mathbb{Z}_{(p)}$. Hence, R is not strongly $g(x)$-invo clean.*

Theorem 2.1. *Let R be a ring and $g(x) \in (x-a)(x-b)C(R)[x]$ with $a, b \in C(R)$. Then the following holds:*

1. *R is strongly $(x-a)(x-b)$-invo clean if and only if R is strongly invo-clean and $b - a \in Inv(R)$.*
2. *If R is strongly invo-clean and $b - a \in Inv(R)$, then R is strongly $g(x)$-invo clean .*

Proof. (1).$\Rightarrow$: Assume that R is strongly $(x-a)(x-b)$-invo clean. Let $r \in R$ such that $r(b-a) + a = v + e$ where $(e-a)(e-b) = 0$, $v \in Inv(R)$ and $ve = ev$. Thus, $r = \frac{e-a}{b-a} + \frac{v}{b-a}$ where $\frac{v}{b-a} \in Inv(R)$, $(\frac{e-a}{b-a})^2 = \frac{e-a}{b-a}$ and $\frac{e-a}{b-a}.\frac{v}{b-a} = \frac{v}{b-a}.\frac{b-a}{e-a}$. Hence, R is strongly invo-clean.
Since a is strongly $(x-a)(x-b)$-invo clean, there exits $v \in Inv(R)$ and $t \in R$ such that $a = t + v$ with $(t-a)(t-b) = 0$ and $tu = ut$. Thus, $t = b$. Therefore, $(b-a) \in Inv(R)$.
$\Leftarrow$: Let $r \in R$. Since R is strongly invo-clean and $b - a \in Inv(R)$, $\frac{(r-a)}{(b-a)} = v + e$ where $v \in Inv(R)$, $e^2 = e$ and $ve = ev$. Thus, $r = [v(b-a) + a] + e(b-a)$ where

$v(b-a) \in Inv(R)$, $[e(b-a)+a-a][e(b-a)+a-b] = 0$ and $[e(b-a)+a]v(b-a) = v(b-a)[e(b-a)+a]$. Then, R is strongly $(x-a)(x-b)$- invo clean.

(2). Follows from (1). □

Corollary 2.1. *Let R be a ring. Then R is strongly invo-clean if and only if R is strongly (x^2+x)-invo clean.*

Proof. In the previous Theorem 2.1 when $a = 0$ and $b = -1$ □

***Remark* 1.** *The equivalence of strongly (x^2+x)-invo clean and invo-clean is a ring property. That is, it holds for a ring R but it may fail for a single element. For example, $1+1 = 2 \in \mathbb{Z}$ is invo-clean but it is not (x^2+x)-invo clean in $\mathbb{Z}$ since $\mathbb{Z}$ has only two involutions 1 and -1.*

Corollary 2.2. *A ring R is strongly invo-clean and $2 \in Inv(R)$ if and only if every element of R is the sum of an involution and a square root of 1.*

Proof. Let $g(x) = (x+1)(x-1) = x^2-1$. Note that the condition that every element of R is the sum of an involution and a square root of 1 is equivalent to R being strongly $g(x)$-invo clean. Therefore, the proof is immediate by Theorem 2.1. □

***Remark* 2.** *Let $g(x) = (x-a)k(x) \in C[x]$. If the equation $k(x) = 0$ has no solution in R, then R cannot be $g(x)$-invo clean (not strongly $g(x)$-invo clean). In fact, suppose R is g(x)-invo clean, then there exist an involution v and a root s of g(x) such that $a = v+s$. Since $g(s) = (s-a)k(s) = 0$ and $s-a \in Inv(R)$, we get $k(s) = 0$, which a contradiction.*

3. General properties of strongly $g(x)$-clean rings

In this section, we study some properties of strongly $g(x)$-invo clean rings and we consider the strongly $(x^n - x)$-invo clean rings.

Let R and S be rings and $\Phi : C(R) \to C(S)$ be a ring homomorphism with $\Phi(1_R) = 1_S$. For $g(x) = \sum\limits_{i=0}^{n} a_i x^i \in C(R)[x]$ and let $g^*(x) = \sum\limits_{i=0}^{n} \Phi(a_i) x^i \in C(S)[x]$. In particular, If $g(x) \in \mathbb{Z}[x]$, then $g^*(x) = g(x)$.

Proposition 3.1. *Let $\Phi : R \to S$ be a ring epimorphism. If R is strongly $g(x)$-invo clean, then S is strongly $g^*(x)$-invo clean.*

Proof. Let $g(x) = \sum_{i=0}^{n} a_i x^i \in C(R)[x]$ then $g^*(x) = \sum_{i=0}^{n} \Phi(a_i) x^i \in C(S)[x]$. As Φ is a ring epimorphism so for any $a \in S$, there exist $r \in R$ such that $\Phi(r) = a$. Since R is strongly $g(x)$-invo clean, there exists $s \in R$ and $v \in Inv(R)$ such that $r = v + s$ and $g(s) = 0$ and $vs = sv$. Then $a = \Phi(r) = \Phi(v + s) = \Phi(v) + \Phi(s)$ with $\Phi(v) \in Inv(S)$, and $g^*(\Phi(s)) = \sum_{i=0}^{n} \Phi(a_i)(\Phi(s))^i = \sum_{i=0}^{n} \Phi(a_i)\Phi(s^i) = \sum_{i=0}^{n} \Phi(a_i s^i) = \Phi(\sum_{i=0}^{n} a_i s^i) = \Phi(g(s)) = \Phi(0) = 0$. Therefore, S is strongly $g^*(x)$-invo clean. □

Proposition 3.2. *If R is a strongly $g(x)$-invo clean ring and I is an ideal of R, then R/I is strongly $g^*(x)$-invo clean where $g^*(x) \in C(R/I)[x]$.*

Proof. Let R be a strongly $g(x)$-invo clean ring and $\Phi : R \to R/I$ defined by $\Phi(r) = r = r + I$. Then, Φ is an epimorphism. Therefore, by Proposition 3.1 R/I is strongly $g^*(x)$-invo clean. □

Corollary 3.1. *Let R be a ring and $g(x) \in C(R)[x]$. If the formal series ring $R[[t]]$ is strongly $g(x)$-invo clean, then R is strongly $g(x)$-invo clean.*

Proof. This is because $\Phi : R[[t]] \to R$ with $\Phi(f) = a_0$ is a ring epimorphism where $f = \sum_{i \geq 0} a_i t^i \in R[[t]]$. □

Proposition 3.3. *Let $\{R_i\}_{i \in I}$be a family rings and $g(x) \in \mathbb{Z}[x]$. Then $R = \prod_{i=1}^{k} R_i$ is strongly $g(x)$-invo clean if and only if R_i is strongly $g(x)$-invo clean for all $i \in I$.*

Proof. $\Rightarrow$: For each $i \in I$, R_i is a homomorphic image of $\prod_{i=1}^{k} R_i$ under the projection homomorphism. Hence, R_i is strongly $g(x)$-invo clean by Proposition 3.1.
$\Leftarrow$: Let $(x_1, x_2, \ldots, x_k) \in \prod_{i=1}^{k} R_i$. For each i, write $x_i = v_i + s_i$ and $v_i s_i = s_i v_i$ where $v_i \in Inv(R_i)$, $g(s_i) = 0$. Let $v = (v_1, v_2, \ldots, v_k)$ and $s = (s_1, s_2, \ldots, s_k)$. Then, it is clear that $v \in Inv(R)$ and $g(s) = 0$ and $vs = sv$. Therefore, $\prod_{i=1}^{k} R_i$ is strongly $g(x)$-invo clean. □

Lemma 3.1. [5] *Let R be a ring and $e \in Id(R)$. Then $Inv(eRe) = (eRe) \cap (\bar{e} + Inv(R))$, where $\bar{e} = 1 - e$.*

Proof. ($\subseteq$) If $v \in Inv(eRe)$, then $v^2 = e$. Since the product of v with $\bar{e}$ is zero, $(v - \bar{e})^2 = e + \bar{e} = 1$, and so $(v - \bar{e}) \in Inv(R)$. Then $v \in \bar{e} + Inv(R)$.
($\supseteq$) If $a = \bar{e} + v \in eRe$ with $v \in Inv(R)$, then $a - \bar{e} = v$, and then $(a - \bar{e})^2 = 1$. Thus, $(ea - e\bar{e})^2 = e$, and so $ea^2 = e$. Therefore $a^2 = e$, and then $a \in Inv(eRe)$. ☐

In [4] Theorem 2.2, for invo-clean rings, if R is an invo-clean ring and $e^2 = e$, then the corner ring eRe is an invo-clean ring.

Theorem 3.1. *Let R be a strongly $(x-a)(x-b)$-invo clean ring with $a, b \in C(R)$. Then for any $e^2 = e \in R$, eRe is $(x - ea)(x - eb)$-invo clean. In particular, if $g(x) \in (x-ea)(x-eb) \in C(R)[x]$ and R is $(x-a)(x-b)$-invo clean with $a, b \in C(R)$, then eRe is strongly $g(x)$-invo clean.*

Proof. By Theorem 2.1, R is $(x - a)(x - b)$-invo clean if and only if R is invo-clean and $(b - a) \in Inv(R)$. If R is invo-clean, then eRe is invo-clean. By Theorem 2.1 and Lemma 3.1, eRe is strongly $(x - ea)(x - eb)$-invo clean. ☐

Proposition 3.4. *Let R be a ring with $2 \in Inv(R)$ and $k \in \mathbb{N}$. Then the following are equivalent:*
(1) *R is strongly invo clean.*
(2) *R is strongly $(x^2 - 2x)$-invo clean.*
(3) *R is strongly $(x^2 + 2x)$-invo clean.*
(4) *R is strongly $(x^2 - 2^k x)$-invo clean.*
(5) *R is strongly $(x^2 - 1)$-invo clean.*
(6) *for any $r \in R$ can be expressed as $r = v + s$ with some $v, s \in Inv(R)$ and $vs = sv$.*

Proof. (1) ⇒ (2) Since R is strongly invo-clean and $r \in R$, $\frac{r}{2} = v+s$ with $v \in Inv(R)$, $s^2 = s$ and $vs = sv$, then $r = 2v+2s$ with $2v \in Inv(R)$, $g(2s) = 0$ and $2v\cdot 2s = 2s\cdot 2v$. Hence, R is strongly $(x^2 - 2x)$-invo clean.

(2) ⇒ (1) Since R is strongly $(x^2 - 2x)$-invo clean, $2r = v + s$ with $v \in Inv(R)$, s is a root of $(x^2 - 2x)$ and $vs = sv$, then $r = \frac{v}{2} + \frac{s}{2}$, where $\frac{v}{2}$ is involution in R, $(\frac{s}{2})^2 = \frac{2s}{2^2} = \frac{s}{2}$ and $\frac{v}{2} \cdot \frac{s}{2} = \frac{s}{2} \cdot \frac{v}{2}$. Therefore, R is strongly invo clean.

(2) $\Rightarrow$ (3) Since R is strongly $(x^2 - 2x)$-invo clean and let $r \in R$, $-r = v + s$ such that $v \in Inv(R)$, $s^2 - 2s = 0$ and $vs = sv$, then $r = (-v) + (-s)$ with $-v \in Inv(R)$, $(-s)^2 + 2(-s) = s^2 - 2s = 0$ and $(-v)(-s) = (-s)(-v)$. Thus, R is strongly $(x^2 + 2x)$-invo clean.

(3) $\Rightarrow$ (2) In Theorem 2.1 when $a = 0$ and $b = -2$.

(3) $\Rightarrow$ (1) Since R is strongly $(x^2 + 2x)$-invo clean, $-2r = v + s$ with $v \in Inv(R)$, $s^2 + 2s = 0$ and $vs = sv$, then $r = \frac{-v}{2} + \frac{-s}{2}$, where $\frac{-v}{2}$ is involution in R, $(\frac{-s}{2})^2 = \frac{-2s}{2^2} = \frac{-s}{2}$ and $\frac{-v}{2} \cdot \frac{-s}{2} = \frac{-s}{2} \cdot \frac{-v}{2}$. Therefore, R is strongly invo clean.

(1) $\Rightarrow$ (4) By Theorem 2.1 let $a = 0$ and $b = 2^k$, Then, R is strongly $(x^2 - 2^k x)$-invo clean.

(4) $\Rightarrow$ (1) As $2 \in Inv(R)$, let $a = 0$ and $b = 2^{2k}$. Then by (1) of Theorem 2.1, R is strongly invo-clean.

(1) $\Rightarrow$ (6) Let $r \in R$, $1 - r = v + s$ where $s^2 = 2s$ (putting $k = 1$ in (4)), $v \in Inv(R)$ and $vs = sv$. Then, $r = v + (1 - s)$ with $-v \in Inv(R)$, $(1 - s)^2 = 1$ and $(-v)(1 - s) = (1 - s)(-v)$.

(6) $\Rightarrow$ (1) Let $r \in R$, $1 - r = v + s$ where $v, s \in Inv(R)$ and $vs = sv$. Then, $r = (-v) + (1 - s)$ with $-v \in Inv(R)$, $(1 - s)^2 = 2(1 - s)$ and $(-v)(1 - s) = (1 - s)(-v)$. Thus, R is strongly invo clean.

(5) $\Rightarrow$ (6) Since R is strongly $(x^2 - 1)$-invo clean, then $r = v + s \in R$ such that $v \in Inv(R)$, $s^2 = 1$ and $vs = sv$.

(6) $\Rightarrow$ (5) Let $r = v + s$ with some $v, s \in Inv(R)$ and $vs = sv$. So, s is a root of $x^2 - 1$. Therefore, R is strongly $(x^2 - 1)$-invo clean.

$\square$

Theorem 3.2. *Let R be a ring, $n \in \mathbb{N}$ and $a, b \in C(R)$. Then R is strongly $(ax^{2n} - bx)$-invo clean ring if and only if R is strongly $(ax^{2n} + bx)$-invo clean.*

Proof. $\Rightarrow$: Suppose R is strongly $(ax^{2n} - bx)$-invo clean, for every $r \in R$, $-r = v + s$ and $vs = sv$ where $(as^{2n} - bs) = 0$ and $v \in Inv(R)$. Then, $r = (-v) + (-s)$ where $(-v) \in Inv(R)$ and $a(-s)^{2n} + b(-s) = as^{2n} - bs = 0$ and $(-v)(-s) = (-s)(-v)$. Hence, R is strongly $(ax^{2n} + bx)$-invo clean.
$\Leftarrow$: Suppose R is strongly $(ax^{2n} + bx)$-invo clean, Then there exists v and s such that $-r = v + s$, $(as^{2n} + bs) = 0$, $v \in Inv(R)$ and $vs = sv$. So, $r = (-v) + (-s)$ satisfies $(-v) \in Inv(R)$, $a(-s)^{2n} - b(-s) = as^{2n} + bs = 0$ and $(-v)(-s) = (-s)(-v)$. Therefore, R strongly $(ax^{2n} - bx)$-invo clean. □

Proposition 3.5. *Let $2 \leq n \in \mathbb{N}$. A ring R is strongly $(x^n - x)$-invo clean ring if for every $r \in R$, $r = v + s$ where $v \in Inv(R)$ and $s^{n-1} = 1$ and $vs = sv$.*

Proof. Let $r = v + s$ where $v \in Inv(R)$, $s^{n-1} = 1$ and $vs = sv$. Then $g(s) = s^n - s = s(1 - 1) = 0$. Thus, s is a root of $(x^n - x)$. Therefore, R is strongly $(x^n - x)$-invo clean. □

Challenging problems

We close the article with the following two problems:

Problem 1. Classify weakly $g(x)$-invo clean rings. Let R be a ring and $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called weakly $g(x)$-invo clean if there exits $v \in Inv(R)$ and s is a root of $g(x)$ such that either $r = v + s$ or $r = v - s$. Describe the structure of these rings. Are they clean? Is the weakly $(x^2 - x)$-invo clean rings are precisely the weakly invo clean rings? Does $g(x)$-invo clean ring considered to be a weakly g(x)-invo clean ring?

Problem 2. What is the behaviour of the matrix rings over (strongly) $g(x)$-invo clean rings? How to identify new families of (strongly) $g(x)$-clean rings through matrix rings and triangular matrix rings?

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