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A NOTE ON GENERALIZED FRAME POTENTIAL

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Abstract. The concept of the frame potential was defined by Benedetto and Fickus [1] and it was showed that the finite unit norm tight frames can be characterized as the minimizers of the the energy functional. The concept was generalized by Carrizo and Heineken [5] and they introduced the concept of mixed frame potential. In the present paper, we further generalize the concept introducing the notion of generalized frame potential and observe that frame potential and mixed frame potential are particular cases of generalized frame potential. We prove some results concerning the generalized frame potential.

1. Introduction

Frames are main tools for use in signal processing, image processing, data compression and sampling theory etc. Today even more uses are being found for the theory such as optics, filter banks, signal detection as well as study of Besov spaces, Banach space theory etc. Earlier, Fourier transform has been a major tool in analysis for over a century. It has a serious lacking for signal analysis in which it hides in its phases information concerning the moment of emission and duration of a signal. In 1946, Gabor [14] filled this gap and formulated a fundamental approach to signal decomposition in terms of elementary signals.

On the basis of this development, in 1952, Duffin and Schaeffer[13] defined frames for Hilbert spaces to study some deep problems in non-harmonic Fourier series. The

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idea of Duffin and Schaeffer did not generate much general interest outside of non-harmonic Fourier series. But after the landmark paper of Daubechies, Grossmann and Meyer [12], in 1986, the theory of frames began to be more widely studied. Stability theorems for Hilbert spaces were studied in [6, 9, 10, 11]. Over the last decade, various other generalizations of frames for Hilbert spaces have been introduced and studied.

One of the key properties of frames is that they provide reconstruction for any vector of the space with the coefficients which may not be unique. Frames in finite-dimensional spaces are used in many applications. Frames in finite-dimensional spaces avoid the approximation problems that come up while truncating infinite frames. Finite tight frames are used frequently in information theory, sampling theory, communication theory etc [2, 3, 4].

In finite dimensional spaces, the concept of frame potential has been introduced by Benedetto and Fickus [1]. It turned out a very important tool regarding applications of frames in finite-dimensional spaces. It measures the orthogonality of a system of vectors. Recently, Carrizo and Heineken [5] introduced the concept of mixed frame potential in finite-dimensional spaces which quantifies the bi-orthogonality, in some sense. The notion of frame potential is closely associated with finite unit norm tight frames. The finite norm tight frames are studied in [7, 8], where physical interpretation of tight frames was given.

In the present paper, generalized frame potential is defined and studied. The frame potential and the mixed frame potential are particular cases of generalized frame potential. We prove the other results related to generalized frame potential.

Throughout the paper, $\mathbb{K}$ will denote $\mathbb{R}$ or $\mathbb{C}$, $\mathcal{H}$ be an infinite dimensional Hilbert space and $\mathcal{H}_d$ be a d-dimensional Hilbert space over $\mathbb{K}$.

Definition 1.1. A sequence $\{f_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2, \ f \in \mathcal{H}. \tag{1.1}$$

These positive constants A and B, respectively, are called lower and upper frame bounds for the frame $\{f_n\}_{n\in\mathbb{N}}$. The inequality (1.1) is called the *frame inequality* for the frame $\{f_n\}_{n\in\mathbb{N}}$. A frame $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is said to be

- *tight* if it is possible to choose $A,\ B$ satisfying inequality (1.1) with $A = B$.
- *Parseval* if it is a tight frame with $A = B = 1$.

If $\{f_n\}_{n=1}^N$ is a frame for $\mathcal{H}_d$, then the bounded linear operator

$$T : \mathbb{K}^N \to \mathcal{H}_d, \quad T\{\alpha_n\}_{n=1}^N = \sum_{n=1}^N \alpha_n f_n, \quad \{\alpha_n\}_{n=1}^N \in \mathbb{K}^N$$

is called the *pre-frame operator* or the *synthesis operator*. The adjoint operator is given by

$$T^* : \mathcal{H}_d \to \mathbb{K}^N, \quad T^*(f) = \{\langle f, f_n\rangle\}_{n=1}^N, \quad f \in \mathcal{H}_d$$

is called the *analysis operator*. By composing T and T^*, we obtain *frame operator*

$$S : \mathcal{H}_d \to \mathcal{H}_d, \quad S(f) = TT^*(f) = \sum_{n=1}^N \langle f, f_n\rangle f_n, \quad f \in \mathcal{H}.$$

Benedetto and Fickus [1] introduced the notion of frame potential. They define frame potential as

Definition 1.2. Let $\{f_n\}_{n=1}^N$ be a frame for $\mathcal{H}_d$. Then the frame potential of the frame $\{f_n\}_{n=1}^N$ is defined as

$$FP\left(\{f_n\}_{n=1}^N\right) = \sum_{m=1}^N \sum_{n=1}^N \langle f_m, f_n\rangle \langle f_n, f_m\rangle.$$

Generalizing the notion of frame potential, Carrizo and Heineken [5] defined the mixed frame potential.

Definition 1.3. Let $\{f_n\}_{n=1}^N$ and $\{g_n\}_{n=1}^N$ be sequences in $\mathcal{H}_d$. Then the mixed frame potential of the $\{f_n\}_{n=1}^N$ and $\{g_n\}_{n=1}^N$ is defined as

$$MFP\left(\{f_n\}_{n=1}^N, \{g_n\}_{n=1}^N\right) = \sum_{m=1}^N \sum_{n=1}^N \langle f_m, g_n\rangle \langle f_n, g_m\rangle.$$

In case $f_n = g_n$, for all $n = 1, 2, \cdots N$, MFP is FP.

2. Generalized Frame Potential

In this section, we further generalize the notion of frame potential and give the following definition

Definition 2.1. Let $\{f_n\}_{n=1}^N$ be a frame for $\mathcal{H}_d$ and Φ be a linear operator on $\mathcal{H}_d$. Then the generalized frame potential of $(\{f_n\}_{n=1}^N)$ with respect to Φ in $B(\mathcal{H})$ or the generalized frame potential of the pair $\big(\{f_n\}_{n=1}^N \subset \mathcal{H}_d, \Phi \in B(\mathcal{H})\big)$ is defined as

$$GFP(\{f_n\}, \Phi) = \sum_{m=1}^N \sum_{n=1}^N \langle f_m, \Phi(f_n)\rangle\langle f_n, \Phi(f_m)\rangle$$

and the alternating generalized frame potential of $(\{f_n\}_{n=1}^N)$ with respect to Φ in $B(\mathcal{H})$ is defined as

$$AGFP(\{f_n\}, \Phi) = \sum_{m=1}^N \sum_{n=1}^N \langle \Phi(f_m), f_n\rangle\langle \Phi(f_n), f_m\rangle.$$

In view of above definition one may observe that:

- if we take $\Phi = Id$, then the generalized frame potential is the frame potential of the sequence $\{f_n\}_{n=1}^N$.
- if we define an operator Φ on $\mathcal{H}_d$ such that

  $$\Phi(f_i) = g_i, \quad i = 1, 2, \cdots N.$$

  Then the generalized frame potential of $\big(\{f_n\}_{n=1}^N \subset \mathcal{H}_d, \Phi \in B(H)\big)$ is the mixed frame potential of $\big(\{f_n\}_{n=1}^N \subset \mathcal{H}_d, \{g_n\}_{n=1}^N \subset \mathcal{H}_d\big)$.

If $\{f_n\}_{n=1}^N$ is the sequence of vectors in $\mathcal{H}_d$ with the analysis operator T^* and Φ be a linear operator on $\mathcal{H}_d$. Then the analysis operator T_Φ^* of $\{\Phi(f_n)_{n=1}^N\}$ is given by

$$T_\Phi^* = T^*\Phi^*.$$

In fact,

$$\begin{aligned} T_\Phi^*(f) &= \{f, \Phi(f_n)\}_{n=1}^N \\ &= \{\Phi^*(f), f_n\}_{n=1}^N = T^*(\Phi^*(f)), \quad f \in \mathcal{H}_d. \end{aligned}$$

Therefore

$$\begin{aligned} TT_\Phi^*(f) &= T(T^*(\Phi^*(f))) \\ &= T(\{\langle \Phi^* f, f_n\rangle\}_{n=1}^N) \\ &= \sum_{n=1}^N \langle f, \Phi(f_n)\rangle f_n. \end{aligned}$$

Similarly

$$\begin{aligned} T_\Phi T^*(f) &= \Phi(T(T^*(f)) \\ &= \Phi\left(\sum_{n=1}^{N}\langle f, f_n\rangle f_n\right) \\ &= \sum_{n=1}^{N}\langle f, f_n\rangle \Phi(f_n). \end{aligned}$$

Further, if Φ is injective (or surjective), then $\{\Phi(f_n)\}_{n=1}^N$ is also a frame with the frame operator $\Phi S \Phi^*$, where S is the frame operator for the frame $\{f_n\}_{n=1}^N$. Furthermore, $\{\Phi(f_n)\}_{n=1}^N$ is a dual frame for the frame $\{f_n\}_{n=1}^N$.
In view of above discussion we have a following result:

Lemma 2.2. *Let $\{f_n\}_{n=1}^N$ be a frame for $\mathcal{H}_d$ and Φ be a linear operator on $\mathcal{H}_d$. Then*

$$GFP(\{f_n\}_{n=1}^N, \Phi) = \mathbf{Tr}((TT_\Phi{}^*)^2),$$

where T^* and T_Φ^* are the analysis operator of $\{f_n\}_{n=1}^N$ and $\{\Phi(f_n)\}_{n=1}^N$.

Proof. Let $\{e_n\}_{n=1}^d$ be an orthogonal basis of $\mathcal{H}_d$. Then

$$\begin{aligned} GFP(\{f_n\}, \Phi) &= \sum_{m=1}^{N}\sum_{n=1}^{N}\langle f_m, \Phi(f_n)\rangle\langle f_n, \Phi(f_m)\rangle \\ &= \sum_{m=1}^{N}\sum_{n=1}^{N}\left\langle \sum_{s=1}^{d}\langle f_m, e_s\rangle e_s, \Phi(f_n)\right\rangle\langle f_n, \Phi(f_m)\rangle \\ &= \sum_{m=1}^{N}\sum_{n=1}^{N}\sum_{s=1}^{d}\langle f_m, e_s\rangle\langle e_s, \Phi(f_n)\rangle\langle f_n, \Phi(f_m)\rangle \\ &= \sum_{s=1}^{d}\left\langle \sum_{m=1}^{N}\langle e_s, f_m\rangle\Phi(f_m), \sum_{n=1}^{N}\langle e_s, \Phi(f_n)\rangle f_n\right\rangle \\ &= \sum_{s=1}^{d}\langle TT_\Phi^* e_s, T_\Phi T^* e_s\rangle \\ &= \sum_{s=1}^{d}\left\langle (TT_\Phi^*)^2 e_s, e_s\right\rangle \\ &= \mathbf{Tr}\left((TT_\Phi^*)^2\right). \end{aligned}$$

If $\{\lambda_n\}_{n=1}^{d}$ denotes the eigenvalues of the operator TT_{Φ}^{*}, together with there multiplicities. Then

$$GFP(\{f_n\}_{n=1}^{N}, \Phi) = \mathbf{Tr}\left((TT_{\Phi}^{*})^2\right) = \sum_{n=1}^{N} \lambda_n^2. \qquad \square$$

Remark 2.3. In view of Lemma (2.2), we have

$$AGFP(\{f_n\}_{n=1}^{N}, \Phi) = \mathbf{Tr}\left((T_{\Phi}^{*}T)^2\right) = \sum_{n=1}^{N} \bar{\lambda}_n^2.$$

Remark 2.4. If $\{f_n\}_{n=1}^{N}$ be a frame for $\mathcal{H}_d$ and Φ be a linear operator on $\mathcal{H}_d$ such that $TT_{\Phi}^{*} = \lambda\mathcal{I}$, where $\mathcal{I}$ is the identity operator and $\lambda \in \mathbb{K}$ then

$$GFP(\{f_n\}, \Phi) = \lambda^2 d.$$

Now we discuss some properties of generalized frame potential in terms of a special set $\mathfrak{S}(\{\alpha_m\}_{m-1}^{N})$ defined as

$$\mathfrak{S}(\{\alpha_m\}_{m=1}^{N}) = \left\{ \left(\{f_n\}_{n=1}^{N} \subset \mathcal{H}_d, \Phi \in L(\mathcal{H}_d) \right) : \langle f_n, \Phi(f_n)\rangle = \alpha_n, \quad n = 1, 2, \cdots N \right\}.$$

Theorem 2.5. *Let $\{f_n\}_{n=1}^{N}$ be a frame for $\mathcal{H}_d$ and Φ be a linear operator on $\mathcal{H}_d$ such that $(\{f_n\}_{n=1}^{N}, \Phi) \in \mathfrak{S}(\{\alpha_n\}_{n=1}^{N})$ i.e., $\langle f_n, \Phi(f_n)\rangle = \alpha_n$, $n = 1, 2, \cdots, N$. Then following statements hold:*

(i) If all the eigenvalues of TT_{Φ}^{*} are real, then $GFP(\{f_n\}_{n=1}^{N}, \Phi)$ and $\sum_{m=1}^{N} \alpha_m$ are real. Further

$$GFP(\{f_n\}_{n=1}^{N}, \Phi) \geq \frac{1}{d}\left(\sum_{m=1}^{N} \alpha_m\right)^2.$$

(ii) If all the eigenvalues of TT_{Φ}^{*} are imaginary, then $GFP(\{f_n\}_{n=1}^{N}, \Phi)$ is real and $\sum_{m=1}^{N} \alpha_m$ is imaginary. Further

$$GFP(\{f_n\}_{n=1}^{N}, \Phi) \leq \frac{1}{d}\left(\sum_{m=1}^{N} \alpha_m\right)^2.$$

(iii) If TT_{Φ}^{*} has only one eigenvalue, then

$$GFP(\{f_n\}_{n=1}^{N}, \Phi) = \frac{1}{d}\left(\sum_{m=1}^{N} \alpha_m\right)^2.$$

Proof. Let $\{\lambda_n\}_{n=1}^{d}$ be the eigenvalues of TT_{Φ}^{*}. Then by Lemma 2.2

$$\begin{aligned} GFP(\{f_n\}_{n=1}^{N}, \Phi) &= \sum_{n=1}^{N} \lambda_n^2 \\ &= \sum_{n=1}^{N} \big((\mathbf{Re}(\lambda_n)^2 - \mathbf{Im}(\lambda_n)^2\big) + 2i \sum_{n=1}^{N} \mathbf{Re}(\lambda_n)\ \mathbf{Im}(\lambda_n). \end{aligned}$$

Let $\{e_n\}_{n=1}^{d}$ be an orthonormal basis for $\mathcal{H}_d$. Then

$$\begin{aligned} \sum_{n=1}^{N} \lambda_n &= \mathbf{Tr}(TT_{\Phi}^{*}) \\ &= \sum_{n=1}^{d} \langle TT_{\Phi}^{*} e_n, e_n \rangle \\ &= \sum_{n=1}^{d} \left\langle \sum_{m=1}^{N} \langle e_n, \Phi(f_m)\rangle f_m, e_n \right\rangle \\ &= \sum_{n=1}^{d} \sum_{m=1}^{N} \langle e_n, \Phi(f_m)\rangle \langle f_m, e_n\rangle \\ &= \sum_{m=1}^{N} \langle f_m, \Phi(f_m)\rangle \\ &= \sum_{m=1}^{N} \alpha_m. \end{aligned}$$

In order to obtain the possible extrema for real and imaginary parts of $GFP : \mathfrak{S}(\{\alpha_n\}_{n=1}^{N}) \to \mathbb{K}$, we study the critical points of the functions $Re : \Lambda \to \mathbb{R}$ and $Im : \Lambda \to \mathbb{R}$, where

$$\Lambda = \left\{ (\lambda_1, \cdots, \lambda_d) \in \mathbb{C}^d (\simeq \mathbb{R}^{2d}) : \sum_{n=1}^{d} \mathbf{Re}(\lambda_n) = \mathbf{Re}\left(\sum_{m=1}^{d} \alpha_m \right) \text{ and } \sum_{n=1}^{d} \mathbf{Im}(\lambda_n) = \mathbf{Im}\left(\sum_{m=1}^{d} \alpha_m \right) \right\}$$

such that

$$\begin{aligned} Re\Big((\lambda_1, \cdots, \lambda_d)\Big) &= Re\Big((\mathbf{Re}(\lambda_1), \cdots,\ \mathbf{Re}(\lambda_d),\ \mathbf{Im}(\lambda_1), \cdots,\ \mathbf{Im}(\lambda_d)) \Big) \\ &= \sum_{n=1}^{N} \big((\mathbf{Re}(\lambda_n)^2 - \mathbf{Im}(\lambda_n)^2\big) \end{aligned}$$

and

$$\begin{aligned} Im\Big((\lambda_1, \cdots, \lambda_d)\Big) &= Im\Big((\mathbf{Re}(\lambda_1), \cdots, \ \mathbf{Re}(\lambda_d), \ \mathbf{Im}(\lambda_1), \cdots, \ \mathbf{Im}(\lambda_d))\Big) \\ &= 2\sum_{n=1}^{d} \mathbf{Re}(\lambda_n) \ \mathbf{Im}(\lambda_n). \end{aligned}$$

Using Lagrange's multiplier method, if $(\lambda_1, \cdots, \lambda_d)$ is a critical point of Re and Im then

$$\lambda_1 = \lambda_2 = \cdots = \lambda_d = \frac{1}{d}\sum_{m=1}^{N} \alpha_m.$$

Further,

- If $\mathbf{Im}(\lambda_1, \cdots, \lambda_d) = 0$, then
$$\min_{\Lambda} \ Re = (\lambda_1, \cdots, \lambda_d) \ \text{ and } \ Im(\lambda_1, \cdots, \lambda_d) = 0.$$
- If $\mathbf{Re}(\lambda_1, \cdots, \lambda_d) = 0$, then
$$\max_{\Lambda} \ Re = (\lambda_1, \cdots, \lambda_d) \ \text{ and } \ Im(\lambda_1, \cdots, \lambda_d) = 0.$$
- If $\mathbf{Re}(\lambda_1, \cdots, \lambda_d) \neq 0$ and $\mathbf{Im}(\lambda_1, \cdots, \lambda_d) \neq 0$, then $(\lambda_1, \cdots, \lambda_d)$ is a saddle point of both the functions Re and Im.

Therefore,

(i) if $(\{f_n\}_{n=1}^N, \Phi) \in \mathfrak{S}(\{\alpha_n\}_{n=1}^N)$ such that every eigenvalue of TT_Φ^* is real then $GFP(\{f_n\}_{n=1}^N, \Phi)$ and $\sum\limits_{m=1}^{N} \alpha_m$ are real. Further
$$GFP(\{f_n\}_{n=1}^N, \Phi) \geq \frac{1}{d}\left(\sum_{m=1}^{N} \alpha_m\right)^2.$$

(ii) if $(\{f_n\}_{n=1}^N, \Phi) \in \mathfrak{S}(\{\alpha_n\}_{n=1}^N)$ such that every eigenvalue of TT_Φ^* is imaginary, then $GFP(\{f_n\}_{n=1}^N, \Phi)$ is real and $\sum\limits_{m=1}^{N} \alpha_m$ is imaginary. Further
$$GFP(\{f_n\}_{n=1}^N, \Phi) \leq \frac{1}{d}\left(\sum_{m=1}^{N} \alpha_m\right)^2.$$

(iii) if TT_Φ^* has only one eigenvalue, then
$$GFP(\{f_n\}_{n=1}^N, \Phi) = \frac{1}{d}\left(\sum_{m=1}^{N} \alpha_m\right)^2.$$
□

Remark 2.6. The bounds in Theorem 2.5(i) and Theorem 2.5(ii) may not be accomplished, but are attained when TT_{Φ}^* has only one eigenvalue which holds if $TT_{\Phi}^* = \left(\frac{1}{d}\sum_{m=1}^{N}\alpha_m\right)\mathcal{I}$.

Theorem 2.7. *Let $\{f_n\}_{n=1}^N$ be a frame for $\mathcal{H}_d$ and Φ be a linear operator on $\mathcal{H}_d$ such that $(\{f_n\}_{n=1}^N, \Phi) \in \mathfrak{S}(\{\alpha_n\}_{n=1}^N)$ and $TT_{\Phi}^* = A.\mathcal{I}$, where $A \in \mathbb{K}$. Then*

$$A = \frac{1}{d}\left(\sum_{m=1}^{N}\alpha_m\right).$$

Proof. Let $\{e_n\}_{n=1}^d$ be an orthonormal basis of $\mathcal{H}_d$. Since $TT_{\Phi}^* = A.\mathcal{I}$, we have

$$\begin{aligned}
\frac{1}{d}\sum_{m=1}^{N}\alpha_m &= \frac{1}{d}\sum_{m=1}^{N}\langle f_m, \Phi(f_m)\rangle \\
&= \frac{1}{d}\sum_{m=1}^{N}\left\langle \sum_{j=1}^{d}\langle f_m, e_j\rangle e_j, \Phi(f_m)\right\rangle \\
&= \frac{1}{d}\sum_{m=1}^{N}\sum_{j=1}^{d}\langle f_m, e_j\rangle\langle e_j, \Phi(f_m)\rangle \\
&= \frac{1}{d}\sum_{j=1}^{d}\left\langle \sum_{m=1}^{N}\langle e_j, \Phi(f_m)\rangle f_m, e_j\right\rangle \\
&= \frac{1}{d}\sum_{j=1}^{d}\langle TT_{\Phi}^*(e_j), e_j\rangle \\
&= A.
\end{aligned}$$

□

Definition 2.8. Let $\{f_n\}_{n=1}^N \subset \mathcal{H}_d$, $\phi : \mathcal{H}_d \to \mathcal{H}_d$ be a linear operator and $c \in \mathbb{K}$. Then $\{f_n\}_{n=1}^d$ is said to ϕ_c- *dual frame* if

$$\sum_{m=1}^{N}\langle f, \phi(f_m)\rangle f_m = cf, \ \ f \in \text{span}\{f_m\}_{m=1}^N.$$

or $$\sum_{m=1}^{N}\langle f, f_m\rangle \phi(f_m) = \overline{c}f, \ \ f \in \text{span}\{\phi(f_m)\}_{m=1}^N.$$

Definition 2.9. Let $\{f_n\}_{n=1}^N$ be a frame for $\mathcal{H}_d$ and Φ be a bounded linear operator on $\mathcal{H}_d$. Then the m^{th}- generalized frame potential of the pair $(f \in \mathcal{H}_d, \Phi \in B(H))$

denoted as $GFP_m(f,\Phi)$ is defined as

$$GFP_m(f,\Phi) = \langle f_m, \Phi(f_m)\rangle^2 + \sum_{\substack{n=1\\ n\neq m}} \langle f_n, \Phi(f)\rangle\langle f, \Phi(f_n)\rangle + GFP(\{f_n\}_{n\neq m}, \Phi).$$

In the next result, we will find certain conditions which are satisfied by generalized frame potential

Theorem 2.10. *Let* $\{\alpha_m\}_{m=1}^N \subseteq \mathbb{K}$. *If* $(\{f_n\},\Phi)$ *be a local extrema or saddle point of the real and imaginary point of* $GFP : \mathfrak{S}(\{\alpha_n\}_{n=1}^N) \to \mathbb{K}$ *then for each* $m = 1, 2, \cdots, N$ *there exist* $c \in \mathbb{K}$ *such that*

$$\sum_{n=1}^N \langle f_m, \Phi(f_n)\rangle f_n = c f_m$$

$$\sum_{\substack{n=1\\ n\neq m}}^N \langle \Phi(f_m), f_n\rangle \Phi(f_n) = \overline{c}\Phi(f_m).$$

Proof. As $(\{f_n\},\Phi)$ is a local extrema or a saddle point of the real or the imaginary part of the generalized frame potential GFP restricted to $\mathfrak{S}(\{\alpha_n\}_{n=1}^N)$, we have (f_m,Φ) is a local extrema or a saddle point of the real or the imaginary part of GFP_m in $\mathfrak{S}(\alpha_m) = \{(f \in \mathcal{H}_d, \Phi \in L(\mathcal{H}_d)) : \langle f, \Phi(f)\rangle = \alpha_m\}$, where

$$GFP_m(f,\Phi) = \alpha_m^2 + \sum_{\substack{n=1\\ n\neq m}}^N \langle f_n, \Phi(f)\rangle\langle f, \Phi(f_n)\rangle + \sum_{\substack{n=1\\ n\neq m}}^N \sum_{\substack{r=1\\ r\neq m}}^N \langle f_n, \Phi(f_r)\rangle\langle \Phi(f_r), f_n\rangle.$$

So, the corresponding constrained problem of several variables can be solved using Lagrange multipliers. Therefore there exist $c_1,\ c_2 \in \mathbb{R}$ such that

(1) $\nabla Re(GFP_m)(f,\Phi)|_{(f_m,\Phi)} = c_1\, \nabla Re(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)} + c_2\, \nabla Im(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)}.$

or there exist $c_3,\ c_4 \in \mathbb{R}$ such that

(2) $\nabla Im(GFP_m)(f,\Phi)|_{(f_m,\Phi)} = c_3\, \nabla Re(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)} + c_4\, \nabla Im(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)}.$

Therefore, from (1), we have the following equations

(i) $\nabla_{Re(f)} Re(GFP_m)(f,\Phi)|_{(f_m,\Phi)} = c_1\ \nabla_{Re(f)} Re(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)} + c_2\, \nabla_{Re(f)} Im(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)}.$

(ii) $\nabla_{Im(f)} Re(GFP_m)(f,\Phi)|_{(f_m,\Phi)} = c_1\ \nabla_{Im(f)} Re(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)} + c_2\, \nabla_{Im(f)} Im(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)}.$

(iii) $\nabla_{Re(\Phi(f))} Re(GFP_m)(f,\Phi)|_{(f_m,\Phi)} = c_1 \ \nabla_{Re(\Phi(f))} Re(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)} + c_2 \ \nabla_{Re(\Phi(f))} Im(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)}$.

(iv) $\nabla_{Im(\Phi(f))} Re(GFP_m)(f,\Phi)|_{(f_m,\Phi)} = c_1 \ \nabla_{Im(\Phi(f))} Re(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)} + c_2 \ \nabla_{Im(\Phi(f))} Im(\langle f, \Phi(f)\rangle)|_{(f_m,\Phi)}$.

Therefore, from (i) and (ii), we have

$$\begin{aligned} Re\left(\sum_{\substack{n=1\\ n\neq m}}^{N} \langle \phi(f_m), f_n\rangle \Phi(f_n)\right) &= c_1 Re(\Phi(f_m)) - c_2 Im(\Phi(f_m)) \\ Im\left(\sum_{\substack{n=1\\ n\neq m}}^{N} \langle \phi(f_m), f_n\rangle \Phi(f_n)\right) &= c_1 Im(\Phi(f_m)) + c_2 Re(\Phi(f_m)). \end{aligned}$$

From (iii) and (iv), we have

$$\begin{aligned} Re\left(\sum_{\substack{n=1\\ n\neq m}}^{N} \langle f_m, \phi(f_n)\rangle f_n\right) &= c_1 Re(f_m) + c_2 Im(f_m) \\ Im\left(\sum_{\substack{n=1\\ n\neq m}}^{N} \langle f_m, \phi(f_n)\rangle f_n\right) &= c_1 Im(f_m) - c_2 Re(f_m). \end{aligned}$$

This gives

$$\begin{aligned} \sum_{\substack{n=1\\ n\neq m}}^{N} \langle \phi(f_m), f_n\rangle \Phi(f_n) &= c_1 \Phi(f_m) + ic_2 \Phi(f_m) \\ &= (c_1 + ic_2)\Phi(f_m) \end{aligned}$$

and

$$\begin{aligned} \sum_{\substack{n=1\\ n\neq m}}^{N} \langle f_m, \phi(f_n)\rangle f_n &= c_1 f_m - ic_2 f_m \\ &= (c_1 - ic_2) f_m. \end{aligned}$$

So we obtain the desired result if we take $c = c_1 + ic_2$. Similarly, from (2), we can have

$$\sum_{\substack{n=1\\ n\neq m}}^{N} \langle \phi(f_m), f_n\rangle \Phi(f_n) = (c_4 - ic_3)\Phi(f_m)$$

and

$$\sum_{\substack{n=1\\n\neq m}}^{N}\langle f_m,\phi(f_n)\rangle f_n=(c_4+ic_3)f_m.$$

Thus, in case if $(\{f_m\}_{m=1}^N,\Phi)$ is a local extrema of the real and the imaginary part of the restricted generalized frame potential, then $c_4=c_1$ and $c_3=c_1$. □

Theorem 2.11. *Let $\{\alpha_m\}_{m=1}^N\subseteq\mathbb{K}$. If $(\{f_n\},\Phi)$ be a local extrema or saddle point of the real and imaginary point of generalized frame potential $GFP:\mathfrak{S}(\{\alpha_n\}_{n=1}^N)\to\mathbb{K}$ then*

(1) for each $m=1,2,\cdots,N$, f_m is the eigenvector of TT_Φ^* and $\Phi(f_m)$ is an eigenvector of $T_\phi T^*$, and the corresponding eigenvalues are conjugates.

(2) for $\{\lambda_j\}_{j=1}^J$ the sequence of distinct eigenvalues of TT_Φ^*, there exists a sequence of indexing sets $\{I_j\}_{j=1}^J=\{1,2,\cdots,N\}$, such that $\{f_m\}_{m\in I_j}$ is Φ_{λ_j} generalized dual frame.

where T^* and T_Φ^* are the analysis operator of $\{f_n\}_{n=1}^N$ and $\{\Phi(f_n)\}_{n=1}^N$.

Proof. (1) Since $(\{f_n\},\Phi)$ be a local extrema or saddle point of the real and imaginary point of $GFP:\mathfrak{S}(\{\alpha_n\}_{n=1}^N)\to\mathbb{K}$ then for each $m=1,2,\cdots,N$ there exist $c\in\mathbb{K}$ such that

$$\sum_{n=1}^{N}\langle f_m,\Phi(f_n)\rangle=cf_m\quad\text{and}\quad\sum_{\substack{n=1\\n\neq m}}^{N}\langle\Phi(f_m),f_n\rangle=\overline{c}\Phi(f_m).$$

This gives

$$\begin{aligned}TT_\Phi^*(f_m)&=\langle f_m,\Phi(f_m)\rangle f_m+\sum_{\substack{n=1\\n\neq m}}^{N}\langle f_m,\Phi(f_n)\rangle f_n\\&=(\alpha_m+c)f_m.\end{aligned}$$

Therefore, f_m is an eigenvalue of TT_Φ^*. Similarly

$$\begin{aligned}T_\Phi T^*(\Phi(f_m))&=\langle\Phi(f_m),f_m\rangle\Phi(f_m)+\sum_{\substack{n=1\\n\neq m}}^{N}\langle\Phi(f_m),f_n\rangle\Phi(f_n)\\&=(\overline{\alpha_m+c})f_m.\end{aligned}$$

Thus, $\Phi(f_m)$ is an eigenvector of $T_\Phi T^*$ and eigenvalues are conjugates.

(2) Let $\{\lambda_j\}_{j=1}^J$ be the sequence of distinct eigenvalues of TT_Φ^*. Since $(TT_\Phi^*)^* = T_\Phi T^*$, therefore the eigenvalues of $T_\Phi T^*$ are the conjugates of the eigenvalues of TT_Φ^*. Let $\{R_j\}_{j=1}^J$ be the set of all right eigenvectors of TT_Φ^* and $\{L_j\}_{j=1}^J$ be the set of all left eigenvectors of TT_Φ^*, i.e. for each $j = 1, \cdots, J$ we have:

$$\begin{aligned} R_j &= \{f \in \mathcal{H}_d : TT_\Phi^* f = \lambda_j f\} = \{f \in \mathcal{H}_d : f^* T_\Phi T^* = \overline{\lambda_j} f^*\} \\ L_j &= \{f \in \mathcal{H}_d : f^* TT_\Phi^* = \lambda_j f^*\} = \{f \in \mathcal{H}_d : T_\Phi T^* f = \overline{\lambda_j} f\}. \end{aligned}$$

We know that if $i \neq j$ then $R_i \perp L_j$. Let $\{I_j\}_{j=1}^J$ be the sequence of indexing sets given by

$$I_j = \left\{ m \in \{1, \cdots, N\} : TT_\Phi^* f_m = \lambda_j f_m \text{ and } T_\Phi T^* \Phi(f_m) = \overline{\lambda_j} \Phi(f_m) \right\}.$$

Take $j \in \{1, \cdots, J\}$ and $f \in R_j$. If $m \notin I_j$ then $m \in I_i$ for some $i \neq j$, hence $\Phi(f_m) \in L_i$ following that $\langle f, \Phi(f_m) \rangle = 0$. This yields

$$\sum_{m \in I_j} \langle f, \Phi(f_m) \rangle f_m = TT_\Phi^* f = \lambda_j f.$$

Analogously we obtain that for $f \in L_j$

$$\sum_{m \in I_j} \langle f, f_m \rangle \Phi(f_m) = T_\Phi T^* f = \overline{\lambda_j} f.$$

Since $\text{span}\{f_m\}_{m \in I_j} \subseteq R_j$, and $\text{span}\{\Phi(f_m)\}_{m \in I_j} \subseteq L_j$, we have

$$\begin{aligned} &\sum_{m \in I_j} \langle f, \Phi(f_m) \rangle f_m = TT_\Phi^* f = \lambda_j f, \quad f \in \text{span}\{f_m\}_{m \in I_j} \text{ and} \\ &\sum_{m \in I_j} \langle f, f_m \rangle \Phi(f_m) = T_\Phi T^* f = \overline{\lambda_j} f, \quad f \in \text{span}\{\Phi(f_m)\}_{m \in I_j}. \end{aligned}$$

Therefore, $\{f_m\}_{m \in I_j}$ is Φ_{λ_j}-generalized dual frames. Moreover, if $\lambda_j \neq 0$ then $\text{span}\{f_m\}_{m \in I_j} = R_j$ and $\text{span}\{\Phi(f_m)\}_{m \in I_j} = L_j$. □

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