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SOLVING CONFORMABLE EVOLUTION EQUATIONS BY AN EXTENDED NUMERICAL METHOD

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Abstract. In this paper, an extension for the tanh-function method is proposed by using an $(\alpha_1, \alpha_2, ..., \alpha_n, \beta)-$ rational transformation method, n is an arbitrary integer. As applications and to illustrate the validity of this method, the (1+3)-dimensional conformable time and space factional Burgers equation, and two other (1+3)-dimensional conformable fractional evolution examples, that are useful for academic purposes, are solved. More kink and generalized traveling wave solutions are obtained and some three-dimensional solution graphs are presented at the end of this paper.

1. Introduction

In recent years, fractional differential equations (FDEs for short), in the sense of Riemann-Louville, Caputo and Grunwald-Letnikov, have played a great role in modeling several real life problems. In fact, FDEs have been used to explain different real word phenomena in numerous fields that include diffusion and dynamics in biology, fluid mechanics, fluid flow, signal processing, and other areas [7, 11, 12, 15, 17, 18, 19, 21, 25, 26, 33].

The phenomena of dissipation, dispersion, reaction and diffusion are very related to the above real word phenomena, and nonlinear FDEs can be successfully used to evaluate them. Wave shapes have an effect on sediment transport, wave skewness and asymmetry have impacts on radar altimetry signals and ship responses to wave

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impacts. Traveling wave solutions are special classes of solutions for nonlinear FDEs. Solitary waves are traveling waves with constant speeds and shapes achieving the origin at distant locations. The appearance of solitary waves in nature is very frequent in solid state physics, chemical kinematics, optical fibers, electrical circuits, elastic media, and many other areas. Consequently, it is important for us to be concerned with searching for traveling wave solutions of nonlinear FDEs to understand the different facts.

Recently, several methods have been proposed to obtain some of the exact solutions and traveling wave solutions to nonlinear FDEs such as, variation iteration method, Adomian decomposition method, and homotopy analysis method, see [1, 5, 6, 16, 22, 23, 24, 27, 34, 35, 36]. However, a general method for solving the FDEs cannot be found. In particular, the exact solutions of FDEs have been very limited before the implementation of the fractional complex transformation by the authors of the paper [10]. After that, many numerical methods have been proposed, such as, the fractional sub-equation method, the $(G'/G)-$expansion method, the exp-function method, the first integral method, the exponential rational function method and the $(G'/G - 1/G)-$expansion method and other remarkable technics, see [8, 9, 29, 30, 31, 32].

The present paper proposes an $(n+1)-$dimensional extended tanh function method to investigate nonlinear conformable fractional differential equations (CFDEs) using some important ideas from the excellent papers [2, 3, 4, 10, 14, 20, 28]. By using Khalil conformable approach [13], that has some important characteristics and it seems more appropriate to describe the behavior of viscoelastic models and other real word phenomena [2, 12, 13], we present recent results on traveling wave solutions for the (1+3)-dimensional conformable time and space fractional Burgers equation and for two other (1+3)-dimensional conformable fractional evolution examples that are useful for academic purposes. We derive new exact solutions that do not exist in the literature. It is very important to note that the idea of the present paper has already been presented with Jumarie's approach, see the paper [10] for more details.

2. Conformable Fractional Derivatives

In this section, we introduce some definitions and properties related to the conformable fractional approach in the sense of Khalil. Fore more details on this important approach, see [13].

Definition 2.1. Let $f : (0, \infty) \to \mathbb{R}$. The conformable fractional derivative of order $0 < \alpha \leq 1$ is defined by

$$(T^{\alpha} f)(t) = \frac{\partial^{\alpha} f(t,x)}{\partial t^{\alpha}} = \lim_{\varepsilon \to 0} \left(\frac{f\left(t+\varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon} \right), \quad t > 0. \tag{2.1}$$

It is to note that when $\alpha = 1$, the above formula is reduced to the standard derivative or order one.

Definition 2.2. The conformable fractional integral of a function $f : (0, \infty) \to \mathbb{R}$ of order $0 < \alpha \leq 1$ is defined as

$$(I^{\alpha} f)(t) = \int_0^t \tau^{\alpha-1} f(\tau)\, d\tau. \tag{2.2}$$

We need also the following properties:

$$I^{\alpha} T^{\alpha} f(t) = f(t) - f(0) \tag{2.3}$$

and

$$(T^{\alpha} f)(t) = t^{1-\alpha} \frac{df(t)}{dt}. \tag{2.4}$$

3. Outline of the Method

In this section, we present the main steps of our proposed extension for the tanh-function method for the case of Khalil fractional theory.

We begin by considering the following nonlinear conformable fractional evolution equation:

$$F\left(u, T_t^{\beta} u, T_t^{2\beta} u, T_{x_i}^{\alpha_i} u, T_t^{\beta}(T_{x_i}^{\alpha_i} u), (T_{x_i}^{2\alpha_i} u), T_t^{\beta}(T_{x_i}^{2\alpha_i} u), ..., i \in \{= 1, ..., n\}\right) = 0, \tag{3.1}$$

where $u := u(x_1, x_2, ..., x_n, t)$ in the unknown function, $T^{\alpha} u$ are conformable partial fractional derivatives of u of order $\alpha, 0 < \alpha \leq 1$ and $T^{2\alpha} u := T^{\alpha}(T^{\alpha} u)$, while F is a polynomial of u and its derivatives.

Then, we introduce the transformation:

$$U(\xi) := u(x_1, x_2, ...x_n, t), \tag{3.2}$$

where,

$$\xi := \frac{k_1 x_1^{\alpha_1}}{\alpha_1} + \frac{k_2 x_2^{\alpha_2}}{\alpha_2} + ... + \frac{k_n x_n^{\alpha_n}}{\alpha_n} - \frac{ct^{\beta}}{\beta}.$$

So, Eq. (3.1) can be easily converted to the following nonlinear ODE:

$$G\Big(U, U', U'', U''', ...\Big) = 0. \tag{3.3}$$

The main steps are the following:

Step 1: We look for the solutions of (3.3) in the form

$$U(\xi) = S(Y) = \sum_{k=0}^{m} a_k Y^k + \sum_{k=1}^{m} b_k Y^{-k} \tag{3.4}$$

wherein,

$$Y := \tanh(\mu\xi), \mu \in \mathbb{R}, \tag{3.5}$$

where, μ is any arbitrary constant and a_k, b_k are constants to be determined later.

Step 2: We balance between the maximum order nonlinear term and the derivative of the maximum order appearing in (3.3), then we determine m. (See for more details [17, 28]).

Step 3: We substitute (3.4) together with (3.5) in (3.3) and using Maple, we find μ, a_k, b_k.

Step 4: We insert the values that have been found in step 3 into Eq. (3.4) along with Eq. (3.5), we construct closed-form traveling wave solutions of (3.3) from which we find the solutions of (3.1).

4. Applications

In this section, we apply the above proposed method to search further comprehensive on exact wave solutions for the (1+3)-dimensional conformable time and space fractional Burgers equation and for two other (1+3)-dimensional conformable time and space fractional evolution examples that are useful for academic purposes.

Example 4.1. *We begin the examples by considering the (1+3)-dimensional conformable Burgers equation:*

$$\frac{\partial^{\beta}u(t,x_1,x_2,x_3)}{\partial t^{\beta}}+au\frac{\partial^{\alpha_1}u(t,x_1,x_2,x_3)}{\partial x_1^{\alpha_1}}-\frac{\partial^{2\alpha_1}u(t,x_1,x_2,x_3)}{\partial x_1^{2\alpha_1}}$$

(4.1)

$$-\frac{\partial^{2\alpha_2}u(t,x_1,x_2,x_3)}{\partial x_2^{2\alpha_2}}-\frac{\partial^{2\alpha_3}u(t,x_1,x_2,x_3)}{\partial x_3^{2\alpha_3}}=0,\qquad 0<\alpha_1,\alpha_2,\alpha_3,\beta\le 1.$$

It is to note that Eq. (4.1) is the generalization of the (1+3)-dimensional classical Burgers equation. Also, the case of $\alpha_1=\alpha_2=\alpha_3=1$, Eq. (4.1) is used to describe the physical processes of propagation of weakly nonlinear acoustic waves through a gas-filled pipe, see [26].

To search for traveling wave solutions for Eq. (4.1), we use the transformation:

$$u\left(t,x_1,x_2,x_3\right)=U\left(\xi\right),\qquad \xi=\frac{k_1x_1^{\alpha_1}}{\alpha_1}+\frac{k_2x_2^{\alpha_2}}{\alpha_2}+\frac{k_3x_3^{\alpha_3}}{\alpha_3}-\frac{ct^{\beta}}{\beta}, \tag{4.2}$$

We have

$$\frac{\partial^{\beta}u(t,x_1,x_2,x_3)}{\partial t^{\beta}}=t^{1-\beta}\frac{\partial u(t,x_1,x_2,x_3)}{\partial t}=t^{1-\beta}\frac{\partial U(\xi)}{\partial t}$$

(4.3)

$$=t^{1-\beta}U_{\xi}\frac{\partial\xi}{\partial t}=t^{1-\beta}U_{\xi}\left(-\frac{\beta c}{\beta}t^{\beta-1}\right)$$

$$=-cU_{\xi}.$$

Therefore,

$$\frac{\partial^{\alpha_i}u(t,x_1,x_2,x_3)}{\partial x_i^{\alpha_i}}=x_i^{1-\alpha_i}\frac{\partial u(t,x_1,x_2,x_3)}{\partial x_i}=x_i^{1-\alpha_i}\frac{\partial U(\xi)}{\partial x_i}$$

(4.4)

$$=k_iU_{\xi},$$

which allows us to obtain

$$\frac{\partial^{2\alpha_i}u(t,x_1,x_2,x_3)}{\partial x_i^{2\alpha_i}}=\frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}\left(\frac{\partial^{\alpha_i}u(t,x_1,x_2,x_3)}{\partial x_i^{\alpha_i}}\right)$$

(4.5)

$$=k_i^2U_{\xi\xi}.$$

Substituting (4.2), (4.3) and (4.5) in (4.1), we get

$$-\left(k_1^2+k_2^2+k_3^2\right)U_{\xi\xi}-cU_{\xi}+ak_1UU_{\xi}=0. \tag{4.6}$$

Consequently,

$$-\left(k_1^2+k_2^2+k_3^2\right)U_\xi-cU+\tfrac{ak_1}{2}U^2=0. \tag{4.7}$$

Hence, we get

$$U\left(\xi\right)=a_0+a_1Y+b_1Y^{-1}. \tag{4.8}$$

Therefore, we have

$$U_\xi=\tfrac{d(U)}{d\xi}=\mu\left(1-Y^2\right)\tfrac{dU}{dY}=\mu\left(1-Y^2\right)\left(a_1-b_1Y^{-2}\right). \tag{4.9}$$

Substituting (4.9) in (4.7), we can write

$$\begin{aligned}&-\left(k_1^2+k_2^2+k_3^2\right)\mu\left(1-Y^2\right)\left(a_1-b_1Y^{-2}\right)\\&-c\left(a_0+a_1Y+b_1Y^{-1}\right)+\tfrac{ak_1}{2}\left(a_0+a_1Y+b_1Y^{-1}\right)^2=0.\end{aligned} \tag{4.10}$$

It yields then that

$$\begin{aligned}&\tfrac{aa_0^2k_1}{2}+ak_1a_1b_1-ca_0-\mu\left(a_1+b_1\right)\left(k_1^2+k_2^2+k_3^2\right)\\&+\left(-ca_1+ak_1a_1a_0\right)Y+\left(a_1\mu\left(k_1^2+k_2^2+k_3^2\right)+\tfrac{ak_1a_1^2}{2}\right)Y^2\\&+\left(-cb_1+ak_1b_1a_0\right)Y^{-1}+\left(b_1\mu\left(k_1^2+k_2^2+k_3^2\right)+\tfrac{ak_1b_1^2}{2}\right)Y^{-2}=0.\end{aligned} \tag{4.11}$$

This allows us to write

$$\begin{aligned}&\tfrac{aa_0^2k_1}{2}+ak_1a_1b_1-ca_0-\mu\left(a_1+b_1\right)\left(k_1^2+k_2^2+k_3^2\right)=0\\&-ca_1+ak_1a_1a_0=0\\&a_1\mu\left(k_1^2+k_2^2+k_3^2\right)+\tfrac{ak_1a_1^2}{2}=0\\&-cb_1+ak_1b_1a_0=0\\&b_1\mu\left(k_1^2+k_2^2+k_3^2\right)+\tfrac{ak_1b_1^2}{2}=0\end{aligned} \tag{4.12}$$

The above system admits the following sets of algebraic solutions:

$$a_0 = \frac{2c}{ak_1}, a_1 = 0, b_1 = 0, \mu = A, \tag{4.13}$$

$$a_0 = \frac{c}{ak_1}, a_1 = 0, b_1 = \frac{c}{ak_1}, \mu = -\frac{c}{2B}, \tag{4.14}$$

$$a_0 = \frac{c}{ak_1}, a_1 = 0, b_1 = -\frac{c}{ak_1}, \mu = \frac{c}{2B}, \tag{4.15}$$

$$a_0 = \frac{c}{ak_1}, a_1 = \frac{c}{ak_1}, b_1 = 0, \mu = -\frac{c}{2B}, \tag{4.16}$$

$$a_0 = \frac{c}{ak_1}, a_1 = -\frac{c}{ak_1}, b_1 = 0, \mu = \frac{c}{2B}, \tag{4.17}$$

$$a_0 = \frac{c}{ak_1}, a_1 = \frac{c}{2ak_1}, b_1 = \frac{c}{2ak_1}, \mu = -\frac{c}{4B}, \tag{4.18}$$

$$a_0 = \frac{c}{ak_1}, a_1 = -\frac{c}{2ak_1}, b_1 = -\frac{c}{2ak_1}, \mu = \frac{c}{4B}, \tag{4.19}$$

where $B = k_1^2 + k_2^2 + k_3^2, A \in \mathbb{R}$.

Consequently, we have the following explicit solutions:

$$u(t, x_1, x_2, x_3) = \frac{2c}{ak_1}, \tag{4.20}$$

$$\begin{aligned} u(t, x_1, x_2, x_3) &= \frac{c}{ak_1} + \frac{c}{ak_1}\left(\tanh \frac{-c}{2B}\left(\frac{k_1 x_1^{\alpha_1}}{\alpha_1} + \frac{k_2 x_2^{\alpha_2}}{\alpha_2} + \frac{k_3 x_3^{\alpha_3}}{\alpha_3} - \frac{ct^\beta}{\beta}\right)\right)^{-1} \\ &= \frac{c}{ak_1} + \frac{c}{ak_1} \coth \frac{-c}{2B}\left(\frac{k_1 x_1^{\alpha_1}}{\alpha_1} + \frac{k_2 x_2^{\alpha_2}}{\alpha_2} + \frac{k_3 x_3^{\alpha_3}}{\alpha_3} - \frac{ct^\beta}{\beta}\right), \end{aligned} \tag{4.21}$$

$$\begin{aligned} u(t, x_1, x_2, x_3) &= \frac{c}{ak_1} - \frac{c}{ak_1}\left(\tanh \frac{c}{2B}\left(\frac{k_1 x_1^{\alpha_1}}{\alpha_1} + \frac{k_2 x_2^{\alpha_2}}{\alpha_2} + \frac{k_3 x_3^{\alpha_3}}{\alpha_3} - \frac{ct^\beta}{\beta}\right)\right)^{-1} \\ &= \frac{c}{ak_1} - \frac{c}{ak_1} \coth \frac{c}{2B}\left(\frac{k_1 x_1^{\alpha_1}}{\alpha_1} + \frac{k_2 x_2^{\alpha_2}}{\alpha_2} + \frac{k_3 x_3^{\alpha_3}}{\alpha_3} - \frac{ct^\beta}{\beta}\right), \end{aligned} \tag{4.22}$$

$$u(t, x_1, x_2, x_3) = \frac{c}{ak_1} + \frac{c}{ak_1} \tanh \frac{-c}{2B}\left(\frac{k_1 x_1^{\alpha_1}}{\alpha_1} + \frac{k_2 x_2^{\alpha_2}}{\alpha_2} + \frac{k_3 x_3^{\alpha_3}}{\alpha_3} - \frac{ct^\beta}{\beta}\right), \tag{4.23}$$

$$u(t, x_1, x_2, x_3) = \frac{c}{ak_1} - \frac{c}{ak_1} \tanh \frac{c}{2B}\left(\frac{k_1 x_1^{\alpha_1}}{\alpha_1} + \frac{k_2 x_2^{\alpha_2}}{\alpha_2} + \frac{k_3 x_3^{\alpha_3}}{\alpha_3} - \frac{ct^\beta}{\beta}\right), \tag{4.24}$$

$$u\left(t,x_1,x_2,x_3\right)=\frac{c}{ak_1}+\frac{c}{2ak_1}\tanh\frac{-c}{4B}\left(\frac{k_1x_1^{\alpha_1}}{\alpha_1}+\frac{k_2x_2^{\alpha_2}}{\alpha_2}+\frac{k_3x_3^{\alpha_3}}{\alpha_3}-\frac{ct^{\beta}}{\beta}\right)$$
$$+\frac{c}{2ak_1}\coth\frac{-c}{4B}\left(\frac{k_1x_1^{\alpha_1}}{\alpha_1}+\frac{k_2x_2^{\alpha_2}}{\alpha_2}+\frac{k_3x_3^{\alpha_3}}{\alpha_3}-\frac{ct^{\beta}}{\beta}\right), \tag{4.25}$$

and

$$u\left(t,x_1,x_2,x_3\right)=\frac{c}{ak_1}-\frac{c}{2ak_1}\tanh\frac{c}{4B}\left(\frac{k_1x_1^{\alpha_1}}{\alpha_1}+\frac{k_2x_2^{\alpha_2}}{\alpha_2}+\frac{k_3x_3^{\alpha_3}}{\alpha_3}-\frac{ct^{\beta}}{\beta}\right)$$
$$-\frac{c}{2ak_1}\coth\frac{c}{4B}\left(\frac{k_1x_1^{\alpha_1}}{\alpha_1}+\frac{k_2x_2^{\alpha_2}}{\alpha_2}+\frac{k_3x_3^{\alpha_3}}{\alpha_3}-\frac{ct^{\beta}}{\beta}\right). \tag{4.26}$$

Figure 1 presents the graph of solution (4.24) for Eq. (4.1) with $a=-3$, $k_1=\frac{9}{10}, k_2=0, k_3=0, c=\frac{9}{10}, \alpha_1=\frac{3}{4}$ and $\beta=\frac{3}{5}$.

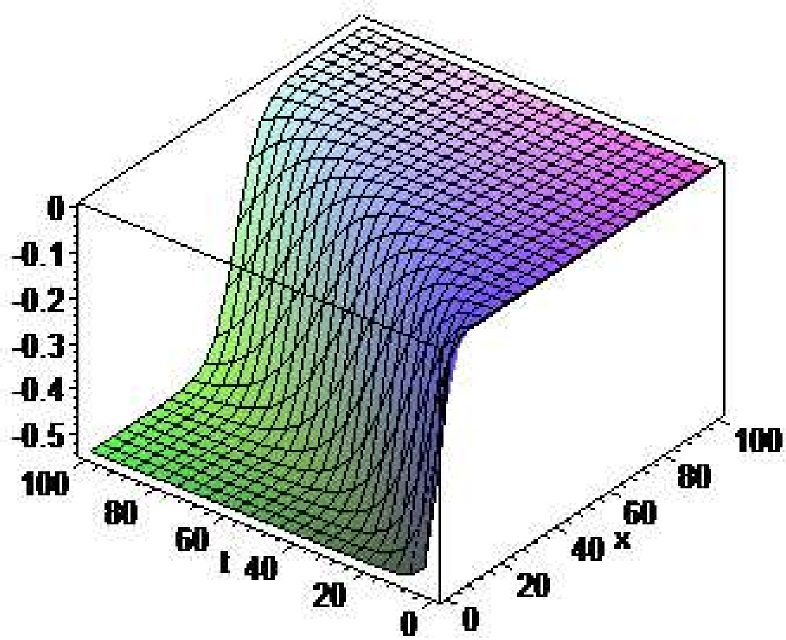

FIGURE 1. 3D plot of traveling wave solution (4.24) of (4.1) Sketched within the intervals $0 \le x_1 \le 100$ and $0 \le t \le 100$

The second example is useful for academic purposes. It is the following.

Example 4.2. *We consider the CFDE with its time and space derivatives:*

$$\frac{\partial^{\beta}u(t,x_1,x_2,x_3)}{\partial t^{\beta}}+\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\left(\frac{\partial^{\alpha_2}u(t,x_1,x_2,x_3)}{\partial x_2^{\alpha_2}}\right)+\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\left(\frac{\partial^{\alpha_3}u(t,x_1,x_2,x_3)}{\partial x_3^{\alpha_3}}\right)$$

$$+\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\left(\frac{\partial^{\alpha_3}u(t,x_1,x_2,x_3)}{\partial x_3^{\alpha_3}}\right)+au\frac{\partial^{\alpha_1}u(t,x_1,x_2,x_3)}{\partial x_1^{\alpha_1}} \tag{4.27}$$

$$+bu\frac{\partial^{\alpha_2}u(t,x_1,x_2,x_3)}{\partial x_2^{\alpha_2}}+du\frac{\partial^{\alpha_3}u(t,x_1,x_2,x_3)}{\partial x_3^{\alpha_3}}=0,\qquad 0<\alpha_1,\alpha_2,\alpha_3,\beta\leq 1.$$

With the same arguments as before, we can consider:

$$u\left(t,x_1,x_2,x_3\right)=U\left(\xi\right),\qquad \xi=\frac{k_1x_1^{\alpha_1}}{\alpha_1}+\frac{k_2x_2^{\alpha_2}}{\alpha_2}+\frac{k_3x_3^{\alpha_3}}{\alpha_3}-\frac{ct^{\beta}}{\beta}. \tag{4.28}$$

Hence, we obtain

$$\frac{\partial^{\beta}u(t,x_1,x_2,x_3)}{\partial t^{\beta}}=-cU_{\xi}$$

$$\frac{\partial^{2\beta}u(t,x_1,x_2,x_3)}{\partial t^{2\beta}}=c^2U_{\xi}$$

$$\frac{\partial^{\alpha_i}u(t,x_1,x_2,x_3)}{\partial x_i^{\alpha_i}}=k_iU_{\xi} \tag{4.29}$$

$$\frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}\left(\frac{\partial^{\alpha_j}u(t,x_1,x_2,x_3)}{\partial x^{\alpha_j}}\right)=k_jk_iU_{\xi\xi},$$

Substituting (4.29) and (4.28) into Eq. (4.27), we have

$$\left(k_1k_2+k_3k_1+k_2k_3\right)U_{\xi\xi}-cU_{\xi}+\left(ak_1+bk_2+dk_3\right)UU_{\xi}=0. \tag{4.30}$$

So, we obtain

$$\left(k_1k_2+k_3k_1+k_2k_3\right)U_{\xi}-cU+\frac{ak_1+bk_2+dk_3}{2}U^2=0. \tag{4.31}$$

Consequently, we get

$$U\left(\xi\right)=U\left(\xi\right)=a_0+a_1Y+b_1Y^{-1}. \tag{4.32}$$

Thanks to (3.5) and (3.6), we have

$$U_{\xi}=\frac{d(U)}{d\xi}=\mu\left(1-Y^2\right)\frac{dU}{dY}=\mu\left(1-Y^2\right)\left(a_1-b_1Y^{-2}\right) \tag{4.33}$$

Substituting (4.33) in (4.31), we find

$$\begin{aligned}&(k_1k_2+k_3k_1+k_2k_3)\,\mu\left(a_1+b_1-a_1Y^2-b_1Y^{-2}\right)-c\left(a_0+a_1Y+b_1Y^{-1}\right)\\&+\tfrac{ak_1+bk_2+dk_3}{2}\left(a_0^2+2a_1b_1+2a_0a_1Y+a_1^2Y^2+2a_0b_1Y^{-1}+b_1^2Y^{-2}\right)=0.\end{aligned}\tag{4.34}$$

Therefore, it yields that

$$\begin{aligned}&a_1B\mu+b_1B\mu+\tfrac{ak_1+bk_2+dk_3}{2}\left(a_0^2+2a_1b_1\right)-ca_0\\&+\left((a+b+d)\,a_0a_1-ca_1\right)Y+\left(\tfrac{ak_1+bk_2+dk_3}{2}a_1^2-Ba_1\mu\right)Y^2\\&+\left((a+b+d)\,a_0b_1-cb_1\right)Y^{-1}+\left(\tfrac{ak_1+bk_2+dk_3}{2}b_1^2-Bb_1\mu\right)Y^{-2}=0.\end{aligned}\tag{4.35}$$

Hence, we have

$$\begin{aligned}&a_1B\mu+b_1B\mu+\tfrac{ak_1+bk_2+dk_3}{2}\left(a_0^2+2a_1b_1\right)-ca_0=0\\&(a+b+d)\,a_0a_1-ca_1=0\\&\left(\tfrac{ak_1+bk_2+dk_3}{2}a_1^2-Ba_1\mu\right)=0\\&(a+b+d)\,a_0b_1-cb_1=0\\&\tfrac{ak_1+bk_2+dk_3}{2}b_1^2-Bb_1\mu=0\end{aligned}\tag{4.36}$$

Solving the above algebraic system, we obtain the following solutions:

$$a_0=\tfrac{2c}{D},a_1=0,b_1=0,\mu=A,\tag{4.37}$$

$$a_0=\tfrac{c}{D},a_1=\tfrac{c}{D},b_1=0,\mu=\tfrac{c}{2B},\tag{4.38}$$

$$a_0=\tfrac{c}{D},a_1=-\tfrac{c}{D},b_1=0,\mu=-\tfrac{c}{2B},\tag{4.39}$$

$$a_0=\tfrac{c}{D},a_1=0,b_1=\tfrac{c}{D},\mu=\tfrac{c}{2B},\tag{4.40}$$

$$a_0=\tfrac{c}{D},a_1=0,b_1=-\tfrac{c}{D},\mu=-\tfrac{c}{2B},\tag{4.41}$$

$$a_0 = \tfrac{c}{D}, a_1 = \tfrac{c}{2D}, b_1 = \tfrac{c}{2D}, \mu = \tfrac{c}{4B}, \tag{4.42}$$

and

$$a_0 = \tfrac{c}{D}, a_1 = -\tfrac{c}{2D}, b_1 = -\tfrac{c}{2D}, \mu = -\tfrac{c}{4B}, \tag{4.43}$$

where $B = k_1^2 + k_2^2 + k_3^2$, $D = ak_1 + bk_2 + dk_3$ and $A \in \mathbb{R}$.

Consequently, we obtain the following generalized solitary wave solutions:

$$u(t, x_1, x_2, x_3) = \tfrac{2c}{D}, \tag{4.44}$$

$$u(t, x_1, x_2, x_3) = \tfrac{c}{D} + \tfrac{c}{D} \tanh \tfrac{c}{2B} \left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta} \right), \tag{4.45}$$

$$u(t, x_1, x_2, x_3) = \tfrac{c}{D} - \tfrac{c}{D} \tanh \left(-\tfrac{c}{2B} \left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta} \right) \right), \tag{4.46}$$

$$\begin{aligned} u(t, x_1, x_2, x_3) &= \tfrac{c}{D} + \tfrac{c}{D} \left(\tanh \tfrac{c}{2B} \left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta} \right) \right)^{-1} \\ &= \tfrac{c}{D} + \tfrac{c}{D} \coth \tfrac{c}{2B} \left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta} \right), \end{aligned} \tag{4.47}$$

$$\begin{aligned} u(t, x_1, x_2, x_3) &= \tfrac{c}{D} - \tfrac{c}{D} \left(\tanh \left(-\tfrac{c}{2B} \left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta} \right) \right) \right)^{-1} \\ &= \tfrac{c}{D} - \tfrac{c}{D} \coth \left(-\tfrac{c}{2B} \left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta} \right) \right), \end{aligned} \tag{4.48}$$

$$\begin{aligned} u(t, x_1, x_2, x_3) &= \tfrac{c}{D} + \tfrac{c}{2D} \tanh \tfrac{c}{4B} \left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta} \right) \\ &\quad + \tfrac{c}{2D} \coth \tfrac{c}{4B} \left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta} \right) \end{aligned} \tag{4.49}$$

and

$$\begin{aligned} u(t, x_1, x_2, x_3) &= \tfrac{c}{D} - \tfrac{c}{2D} \tanh \tfrac{-c}{4B} \left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta} \right) \\ &\quad - \tfrac{c}{2D} \coth \tfrac{-c}{4B} \left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta} \right). \end{aligned} \tag{4.50}$$

Figure 2 presents the graph of solution (4.49) for Eq. (4.27) with $a = 3, b = -2$, $d = -3, k_1 = 3, k_2 = 0, k_3 = 0, c = 5, \alpha_1 = \frac{3}{4}$ and $\beta = \frac{7}{10}$.

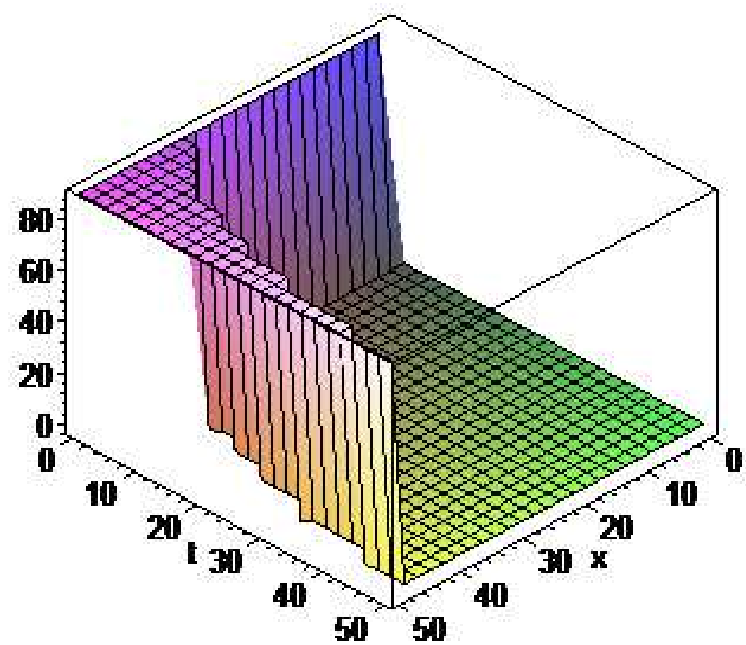

FIGURE 2. 3D plot of traveling wave solution (4.49) of (4.27) Sketched within the intervals $0 \leq x_1 \leq 50$ and $0 \leq t \leq 50$

Example 4.3. *The 3rd example is the following nonlinear CFDE which is also useful for academic purposes:*

$$\begin{aligned} &2\frac{\partial^{\beta}}{\partial t^{\beta}}\left(\frac{\partial^{\alpha_2}u(t,x_1,x_2,x_3)}{\partial x_2^{\alpha_2}}\right)+3\left(\frac{\partial^{\alpha_2}u(t,x_1,x_2,x_3)}{\partial x_2^{\alpha_2}}\right)\left(\frac{\partial^{2\alpha_1}u(t,x_1,x_2,x_3)}{\partial x_1^{2\alpha_1}}\right)\\ &+3\left(\frac{\partial^{\alpha_1}u(t,x_1,x_2,x_3)}{\partial x_1^{\alpha_1}}\right)\left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}u(t,x_1,x_2,x_3)}{\partial x_2^{\alpha_2}}\right)-3\left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_3}u(t,x_1,x_2,x_3)}{\partial x_3^{\alpha_3}}\right)\\ &+\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\left(\frac{\partial^{3\alpha_1}u(t,x_1,x_2,x_3)}{\partial x_1^{3\alpha_1}}\right)=0, \qquad 0<\alpha_1,\alpha_2,\alpha_3,\beta\leq 1. \end{aligned} \tag{4.51}$$

We take

$$u\left(t,x_1,x_2,x_3\right)=U\left(\xi\right), \qquad \xi=\frac{k_1x_1^{\alpha_1}}{\alpha_1}+\frac{k_2x_2^{\alpha_2}}{\alpha_2}+\frac{k_3x_3^{\alpha_3}}{\alpha_3}-\frac{ct^{\beta}}{\beta}, \tag{4.52}$$

So, we have

(4.53)
$$\begin{aligned}
&\frac{\partial^{\beta} u(t,x_1,x_2,x_3)}{\partial t^{\beta}} = -cU_{\xi}\\
&\frac{\partial^{\beta}}{\partial t^{\beta}}\left(\frac{\partial^{\alpha_i} u(t,x_1,x_2,x_3)}{\partial x_i^{\alpha_i}}\right) = -ck_i U_{\xi\xi}\\
&\frac{\partial^{\alpha_i} u(t,x_1,x_2,x_3)}{\partial x_i^{\alpha_i}} = k_i U_{\xi}\\
&\frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}\left(\frac{\partial^{\alpha_j} u(t,x_1,x_2,x_3)}{\partial x^{\alpha_j}}\right) = k_j k_i U_{\xi\xi}.
\end{aligned}$$

Substituting (4.53) and (4.52) into (4.51), we can write

$$6k_1^2 k_2 U_{\xi} U_{\xi\xi} - (3k_1k_3 + 2ck_2)\, U_{\xi\xi} + k_1^3 k_2 U_{\xi\xi\xi\xi} = 0, \tag{4.54}$$

and consequently, we obtain

$$k_1^3 k_2 U_{\xi\xi\xi} - (3k_1k_3 + 2ck_2)\, U_{\xi} + 3k_1^2 k_2 \left(U_{\xi}\right)^2 = 0. \tag{4.55}$$

Taking

$$U(\xi) = U(\xi) = a_0 + a_1 Y + b_1 Y^{-1}, \tag{4.56}$$

then, by (3.5) and (3.6), we have

(4.57)
$$\begin{aligned}
&U_{\xi} = \frac{d(U)}{d\xi} = \mu\left(1-Y^2\right)\frac{dU}{dY} = a_1\mu + b_1\mu - a_1\mu Y^2 - b_1\mu Y^{-2}.\\
&U_{\xi\xi} = \frac{d^2(U)}{d\xi^2} = \mu^2\left(1-Y^2\right)\left(-2Y\frac{dU}{dY} + \left(1-Y^2\right)\frac{d^2U}{dY^2}\right)\\
&U_{\xi\xi\xi} = \frac{d^3(U)}{d\xi^3} = \mu^3\left(1-Y^2\right)\begin{pmatrix} 2\left(3Y^2-1\right)\frac{dU}{dY} \\ -6Y\left(1-Y^2\right)\frac{d^2U}{d\xi^2} + \left(1-Y^2\right)^2\frac{d^3U}{dY^3}\end{pmatrix}\\
&= -2a_1\mu^3 + 2b_1\mu^3 + 8a_1\mu^3Y^2 + 4b_1\mu^3Y^{-2} - 6a_1\mu^3Y^4 - 6b_1Y^{-4}\mu^3.
\end{aligned}$$

Substituting (4.57) in (4.55), we obtain

$$\begin{aligned}
&k_1^3k_2\left(-2a_1\mu^3+2b_1\mu^3+8a_1\mu^3Y^2+4b_1\mu^3Y^{-2}-6a_1\mu^3Y^4-6b_1Y^{-4}\mu^3\right)\\
&+3k_1^2k_2\left(a_1+b_1\right)^2\mu^2+6k_1^2k_2a_1b_1\mu^2+3k_1^2k_2b_1^2\mu^2Y^{-4}+3k_1^2k_2a_1^2\mu^2Y^4\\
&-6k_1^2k_2\left(a_1+b_1\right)b_1\mu^2Y^{-2}-6k_1^2k_2\left(a_1+b_1\right)a_1\mu^2Y^2\\
&-\left(3k_1k_3+2ck_2\right)\left(a_1\mu+b_1\mu-b_1\mu Y^{-2}-a_1\mu Y^2\right)=0.
\end{aligned}\tag{4.58}$$

Therefore, it yields that

$$\begin{aligned}
&-2k_1^3k_2a_1\mu^3+2k_1^3k_2b_1\mu^3+3k_1^2k_2\left(a_1+b_1\right)^2\mu^2\\
&+6k_1^2k_2a_1b_1\mu^2-\left(3k_1k_3+2ck_2\right)\left(a_1+b_1\right)\mu\\
&+\left(8k_1^3k_2a_1\mu^3-6k_1^2k_2a_1\left(a_1+b_1\right)\mu^2+a_1\left(3k_1k_3+2ck_2\right)\mu\right)Y^2\\
&+\left(4k_1^3k_2b_1\mu^3-6k_1^2k_2\left(a_1+b_1\right)b_1\mu^2+\left(3k_1k_3+2ck_2\right)b_1\mu\right)Y^{-2}\\
&+\left(3k_1^2k_2a_1^2\mu^2-6k_1^3k_2a_1\mu^3\right)Y^4+\left(3k_1^2k_2b_1^2\mu^2-6b_1k_1^3k_2\mu^3\right)Y^{-4}=0.
\end{aligned}\tag{4.59}$$

So, we can write

$$\begin{aligned}
&-2k_1^3k_2a_1\mu^3+2k_1^3k_2b_1\mu^3+3k_1^2k_2\left(a_1+b_1\right)^2\mu^2\\
&+6k_1^2k_2a_1b_1\mu^2-\left(3k_1k_3+2ck_2\right)\left(a_1+b_1\right)\mu=0\\
&8k_1^3k_2a_1\mu^3-6k_1^2k_2a_1\left(a_1+b_1\right)\mu^2+a_1\left(3k_1k_3+2ck_2\right)\mu=0\\
&4k_1^3k_2b_1\mu^3-6k_1^2k_2\left(a_1+b_1\right)b_1\mu^2+\left(3k_1k_3+2ck_2\right)b_1\mu=0\\
&3k_1^2k_2a_1\mu^2\left(a_1-2k_1\mu\right)=0\\
&3k_1^2k_2b_1\mu^2\left(b_1-2k_1\mu\right)=0.
\end{aligned}\tag{4.60}$$

Based on this system, we get:

$$a_0 = D, a_1 = 0, b_1 = 0, \mu = E, \tag{4.61}$$

$$a_0 = D, a_1 = 0, b_1 = \sqrt{\tfrac{Bk_1}{2A}}, \mu = \tfrac{1}{4}\sqrt{\tfrac{2B}{Ak_1}}, \tag{4.62}$$

$$a_0 = D, a_1 = \sqrt{\tfrac{Bk_1}{A}}, b_1 = 0, \mu = \tfrac{1}{2}\sqrt{\tfrac{B}{Ak_1}}, \tag{4.63}$$

where $A = k_1^2 k_2,\ B = 3k_1k_3 + 2ck_2$ and $D, E \in \mathbb{R}$.

Therefore, we have the following solutions:

$$u(t, x_1, x_2, x_3) = D \tag{4.64}$$

$$u(t, x_1, x_2, x_3) = D + \sqrt{\tfrac{Bk_1}{2A}} \coth\left(\tfrac{1}{4}\sqrt{\tfrac{2B}{Ak_1}}\left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta}\right)\right). \tag{4.65}$$

$$u(t, x_1, x_2, x_3) = D + \sqrt{\tfrac{Bk_1}{A}} \tanh\left(\tfrac{1}{2}\sqrt{\tfrac{B}{Ak_1}}\left(\tfrac{k_1 x_1^{\alpha_1}}{\alpha_1} + \tfrac{k_2 x_2^{\alpha_2}}{\alpha_2} + \tfrac{k_3 x_3^{\alpha_3}}{\alpha_3} - \tfrac{ct^\beta}{\beta}\right)\right). \tag{4.66}$$

Conclusion

We have presented an $(n+1)-$dimensional extended tanh function method for solving the (1+3)-dimensional conformable time and space fractional Burgers equations. Using the same method, two other (1+3)-dimensional conformable time-and-space-fractional evolution equations, that are useful for academic purposes, have also been solved. The work on the proposed examples has allowed us to derive many kink and traveling wave solutions. At the end of the paper, we have plotted the dynamics of some traveling wave solutions in terms of time and 1-space coordinates to complete the study and to confirm the power of this method to handle many other nonlinear conformable fractional evolution equations in the sense of R. Khalil of various forms.

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References

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., 279, 2015, 57-66.

[2] M.A.E. Abdelrahman, H.A. Alkhidhr, *Closed-form solutions to the conformable space-time fractional simplified MCH equation and time fractional Phi-4 equation,* Results in Physics, 18, 2020, 103-114.

[3] E. Aksoy, O. Guner, A. Bekir, A.C. Cevikel, *Exact solutions of the (3+1)-dimensional space-time fractional Jimbo-Miwa equation,* AIP Conference Proceedings, 1738, 290014, 2016, doi: 10.1063/1.4952086.

[4] A. Bekir, *On traveling wave solutions to combined KdV-mKdV equation and modifed Burgers-KdV equation,* Communications in Nonlinear Science and Numerical Simulation, 14(4), 2009, 1038-1042.

[5] S.A. Elwakil, S.K. El-Labany, M.A. Zahran, R. Sabry, *Modified extended tanh-function method and its applications to nonlinear equations,* Applied Mathematics and Computation, 161, 2005, 403-412.

[6] E. Fan, *Extended tanh-function method and its applications to nonlinear equations*, Physics Letters A, 277, 2000, 212-218. 2000.

[7] A.K. Golmankhaneh and D. Baleanu, *On nonlinear fractional Klein–Gordon equation*, Sign. Process., 91, 2011, 446-451.

[8] O. Guner, A. Bekir, *Traveling wave solutions for time-dependent coefficient nonlinear evolution equations,* Waves in Random and Complex Media, 25(3), 2015, 342-349.

[9] O. Guner, A. Bekir, K. Hosseini, *Solitary wave solutions of nonlinear conformable time-fractional Boussinesq equations,* Acta Physica Polonica A, 136(1), 2019, 135-140.

[10] J.H. He, S.K. Elagan, Z.B. Li, *Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus*, Physics Letters A, 376(4), 2012, 257-259.

[11] K. Hosseini, A. Bekir, M. Kaplan, O. Guner, *On a new technique for solving the nonlinear conformable time-fractional differential equations,* Opt. Quant. Electron, 49(343), 2017, 1-12.

[12] M. Inc, H. Rezazadeh, J. Vahidi, M. Eslami, M.A. Akinlar, M. Nasir Ali, Y.M. Chu, *New solitary wave solutions for the conformable Klein-Gordon equation with quantic nonlinearity*, AIMS Mathematics, 5(6), 2020, 6972-6984.

[13] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, 264, 2014, 65-70.

[14] M. Kaplan, A. Akbulut, *Application of two different algorithms to the approximate long water wave equation with conformable fractional derivative,* , Arab Journal of Basic and Applied Sciences, 25(2), 2018, 77-84.

[15] A. Kurt, A. Tozar, O. Tasbozan, *Applying the new extended direct algebraic method to solve the equation of obliquely interacting waves in shallow waters*, Journal of Ocean University of China, 19, 2020, 772-780.

[16] B. Lu, *The first integral method for some time fractional differential equations*, J. Math. Anal. Appl., 395, 2021, 684-693.

[17] W. Malfliet, W. Hereman, *The tanh method: I. Exact solutions of nonlinear evolution and wave equations,* Phys. Scripta, 54, 1996, 563-568.

[18] W. Malfliet, W. Hereman, *The tanh method: II. Perturbation technique for conservative systems,* Phys. Scripta, 54, 1996, 569-575.

[19] I. Podlubny, *Fractional Differential Equations,* Academic Press, San Diego, 1999.

[20] Y. Pandir, A. Yildirim, *Analytical approach for the fractional differential equations by using the extended tanh method*, Waves in Ranom and Compex Media, 28(3), 2018, 399-410.

[21] H. Rezazadeh, A. Kurt, A. Tozar, O. Tasbozan, S.M. Mirhosseini-Alizamini, *Wave behaviors of Kundu-Mukherjee-Naskar model arising in optical fiber communication systems with complex structure,* Optical and Quantum Electronics, 53(6), 2021, 1-11.

[22] M. Shateri, D.D. Ganji, *Solitary wave solutions for a time-fraction generalized Hirota-Satsuma coupled KdV equation by a new analytical technique*, Int. J. Diff. Eqs., Art. ID 954674, 2010, 954-674.

[23] O. Tasbozan, Y. Cenesiz, A. Kurt, D. Baleanu, *New analytical solutions for conformable fractional PDEs arising in mathematical physics by exp-function method*, Open Phys. 15, 2017, 647-651.

[24] F. Tascan, A. Bekir, M. Koparan, *Travelling wave solutions of nonlinear evolutions by using the first integral method*, Commun. Nonlin. Sci. Numer. Simul., 14, 2009, 1810-1815.

[25] A. Tozar, O. Tasbozan, A. Kurt, *Obtaining optical solitons via resonant nonlinear Schrondinger equation for optical fiber application,* Optical and Quantum Electronics, 53(6), 2021, 1-8.

[26] M.H. Uddin, M.A. Arefin, M. Ali Akbar, M. Inc, *New explicit solutions to the fractional-Order Burgers equation,* Mathematical Problems in Engineering, Art. ID 6698028, 2021, 11 pages.

[27] A.M. Wazwaz, *The tanh method for travelling wave solutions of nonliear equations,* Applied Mathematics and Computation, 154, 2004, 713-723.

[28] A.M. Wazwaz, *New solitary wave solutions to modified forms of Degasperis-Procesi andCamassa-Holm equations*, Applied Mathematics and Computations, 186(1), 2007, 130-141.

[29] A.M. Wazwaz, *The extended tanh method for new solition solutions for many forms of the fifth order KdV equations*, Applied Mathematics and Computations, 184(2), 2007, 1002-1014.

[30] A.M. Wazwaz, *Analytic study on Burgers, Fisher, Huxely equations and combined forms of these equations*, Appl. Math. Comput., 195, 2008, 754-761.

[31] A.M. Wazwaz, *Travelling wave solutions of generalized forms of Burgers, Burgers KdV and Burgers-Huxely equations*, Appl. Math. Comput., 169, 2005, 639-656.

[32] A.M. Wazwaz, *The tanh method: solitons and periodic solutions for the Dodd-Bullough-Mikhailov and Tzitzeica-Dodd-Bulough equations.* Chaos, Solitons and Fractals, 25(1), 2005, 55-63.

[33] Z. Wenjing, X. Yonghui, B. Zhang, B. Yuzhen, *Exact traveling wave solutions and bifurcations of the time-fractional differential equations with applications*, International Journal of Bifurcation and Chaos, 29(3), 2019, 201-209.

[34] M. Willy and W. Hereman, *The tanh method: exact solutions of nonlinear evolution and wave equations,* Physica Scripta, 54, 1996, 563-568.

[35] M. Willy, *The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations,* Journal of Computational and Applied Mathematics, 164-165(1), 2004, 529-541.

[36] E. Yusufoglu and A. Bekir, *On the extended tanh method applications of nonliear equations,* International Journal of Nonlinear Science, 4, 2007, 10-16.

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