Jordan Journal of Mathematics and Statistics (JJMS), 15(1), 2022, pp 1 - 14
DOI: https://doi.org/10.47013/15.1.1

MAXIMAL IDEALS OF TRANSITIVE BE-ALGEBRAS

M. BALA PRABHAKAR [(1)], S. KALESHA VALI [(2)] AND M. SAMBASIVA RAO [(3)]

Abstract. The notion of maximal ideals is introduced in transitive BE-algebras. Some equivalent conditions are derived for a proper ideal of BE-algebra to become a maximal ideal. The concept of semi-simple BE-algebras is introduced and its properties are studied in terms of maximal ideals of BE-algebras.

1. Introduction

The concept of BE-algebras was introduced and extensively studied in [8]. The class of BE-algebras was introduced as a generalization of the class of BCK-algebras of K. Iseki and S. Tanaka [6]. Some properties of filters of BE-algebras were studied by S.S. Ahn and Y.H. Kim in [1] and by B.L. Meng in [9]. The notion of dual ideals in BCK-algebras was introduced by E.Y. Deeba [4] in 1979. Later 2000, P. Sun [12] investigated the homomorphism theorems via dual ideals in bounded BCK-algebras. In [10], J. Meng introduced the notion of BCK-filters in BCK-algebras and presented a description of the BCK-filter generated by a set. In the paper[10], he discussed prime decompositions and irreducible decompositions. In [7], Y.B. Jun, S.M. Hong, and J. Meng, considered the fuzzification of the concept of BCK-filters, and investigate their properties.

In this work, the notion of maximal ideals is introduced in transitive BE-algebras. A necessary and sufficient condition is derived for a proper ideal of BE-algebra to become a maximal ideal. The concept of semi-simple BE-algebras is introduced and its properties are studied in terms of maximal ideals of BE-algebras.

2010 *Mathematics Subject Classification.* 03G25.

Key words and phrases. Transitive BE-algebra, ideal, maximal ideal, semi-simple BE-algebra.
Copyright © Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.
Received: May 17, 2020 Accepted: Dec. 7, 2021 .

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1], [2], [3], [8], [9] and [11] for the ready reference of the reader.

Definition 2.1. [8] An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra, if it satisfies the following properties:

(1) $x * x = 1$,

(2) $x * 1 = 1$,

(3) $1 * x = x$,

(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra X is called self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called transitive if $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. Every self-distributive BE-algebra is transitive. A BE-algebra X is called commutative if $(x * y) * y = (y * x) * x$ for all $x, y \in X$. We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. Clearly, $\leq$ is reflexive. If X is commutative, then the relation $\leq$ is both anti-symmetric, transitive and so it is a partial order on X.

Theorem 2.1. [9] Let X be a transitive BE-algebra and $x, y, z \in X$. Then

(1) $1 \leq x$ implies $x = 1$,

(2) $y \leq z$ implies $x * y \leq x * z$ and $z * x \leq y * x$.

Definition 2.2. [8] A non-empty subset F of a BE-algebra X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,

(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

[1]For any $a \in X$, $\langle a \rangle = \{x \in X \mid a^n * x = 1 \text{ for some } n \in \mathbb{N}\}$ is called the principal filter generated a. If X is self-distributive, then $\langle a \rangle = \{x \in X \mid a * x = 1\}$. For a commutative BE-algebra, define $x \vee y = (y * x) * x$ for any $x, y \in X$. Then

$x \vee y = y \vee x$ and the suprimum of x and y is $x \vee y$ for all $x, y \in X$. Hence $(X, \vee)$ become a semilattice which is called a BE-semilattice.

A BE-algebra X is called bounded [3], if there exists an element 0 satisfying $0 \leq x$ (or $0 * x = 1$) for all $x \in X$. Define an unary operation N on a bounded BE-algebra X by $xN = x * 0$ for all $x \in X$.

Theorem 2.2. [3] *Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) $1N = 0$ *and* $0N = 1$,

(2) $x \leq xNN$,

(3) $x * yN = y * xN$.

An element x of a bounded BE-algebra X is called *dense* [11] if $xN = 0$. We denote the set of all dense elements of a BE-algebra X by $\mathcal{D}(X)$. A BE-algebra X is called a *dense BE-algebra* if every non-zero element of X is dense (that is $xN = 0$ for all $0 \neq x \in X$). Let X and Y be two bounded BE-algebras, then a homomorphism $f : X \to Y$ is called bounded[2], if $f(0) = 0$. If f is a bounded homomorphism, then it is easily observed that $f(xN) = f(x)N$ for all $x \in X$. For any bounded homomorphism $f : X \to Y$, define the *dual kernel* of the homomorphism f as $Dker(f) = \{x \in X \mid f(x) = 0\}$. It is easy to check that $Dker(f) = \{0\}$ whenever f is an injective homomorphism.

3. Maximal ideals

In this section, some properties of ideals of a transitive BE-algebras are studied and the notion of maximal ideals is introduced in transitive BE-algebras. Some properties of maximal ideals are studied. The notion of semi-simple BE-algebra is introduced and characterized in terms of maximal ideals.

Definition 3.1. A non-empty subset I of a BE-algebra X is called an *ideal* of X if it satisfies the following conditions for all $x, y \in X$:

(I1) $0 \in I$,

(I2) $x \in I$ and $(xN * yN)N \in I$ imply that $y \in I$.

Obviously the single-ton set $\{0\}$ is an ideal of a BE-algebra X. For, suppose $x \in \{0\}$ and $(xN * yN)N \in \{0\}$ for $x, y \in X$. Then $x = 0$ and $yNN = (0N * yN)N \in \{0\}$. Hence $y \leq yNN = 0 \in \{0\}$. Thus $\{0\}$ is an ideal of X. In the following example, we observe non-trivial ideals of a BE-algebra.

Example 3.1. *Let $X = \{1, a, b, c, d, 0\}$. Define an operation $*$ on X as follows:*

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Clearly, $(X, *, 0, 1)$ is a bounded BE-algebra. It can be easily verified that the set $I = \{0, c, d\}$ is an ideal of X. However, the set $J = \{0, a, b, d\}$ is not an ideal of X, because $a \in J$ and $(aN * cN)N = (d * b)N = aN = d \in J$ but $c \notin J$.

Lemma 3.1. *Let X be a transitive BE-algebra X. For any $x, y, z \in X$, we have:*

(1) *$xNNN \leq xN$,*

(2) *$x * y \leq yN * xN$,*

(3) *$x * yN \leq xNN * yN$,*

(4) *$(x * yNN)NN \leq x * yNN$,*

(5) *$(xN * yN)NN \leq xN * yN$,*

(6) *$x \leq y$ implies $yN \leq xN$,*

(7) *$x \leq y$ implies $y * zN \leq x * zN$.*

Proof. (1). Let $x \in X$. Then $1 = (x * 0) * (x * 0) = x * ((x * 0) * 0) = x * xNN \leq x * xNNNN = xNNN * xN$. Hence $xNNN * xN = 1$, which gives $xNNN \leq xN$.

(2). Let $x, y \in X$. Since X is transitive, then $yN = y * 0 \leq (x * y) * (x * 0) = (x * y) * xN$. Hence $1 = yN * yN \leq yN * ((x * y) * xN) = (x * y) * (yN * xN)$. Thus $(x * y) * (yN * xN) = 1$. Therefore, $x * y \leq yN * xN$.

(3). Let $x, y \in X$. Then $x * yN = y * xN \leq y * xNNN = xNN * yN$.

(4). Let $x, y \in X$. Clearly, $(x * yNN)N \leq (x * yNN)NNN$. Since X is transitive, then $yN * (x * yNN)N \leq yN * (x * yNN)NNN$ and so $x * (yN * (x * yNN)N) \leq x * (yN * (x * yNN)NNN)$. Hence

$$\begin{aligned} 1 &= (x * yNN) * (x * yNN) \\ &= x * ((x * yNN) * yNN) \\ &= x * (yN * (x * yNN)N) \\ &\leq x * (yN * (x * yNN)NNN) \\ &= x * ((x * yNN)NN * yNN) \\ &= (x * yNN)NN * (x * yNN). \end{aligned}$$

Thus $(x * yNN)NN * (x * yNN) = 1$. Therefore, $(x * yNN)NN \leq (x * yNN)$.

(5). From (4), it can be easily verified.

(6). Let $x, y \in X$ be such that $x \leq y$. Then by (2), $1 = x * y \leq yN * xN$. Hence $yN * xN = 1$. Therefore, $yN \leq xN$.

(7). Let $x, y \in X$ be such that $x \leq y$. Then by (6), $yN \leq xN$. Since X is transitive, then $z * yN \leq z * xN$. Therefore, $y * zN \leq x * zN$. □

Proposition 3.1. *Let I be an ideal of a transitive BE-algebra X. Then we have:*

(1) *For any $x, y \in X, x \in I$ and $y \leq x$ imply $y \in I$,*
(2) *For any $x, y \in X, xN = yN, x \in I$ imply $y \in I$,*
(3) *For any $x \in X, x \in I$ if and only if $xNN \in I$.*

Proof. (1). Let $x, y \in X$. Suppose $x \in I$ and $y \leq x$. Then $xN \leq yN$, which implies $xN * yN = 1$. Hence $(xN * yN)N = 0 \in I$. Since $x \in I$, then $y \in I$.

(2). Let $x, y \in X$. Assume that $xN = yN$. Suppose $x \in I$. Then $(xN * yN)N = 1N = 0 \in I$. Since I is an ideal of X, then $y \in I$.

(3). Let $x \in X$. Suppose $x \in I$. Then $(xN * xNNN)N = (xNN * xNN)N = 1N = 0 \in I$. Since $x \in I$, it yields $xNN \in I$. Conversely, let $xNN \in I$ for any $x \in X$. Since $x \leq xNN$, by property (1) we get that $x \in I$. □

We denote by $\mathcal{I}(X)$ the set of all ideals of a BE-algebra X and $\mathcal{F}(X)$ the set of all filters of X. Let A be a non-empty subset of X, then the set

$$[A] = \bigcap\{I \in \mathcal{I}(X) \mid A \subseteq I\}$$

is called the ideal generated by A, denoted $[A]$. In the following, we characterize the elements of a principal ideal generated by a set.

Theorem 3.1. *Let X be a transitive BE-algebra and $\emptyset \neq A \subseteq X$. Then*

$$[A] = \{x \in X \mid a_1N * (a_2N * (\cdots(a_nN * xN)\cdots)) = 1 \text{ for some } a_1, a_2, \ldots, a_n \in A \text{ and } n \in \mathbb{N}\}.$$

Proof. It is enough to show that $[A]$ is the smallest ideal of X containing the set A. Clearly, $0 \in [A]$. Let $x \in [A]$ and $(xN * yN)N \in [A]$. Then there exist $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in A$ such that $a_1N * (a_2N * (\cdots(a_nN * xN)\cdots)) = 1$ and $b_1N * (b_2N * (\cdots(b_mN * (xN * yN)NN)\cdots)) = 1$. Hence

$$\begin{aligned}
1 &= b_mN * (\cdots * (b_1N * (xN * yN)NN)\cdots) \\
&\leq b_mN * (\cdots * (b_1N * (xN * yN))\cdots) \\
&= b_mN * (\cdots * (xN * (b_1N * yN))\cdots) \\
&\quad \cdots \\
&\quad \cdots \\
&= xN * (b_mN * (\cdots * (b_1N * yN))\cdots).
\end{aligned}$$

Hence $xN \leq b_mN * (\cdots * (b_1N * yN)\cdots)$. Since X is transitive, then $1 = a_nN * (\cdots * (a_1N * xN)\cdots) \leq a_nN * (\cdots * (a_1N * (b_mN * (\cdots * (b_1N * yN)\cdots)))\cdots)$. Hence

$$a_nN * (\cdots * (a_1N * (b_mN * (\cdots * (b_1N * yN)\cdots)))\cdots) = 1$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in A$. From the structure of $[A]$, it yields that $y \in [A]$. Therefore, $[A]$ is an ideal of X. For any $x \in A$, we get $xN * (\cdots * (xN * xN)\cdots) = 1$. Hence $x \in [A]$. Therefore, $A \subseteq [A]$.

Let I be an ideal of X containing A. Let $x \in [A]$. Then there exists $a_1, a_2, \ldots, a_n \in A \subseteq I$ such that $a_nN * (\cdots * (a_1N * xN)\cdots) = 1$. Hence $(a_nN * (\cdots * (a_1N * xN)\cdots)NN)N \leq (a_nN * (\cdots * (a_1N * xN)\cdots))N = 0 \in I$. Thus by Proposition 3.1(1), we get $(a_nN * (\cdots * (a_1N * xN)\cdots)NN)N \in I$. Since $a_n \in I$ and I is an ideal, then $(a_{n-1}N * (\cdots * (a_1N * xN)\cdots))N \in I$. We continue in this manner, we finally get $x \in I$. Hence $[A] \subseteq I$. Therefore, $[A]$ is the smallest ideal containing A. $\square$

For $A = \{a\}$, we then denote $[\{a\}]$, briefly by $[a]$. We call this ideal by *principal ideal generated by* a and is represented by $[a] = \{x \in X \mid (aN)^n * xN = 1 \text{ for some } n \in \mathbb{N}\}$. The following is a direct consequence of the above theorem:

Corollary 3.1. *Let X be a transitive BE-algebra. For any $a, b \in X$, and $A, B \subseteq X$, we have*

(1) $[0] = \{0\}$,

(2) $[X] = X$ *and* $[1] = X$,

(3) $A \subseteq B$ *implies* $[A] \subseteq [B]$,

(4) $a \leq b$ *implies* $[a] \subseteq [b]$,

(5) *if A is an ideal, then* $[A] = A$,

(6) *if A is an ideal and $a \in A$, then* $[a] \subseteq A$.

Proof. (1). Let $x \in [0]$. Then $(0N)^n * xN = 1$ for some $n \in \mathbb{N}$. Hence $xN = 1$. Thus $x \leq xNN = 1N = 0$. Therefore, $x = 0$, which means $[0] = \{0\}$.

(2). For all $x \in X$, we get $1N * xN = 1 = 0 * xN = 1$. Hence $[1] = X$.

(3). Suppose $A \subseteq B$ and let $x \in [A]$ then $a_1N * (a_2N * (\cdots(a_nN * xN)\cdots)) = 1$ for some $a_1, a_2, \ldots, a_n \in A$ and $n \in \mathbb{N}$. Since $A \subseteq B$ implies $a_1N * (a_2N * (\cdots(a_nN * xN)\cdots)) = 1$ for some $a_1, a_2, \ldots, a_n \in B$ and $n \in \mathbb{N}$, we get $x \in [B]$ and hence $[A] \subseteq [B]$

(4). Suppose $a \leq b$. By Lemma 3.1(6), we get $bN \leq aN$. Again by Lemma 3.1(7), we get $aN * xN \leq bN * xN$ for any $x \in X$. Similarly, we can get $(aN)^n * xN \leq (bN)^n * xN$ for $n \in \mathbb{N}$. Let $x \in [a]$ then $(aN)^n * xN = 1$. Thus $1 = (aN)^n * xN \leq (bN)^n * xN$. Hence $(bN)^n * xN = 1$, which gives $x \in [b]$. Therefore, $[a] \subseteq [b]$.

(5). From the construction of $[A]$, it is obvious.

(6). Let A be an ideal and $a \in A$. Suppose $x \in [a]$. Then there exists $n \in \mathbb{N}$ such that $(aN)^n * xN = 1$. Thus $1 = aN * ((aN)^{n-1} * xN) \leq aN * ((aN)^{n-1} * xN)NN$. Hence $aN * ((aN)^{n-1} * xN)NN = 1$, which gives $(aN * ((aN)^{n-1} * xN)NN)N = 0 \in A$. Since $a \in A$ and A is an ideal, then $((aN)^{n-1} * xN)N \in A$. Now,

$$
\begin{aligned}
(aN * ((aN)^{n-2} * xN)NN)N &\leq (aN * ((aN)^{n-2} * xN))N \\
&= ((aN)^{n-1} * xN)N \in A.
\end{aligned}
$$

Which yields $(aN*((aN)^{n-2}*xN)NN)N \in A$. Since $a \in A$, then $(aN)^{n-2}*xN)N \in A$. We continue in this manner, we finally get $x \in A$. Therefore, $[a] \subseteq A$. □

Corollary 3.2. *Let X be a transitive BE-algebra and $a \in X$. For any $A \subseteq X$, the set $[A \cup \{a\}]$ is the smallest ideal of X that contains both A and a.*

Corollary 3.3. *If X is self-distributive and $a \in X$. Then*

$$[a] = \{x \in X \mid aN * xN = 1\}.$$

Proposition 3.2. *Let X be a transitive BE-algebra and I is an ideal of X. For any $a \in X$,*

$$[I \cup \{a\}] = \{x \in X \mid ((aN)^n * xN)N \in I \text{ for some } n \in \mathbb{N}\}.$$

Proof. Let us consider, $B = \{x \in X \mid ((aN)^n * xN)N \in I$ for some $n \in \mathbb{N}\}$. It is enough to show that B is the smallest ideal of X containing both I and a. Clearly, $0 \in B$. Let $x, y \in X$ be such that $x \in B$ and $(xN * yN)N \in B$. Then there exists $m, n \in \mathbb{N}$ such that $((aN)^n * xN)N \in I$ and $((aN)^m * (xN * yN)NN)N \in I$. By Lemma 3.1(5), we have

$$(aN)^m * (xN * yN)NN \leq (aN)^m * (xN * yN) = xN * ((aN)^m * yN).$$

By Lemma 3.1(6), we get $(xN * ((aN)^m * yN))N \leq ((aN)^m * (xN * yN)NN)N \in I$. By applying the transitivity of X and Lemma 3.1(2), we get

$$\begin{aligned} xN * ((aN)^m * yN) &\leq ((aN)^n * xN) * ((aN)^n * ((aN)^m * yN)) \\ &\leq ((aN)^n * xN)NN * ((aN)^{n+m} * yN)NN. \end{aligned}$$

Hence $(((aN)^n * xN)NN * ((aN)^{n+m} * yN)NN)N \leq (xN * ((aN)^m * yN))N \in I$. Since $((aN)^n * xN)N \in I$ and I is an ideal, then $((aN)^{n+m} * yN)N \in I$. Thus $y \in B$. Therefore, B is an ideal of X. Let $x \in I$. Clearly, $aN * xN \leq (aN * xN)NN$. Then by Lemma 3.1(6),

$$\begin{aligned} (xN * (aN * xN)NN)N &\leq (xN * (aN * xN))N \\ &= (aN * (xN * xN))N \\ &= (aN * 1)N \\ &= 0. \end{aligned}$$

Hence $(xN * (aN * xN)NN)N = 0 \in I$. Since $x \in I$ and I is an ideal, then $(aN * xN)N \in I$. Thus $x \in B$. Since $(aN * aN)N = 0 \in I$, then $a \in B$. Therefore, B is an ideal of X containing both I and a.

Suppose K is an ideal of X such that $I \subseteq K$ and $a \in K$. Let $x \in B$. Then $((aN)^n * xN)N \in I \subseteq K$ for some $n \in \mathbb{N}$. Then

$$(aN)^n * xN = aN * ((aN)^{n-1} * xN) \leq aN * ((aN)^{n-1} * xN)NN.$$

Hence $(aN * ((aN)^{n-1} * xN)NN)N \leq ((aN)^n * xN)N \in K$. Since $a \in K$, then $((aN)^{n-1} * xN)N \in K$. We continue in this manner, finally we get $x \in K$. Hence $B \subseteq K$. Thus B is the smallest ideal of X containing both I and a. □

Corollary 3.4. *Let X be a self-distributive BE-algebra and I is an ideal of X. Then for any $a \in X$, $[I \cup \{a\}] = \{x \in X \mid (aN * xN)N \in I\}$.*

Definition 3.2. An ideal I of a BE-algebra X is said to be *proper* if $I \neq X$.

Definition 3.3. A proper ideal M of a BE-algebra X is said to be *maximal*, if M is not properly contained in any other proper ideal of X (that is $M \subseteq I \subseteq X$ implies $M = I$ or $I = X$ for any ideal I of X).

Example 3.2. *Let $X = \{0, a, b, c, d, 1\}$. Define an operation $*$ on X as follows:*

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	1	1	d	d
b	1	c	1	c	d	c
c	1	b	b	1	d	b
d	1	a	b	c	1	a
0	1	1	1	1	1	1

Clearly, $(X, *, 0, 1)$ is a bounded BE-algebra. It is easy to check that $I_1 = \{0\}$, $I_2 = \{0, a\}$, $I_3 = \{0, b\}$, $I_4 = \{0, c\}$, $I_5 = \{0, a, b\}$ and $I_6 = \{0, a, c\}$ are ideals of X in which I_2, I_3, I_4, I_5 and I_6 are proper ideals. Also here we can easily observe that I_5 and I_6 are only maximal ideals of X.

Theorem 3.2. *A proper ideal M of a transitive BE-algebra X is maximal if and only if $[M \cup \{x\}] = X$ for any $x \in X - M$.*

Proof. Let M be a proper ideal of X. Assume that M is maximal. Let $x \in X - M$. Suppose $[M \cup \{x\}] \neq X$. Choose $a \in X$ such that $a \notin [M \cup \{x\}]$. Hence $M \subseteq [M \cup \{x\}] \subset X$. Since M is maximal, then $M = [M \cup \{x\}]$. Hence $x \in M$, which is a contradiction. Therefore, $[M \cup \{x\}] = X$.

Conversely, assume the condition. Suppose there exists an ideal I of X such that $M \subseteq I \subseteq X$. Let $M \neq I$. Then $M \subset I$. Choose $x \in I$ such that $x \notin M$. By the assumed condition, we get $[M \cup \{x\}] = X$. If $a \in X$, then $a \in [M \cup \{x\}]$. Hence $((xN)^n * aN)N \in M \subseteq I$ for some $n \in \mathbb{N}$. Then

$$(xN)^n * aN = xN * ((xN)^{n-1} * aN) \leq xN * ((xN)^{n-1} * aN)NN.$$

By Lemma 3.1(6) and Proposition 3.1(1), we get $(xN * ((xN)^{n-1} * aN)NN)N \leq ((xN)^n * aN)N \in I$. Since $x \in I$, implies $((xN)^{n-1} * aN)N \in I$. We continue in this manner, finally we get $a \in I$. Hence $I = X$. Therefore, M is a maximal ideal of X. □

Example 3.3. *Consider the BE-algebra, $X = \{0, a, b, c, d, 1\}$ given in Example 3.2. Here, the set $I_5 = \{0, a, b\}$ is a maximal ideal of X. Take, $c \in X$ then clearly $[I_5 \cup \{c\}] = X$ for $c \in X - I_5$. Similarly, $[I_5 \cup \{d\}] = X$ for $d \in X - I_5$.*

Theorem 3.3. *Let X be a BE-algebra and I is an ideal of X. Then*

(1) *I is proper if and only if $1 \notin I$.*

(2) *each proper ideal is contained in a maximal ideal.*

Proof. (1) Assume that I is proper. Then $I \neq X$. Suppose $1 \in I$. For $x \in X$, we have $(1N * xN)N = 1N = 0 \in I$. Since $1 \in I$ and I is an ideal, then $x \in I$. Hence $X \subseteq I$. Thus $I = X$, which is a contradiction. The converse is clear.

(2) By the Zorn's lemma, it follows immediately. □

Theorem 3.4. *Every BE-algebra contains at least one maximal ideal.*

Proof. Since $\{0\}$ is a proper ideal of X, it is clear by above Theorem 3.3. □

Proposition 3.3. *Let I be a proper ideal of a self-distributive BE-algebra X. Then I is maximal if and only if for any $x \in X$,*

$$x \notin I \text{ implies } xN \in I.$$

Proof. Let I be a proper ideal of X. Assume that I is maximal. Let $x \notin I$. Then $[I \cup \{x\}] = X$. Hence $1 \in [I \cup \{x\}]$. Since X is self-distributive, then $xNNN = (xN * 1N)N \in I$. Since $xN \leq xNNN$, then $xN \in I$.

Conversely, assume the condition. Suppose I is not maximal. Then there exists a proper ideal Q of X such that $I \subset Q$. Choose $x \in Q - I$. Then $x \notin I$. By the assumed condition, we get $xN \in I \subseteq Q$. Since $xNNN \leq xN$, then $(xN * 1N)N = xNNN \in Q$. Since $x \in Q$ and Q is an ideal, then $1 \in Q$ which is contradiction to that Q is proper. Therefore, I is a maximal ideal of X. □

Definition 3.4. Let X be a BE-algebra. Then the *radical* of X, denoted as $rad(X)$, defined as,

$$rad(X) = \cap\{I \mid I \in Max(X)\}$$

where $Max(X)$ is the family of all maximal ideals of X.

It is clear that $rad(X)$ is always exists for a BE-algebra. In the contemporary algebra, the following is a standard terminology. We say that a BE-algebra is semi-simple if $rad(X) = \{0\}$. We first observe the non-trivial examples:

Example 3.4. *Let $X = \{0, a, b, 1\}$. Define an operation $*$ on X as follows:*

$*$	1	a	b	0
1	1	a	b	0
a	1	1	b	b
b	1	a	1	a
0	1	1	1	1

Clearly, $(X, *, 0, 1)$ is a bounded BE-algebra. It can be easily verified that the sets $I_1 = \{0\}, I_2 = \{0, a\}, I_3 = \{0, b\}$ are ideals of X in which I_2 and I_3 are the only maximal ideals. Hence $rad(X) = I_2 \cap I_3 = \{0\}$. Therefore, X is semi-simple.

Example 3.5. *Let $X = \{0, a, b, c, d, 1\}$. Define an operation $*$ on X as follows:*

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	1	1	d	d
b	1	c	1	c	d	c
c	1	b	b	1	d	b
d	1	a	b	c	1	a
0	1	1	1	1	1	1

Clearly, $(X, *, 0, 1)$ is a bounded BE-algebra. It is easy to check that $I_1 = \{0\}$, $I_2 = \{0, a\}$, $I_3 = \{0, b\}$, $I_4 = \{0, c\}$, $I_5 = \{0, a, b\}$ and $I_6 = \{0, a, c\}$ are ideals of X in which I_5 and I_6 are only maximal ideals of X. Hence $rad(X) = I_5 \cap I_6 = I_2 \neq \{0\}$. Therefore, X is not semi-simple.

Theorem 3.5. *A transitive BE-algebra X is semi-simple if and only if for each $0 \neq x \in X$, there exists a proper ideal I of X such that $[I \cup \{x\}] = X$.*

Proof. Assume that X is semi-simple. Then $\bigcap\limits_{I \in Max(X)} I = \{0\}$. Let $0 \neq x \in X$. Then there exists a maximal ideal I of X such that $x \notin I$ (otherwise, if every maximal ideal contains x, then $0 \neq x \in \bigcap\limits_{I \in Max(X)} I = \{0\}$). Since I is maximal, then $[I \cup \{x\}] = X$.

Conversely, assume the condition. Suppose $\bigcap\limits_{I \in Max(X)} I \neq \{0\}$. Choose $0 \neq x \in \bigcap\limits_{I \in Max(X)} I$. By the assumed condition, there exists a proper ideal I of X such that $[I \cup \{x\}] = X$. Hence $x \notin I$. Consider, $\mathfrak{T} = \{J \mid J$ is an ideal of $X, x \notin J$ and $I \subseteq J\}$. Clearly, $I \in \mathfrak{T}$ and $\mathfrak{T} \neq \emptyset$. Clearly, $\mathfrak{T}$ is a partially ordered set, with the set inclusion, in which every chain has an upper bound. By the Zorn's lemma, $\mathfrak{T}$ has a maximal element say I_0. Then $x \notin I_0$ and $I \subseteq I_0$. Suppose there exists a proper ideal M of X such that $I \subseteq I_0 \subset M \subseteq X$. By the maximality of M, we get $x \in M$. Hence $X = [I \cup \{x\}] \subset [M \cup \{x\}] = M$. Thus I_0 is a maximal ideal of X and $x \notin I_0$, which is a contradiction. Therefore, $\bigcap\limits_{I \in Max(X)} I = \{0\}$, which means that X is semi-simple. □

Example 3.6. *Consider the BE-algebra, $X = \{0, a, b, 1\}$ given in Example 3.4. Here, the sets $I_2 = \{0, a\}$, $I_3 = \{0, b\}$ are proper ideals of X. Take, $0 \neq b \in X$*

then clearly $[I_2 \cup \{b\}] = X$. *Similarly,* $[I_3 \cup \{a\}] = X$ *for* $0 \neq a \in X$. *Hence* X *is semi-simple.*

Theorem 3.6. *Let* X *be a self-distributive* BE*-algebra. Then for every* $0 \neq x \in X$ *there exists a maximal ideal* I *of* X *such that* $x \notin I$.

Proof. Let $0 \neq x \in X$. We first claim that $[xN]$ is a proper ideal of X. Suppose $1 \in [xN]$. Since X is self-distributive, then $xNNN = xNN * 1N = 1$. Hence $x \leq xNN \leq xNNNN = 0$. Thus $x = 0$, which is a contradiction. Therefore, $[xN]$ is a proper ideal of X. Then there exists a maximal ideal I of X such that $[xN] \subseteq I$. Suppose $x \in I$. Then $(xN * 1N)N = xNNN \leq xN \in [xN] \subseteq I$. Hence $(xN * 1N)N \in I$. Since $x \in I$, then $1 \in I$, which is a contradiction. Therefore, I is a maximal ideal of X such that $x \notin I$. □

Corollary 3.5. *Every self-distributive* BE*-algebra is semi-simple.*

Acknowledgement

The authors would like to thank the editor and the referees for their valuable suggestions to improve this work.

References

[1] S.S. Ahn, Y.H. Kim and J.M. Ko, *Filters in commutative BE-algebras*, Commun. Korean. Math. Soc., **27**, no.2 (2012), 233–242.

[2] R. Borzooei and A.B. Saeid, Involutory $BE - algebras$, *Journal of Mathematics and App.*, **37**(2014), 13–26.

[3] Z. Ciloglu and Y. Ceven, *Commutative and bounded BE-algebras*, Algebra, Volume 2013(2013), Article ID 473714, 1–5 pages.

[4] E.Y. Deeba, *A characterization of complete BCK-algebras*, Math. Seminar Notes, **7** (1979), 343–349.

[5] E.Y. Deeba, *Filter theory of BCK-algebras*, Math. Japon., **25** (1980), 631–639.

[6] K. Iseki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon., **23**, no.1 (1979), 1–26.

[7] Y.B. Jun, S.M. Hong, and J. Meng, *Fuzzy BCK-filters*, Mathematica Japonica, **47**, no. 1 (1998), 45–49.

[8] H.S. Kim and Y.H. Kim, *On BE-algebras*, Sci. Math. Jpn., **66**, no.1 (2006), 1299–1302.

[9] B.L. Meng, *On filters in BE-algebras*, Sci. Math. Japon, Online, e-2010, 105–111.

[10] J. Meng, *BCK-filters*, Mathematica Japonica, **44** (1996), 119–129.

[11] M. B. Prabhakar, S. K. Vali and M. Sambasiva Rao, *Closed and Dense Elements of BE-algebras*, Journal of the Chungcheong Mathematical Society, **32**, no. 1 (2019), 53–67.

[12] P. Sun, *Homomorphism theorems on dual ideals in BCK-algebras*, Soo. J. Math., **26**, no.3 (2000), 309–316.

(1) [Corresponding author]Department of Mathematics, Aditya Engineering College(A), Surampalem, Kakinada, Andhra Pradesh, India-533 437.

Email address: prabhakar_mb@yahoo.co.in

(2) Department of Mathematics, JNTUK University College of Engineering,Vizianagaram, Andhra Pradesh, India-535 003.

Email address: valijntuv@gmail.com

(3) Department of Mathematics, MVGR College of Engineering(A), Vizianagaram, Andhra Pradesh, India-535 005.

Email address: mssraomaths35@rediffmail.com