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WEAK SOLUTIONS OF COUPLED CAPUTO TYPE MODIFICATION OF THE ERDÉLYI-KOBER IMPLICIT FRACTIONAL DIFFERENTIAL SYSTEMS WITH RETARDED AND ADVANCED ARGUMENTS

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Abstract. In this paper, we investigate the existence of weak solutions for some coupled systems of fractional Caputo-type modification of the Erdélyi-Kober differential equations with retardation and anticipation. Our approach is based on Mönch's fixed point theorem associated with the technique of measure of weak noncompactness. Finally, an example of our results is provided.

1. Introduction

The significance of fractional differential equations in expressing various phenomena in a number of scientific domains cannot be overstated. They have use in viscoelasticity, electrochemistry, control, porous media, electromagnetism, and other domains. For more information, we recommend that the reader consult the monographs [1, 2, 3, 20], the most recent research papers [23, 24] and the sources within. Coupled systems of fractional differential equations, on the other hand, appear in a variety of problems. Several researchers have studied the coupled system of nonlinear fractional differential equations in recent years. We direct the reader to the article [7] for a short example of such work. Details on the Erdélyi-Kober fractional operators and their properties may be found in [19, 17, 18].

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Numerical approaches for Riemann-Liouville and Caputo fractional derivative operators are studied in [9, 10]. Implicit differential equations have been considered by many authors [4, 11]. Our investigation relies upon Mönch's fixed point theorem combined with the technique of measures of weak noncompactness. This technique was introduced by De Blasi [12].

In [8, 16], the authors studied the existence and uniqueness of weak solutions for boundary value problem for Caputo and Hadamard-type fractional differential equations and Pettis-Hadamard with retardation and anticipation. In [6], the authors investigated a coupled Pettis-Hadamard fractional differential system with retarded and advanced argument given by

$$\begin{cases} (\mathfrak{p}(\vartheta), \mathfrak{q}(\vartheta)) = (\delta_1(\vartheta), \delta_2(\vartheta)), \ \vartheta \in [1-\lambda_1, 1], \ \lambda_1 > 0 \\ ({}^H D_1^{\zeta_1} \mathfrak{p})(\vartheta) = \Psi_1(\vartheta, \mathfrak{p}(\vartheta), \mathfrak{q}(\vartheta)) \\ ({}^H D_1^{\zeta_2} \mathfrak{q})(\vartheta) = \Psi_2(\vartheta, \mathfrak{p}(\vartheta), \mathfrak{q}(\vartheta)) \\ (\mathfrak{p}(\vartheta), \mathfrak{q}(\vartheta)) = (\tilde{\delta}_1(\vartheta), \tilde{\delta}_2(\vartheta)), \ \vartheta \in [e, e+\lambda_2], \ \lambda_2 > 0, \end{cases} \quad ; \ \vartheta \in \Theta := [1, e],$$

where $\lambda_1, \lambda_2 > 0, \zeta_j \in (1, 2]$, $(\Xi, \|\cdot\|)$ is a real Banach space and $\Psi_j : \Theta \times C([-\lambda_1, \lambda_2], \Xi)^2 \to \Xi$ is a given function, $\delta \in C([1-\lambda_1, 1], \Xi)$ with $\delta_j(1) = 0$ and $\tilde{\delta}_j \in C([e, e+\lambda_2], \Xi)$ with $\tilde{\delta}_j(e) = 0, \quad j = 1, 2$.

In [5], Abbas *et al.* considered the following fractional differential systems:

$$\begin{cases} {}^H D_1^{\zeta_1,\zeta_2} \mathfrak{p}(\vartheta) = \Psi_1(\vartheta, \ \mathfrak{p}(\vartheta), \ \mathfrak{q}(\vartheta), \ {}^H D_1^{\zeta_1,\zeta_2} \mathfrak{p}(\vartheta), \ {}^H D_1^{\zeta_1,\zeta_2} \mathfrak{q}(\vartheta)) \\ {}^H D_1^{\zeta_1,\zeta_2} \mathfrak{q}(\vartheta) = \Psi_2(\vartheta, \ \mathfrak{p}(\vartheta), \ \mathfrak{q}(\vartheta), \ {}^H D_1^{\zeta_1,\zeta_2} \mathfrak{p}(\vartheta), \ {}^H D_1^{\zeta_1,\zeta_2} \mathfrak{q}(\vartheta)) \end{cases} \quad \vartheta \in \Theta := [1, \bar{\kappa}],$$

with the initial conditions

$$\begin{cases} ({}^H \mathcal{J}^{1-\gamma} \mathfrak{p})(\vartheta) \mid_{\vartheta=1} = \delta_1 \\ ({}^H \mathcal{J}^{1-\gamma} \mathfrak{q})(\vartheta) \mid_{\vartheta=1} = \delta_2, \end{cases}$$

where $\bar{\kappa} > 1, \vartheta \in \Theta = [1, \bar{\kappa}], \zeta_1 \in (0, 1), \zeta_2 \in [0, 1], \ \gamma = \zeta_1 + \zeta_2 - \zeta_1\zeta_2, \ \Psi_j : \Theta \times \Xi^4 \to \Xi; \ j = 1, 2$ are given continuous functions, $(\Xi, \|\cdot\|_\Xi)$ is a Banach space, ${}^H D_1^{\zeta_1,\zeta_2}$ is the Hilfer-Hadamard fractional derivative of order ζ_1 and type ζ_2, and ${}^H \mathcal{J}^{1-\gamma}$ is the Hadamard fractional operator of order $1 - \gamma$.

Motivated by the works mentioned above, in this paper, we consider the problem:

$$(1.1)\qquad \begin{cases} {}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{p}(\vartheta) = \Psi_1(\vartheta,\ \mathfrak{p}^{\vartheta},\ \mathfrak{q}^{\vartheta},\ {}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{p}(\vartheta),\ {}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{q}(\vartheta))) \\ {}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{q}(\vartheta) = \Psi_2(\vartheta,\ \mathfrak{p}^{\vartheta},\ \mathfrak{q}^{\vartheta},\ {}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{p}(\vartheta),\ {}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{q}(\vartheta)) \end{cases} \quad \vartheta \in \Theta := [\kappa, \bar{\kappa}],$$

$$(1.2)\qquad \begin{cases} (\mathfrak{p}(\vartheta), \mathfrak{q}(\vartheta)) = (\delta_1(\vartheta), \delta_2(\vartheta)),\ \vartheta \in [\kappa - \lambda_1, \kappa],\ \lambda_1 > 0, \\ (\mathfrak{p}(\vartheta), \mathfrak{q}(\vartheta)) = (\tilde{\delta}_1(\vartheta), \tilde{\delta}_2(\vartheta)),\ \vartheta \in [\bar{\kappa}, \bar{\kappa} + \lambda_2],\ \lambda_2 > 0, \end{cases}$$

where ${}^{\rho}_{c}D^{\zeta}_{\kappa^+}$ is the Caputo-type fractional derivative defined in the sequel and $(\Xi, \|\cdot\|_{\Xi})$ is a Banach space with dual Ξ^*, such that Ξ is the dual of a weakly compactly generated Banach space X, $\Psi_j : \Theta \times C([-\lambda_1, \lambda_2], \Xi)^2 \times \Xi^2 \to \Xi$ is a given function, $\delta_j \in C([\kappa-\lambda_1, \kappa], \Xi)$ with $\delta_j(\kappa) = 0$ and $\tilde{\delta}_j \in C([\bar{\kappa}, \bar{\kappa}+\lambda_2], \Xi)$ with $\tilde{\delta}_j(\bar{\kappa}) = 0,\ j = 1, 2$. By $\mathfrak{p}^{\vartheta}$ we denote the element of $C([-\lambda_1, \lambda_2])$ given by:

$$\mathfrak{p}^{\vartheta}(\varrho) = \mathfrak{p}(\vartheta + \varrho) : \varrho \in [-\lambda_1, \lambda_2].$$

2. Preliminaries

We present here the definitions of Erdélyi-Kober fractional integral and Erdélyi-Kober fractional derivative and then some auxiliary results that will be used to prove our main results.

Consider the Banach spaces of continuous functions $C([-\lambda_1, \lambda_2], \Xi)$ with the norm

$$\|\mathfrak{p}\|_{[-\lambda_1,\lambda_2]} = \sup\{\|\mathfrak{p}(\vartheta)\|_{\Xi} : -\lambda_1 \le \vartheta \le \lambda_2\},$$

and $C([\kappa, \bar{\kappa}], \Xi)$ with the norm

$$\|\mathfrak{p}\|_{[\kappa,\bar{\kappa}]} = \sup\{\|\mathfrak{p}(\vartheta)\|_{\Xi} : \kappa \le \vartheta \le \bar{\kappa}\}.$$

Also, let $E_1 = C([\kappa - \lambda_1, \kappa], \Xi)$, $E_2 = C([\bar{\kappa}, \bar{\kappa} + \lambda_2], \Xi)$,
and

$$AC^1(\Theta) := \{\mathfrak{p} : \Theta \longrightarrow \Xi : \mathfrak{p}' \in AC(\Theta)\},$$

where

$$\mathfrak{p}'(\vartheta) = \vartheta \frac{d}{d\vartheta}\mathfrak{p}(\vartheta), \quad \vartheta \in \Theta.$$

$AC(\Theta, \Xi)$ is the space of absolutely continuous functions on Θ.

Consider the space $\Phi = \{\mathfrak{p} : [\kappa - \lambda_1, \bar{\kappa} + \lambda_2] \longmapsto \Xi : \mathfrak{p} \mid_{[\kappa-\lambda_1,\kappa]} \in C([\kappa - \lambda_1, \kappa]), \mathfrak{p} \mid_{[\kappa,\bar{\kappa}]} \in AC^1([\kappa, \bar{\kappa}])$

$$\text{and } \mathfrak{p} \mid_{[\bar{\kappa},\bar{\kappa}+\lambda_2]} \in C([\bar{\kappa}, \bar{\kappa} + \lambda_2])\}$$

with the norms

$$\|\mathfrak{p}\|_{[\kappa-\lambda_1,\kappa]} = \sup\{\|\mathfrak{p}(\vartheta)\|_{\Xi} : \kappa - \lambda_1 \leq \vartheta \leq \kappa\},$$

$$\|\mathfrak{p}\|_{[\bar{\kappa},\bar{\kappa}+\lambda_2]} = \sup\{\|\mathfrak{p}(\vartheta)\|_{\Xi} : \bar{\kappa} \leq \vartheta \leq \bar{\kappa} + \lambda_2\},$$

$$\|\mathfrak{p}\|_{\Phi} = \sup\{\|\mathfrak{p}(\vartheta)\|_{\Xi} : \kappa - \lambda_1 \leq \vartheta \leq \bar{\kappa} + \lambda_2\}.$$

Let $\overline{\Phi} := \Phi \times \Phi$ be the product space with the norm

$$\|(\mathfrak{p}, \mathfrak{q})\|_{\overline{\Phi}} := \|\mathfrak{p}\|_{\Phi} + \|\mathfrak{q}\|_{\Phi}.$$

Let$(\Xi, \omega) = (\Xi, \sigma(\Xi, \Xi^*))$ be the Banach space Ξ with weak topology.

Consider the space $X^p_c(\kappa, \bar{\kappa})$,$(c \in \mathbb{R}, 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions $\mathfrak{p}$ on $[\kappa, \bar{\kappa}]$ for which $\|\mathfrak{p}\|_{X^p_c} < \infty$, where the norm is defined by :

$$\|\mathfrak{p}\|_{X^p_c} = \left(\int_{\kappa}^{\bar{\kappa}} |\vartheta^c \mathfrak{p}(\vartheta)|^p \frac{d\vartheta}{\vartheta}\right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, c \in \mathbb{R}).$$

In particular, where $c = \frac{1}{p}$ the space $X^p_c(\kappa, \bar{\kappa})$ coincides with $L^p(\kappa, \bar{\kappa})$, i.e.

$$X^p_{\frac{1}{p}}(\kappa, \bar{\kappa}) = L^p(\kappa, \bar{\kappa}).$$

Denote by $L^\infty(\Theta, \mathbb{R})$, the Banach space of essentially bounded measurable functions $\mathfrak{p} : \Theta \longrightarrow \mathbb{R}$ equipped with the norm

$$\|\mathfrak{p}\|_{L^\infty} = \inf\{c > 0; \ \ |\mathfrak{p}(x)| \leq c \text{ a.e. on } \ \Theta\}.$$

Definition 2.1. *A Banach space X is said to be weakly compactly generated (WCG) if it contains a weakly compact set whose linear span is dense in X.*

Definition 2.2. *A function $\bar{\Psi} : \Xi \longrightarrow \Xi$ is said to be weakly sequentially continuous if $\bar{\Psi}$ takes each weakly convergent sequence in Ξ to weakly convergent sequence in Ξ (i.e. for any $(x_n)_n$ in Ξ with $x_n \longrightarrow x$ in (Ξ, ω), $\bar{\Psi}(x_n) \longrightarrow \bar{\Psi}(x)$ in (Ξ, ω)).*

Definition 2.3. *([21]) The function $x : J \longrightarrow \Xi$ is said to be Pettis integrable on J if and only if there is an element $\mathfrak{p}_\Theta \in \Xi$ corresponding to each $\Theta \subset J$ such that $\varphi(\mathfrak{p}_\Theta) = \int_\Theta \varphi(\mathfrak{p}(\varrho))d\varrho$ for all $\varphi \in \Xi^*$, where the integral on the right is supposed to exist in the sense of Lebesgue. We have $\mathfrak{p}_\Theta = \int_\Theta \varphi(\mathfrak{p}(\varrho))d\varrho$. Let $P(J, \Xi)$ be the space of all Ξ-valued Pettis integrable functions in the interval J, and let $L_1(\Theta, \Xi)$ be the Banach space of Bochner-integrable measurable functions $\mathfrak{p} : \Theta \longrightarrow \Xi$. Define the class*

$$P_1(J, \Xi) = \{\mathfrak{p} \in P(J, \Xi) : \varphi(\mathfrak{p}) \in L_1(\Theta, \mathbb{R}) \quad for \quad every \quad \varphi \in \Xi^*\}.$$

The space $P_1(J, \Xi)$ is normed by

$$\|\mathfrak{p}\|_{P_1} = \sup_{\varphi \in \Xi^*, \|\varphi\| \leq 1} \int_\kappa^{\bar{\kappa}} |\varphi(\mathfrak{p}(x))| d\upsilon x,$$

where υ is the Lebesgue measure on J.

Proposition 2.1. *([21]) If $\mathfrak{p} \in P_1(\Theta, \Xi)$ and $\bar{\Psi}$ is a measurable and essentially bounded real-valued function, then $\mathfrak{p}\bar{\Psi} \in P(\Theta, \Xi)$. In what follows, the symbol " $\int$ " denotes the Pettis integral.*

Definition 2.4. *([19]) Let $\zeta \in \mathbb{R}, c \in \mathbb{R}$ and $g \in X_c^p(\kappa, \bar{\kappa})$, the Erdélyi-Kober fractional integral of order ζ is given by :*

$$({}^\rho\mathcal{J}_{\kappa^+}^\zeta g)(\vartheta) = \frac{1}{\Gamma(\zeta)} \int_a^\vartheta \varrho^{\rho-1} \left(\frac{\vartheta^\rho - \varrho^\rho}{\rho} \right)^{\zeta-1} g(\varrho) d\varrho, \qquad \vartheta > \kappa, \rho > 0 \tag{2.1}$$

where Γ is the Euler gamma function defined by: $\Gamma(\zeta) = \displaystyle\int_0^\infty \vartheta^{\zeta-1} e^{-\vartheta} d\vartheta, \ \zeta > 0.$
Let $g \in P_1(\Theta, \Xi)$. For every $\varphi \in \Xi^*$, we have

$$\varphi({}^\rho\mathcal{J}_{\kappa^+}^\zeta g(\vartheta)) = {}^\rho\mathcal{J}_{\kappa^+}^\zeta \varphi(g(\vartheta)) \quad for \quad a.e. \quad \vartheta \in \Theta.$$

Definition 2.5. *([19]) The generalized fractional derivative is given for $0 \leq \kappa < \vartheta$, by:*

$$({}^\rho D_{\kappa^+}^\zeta g)(\vartheta) = \frac{\rho^{1-n+\zeta}}{\Gamma(n-\zeta)} \left(\varrho^{1-\rho} \frac{d}{d\varrho} \right)^n \int_a^\vartheta \frac{\varrho^{\rho-1}}{(\vartheta^\rho - \varrho^\rho)^{1-n+\zeta}} g(\varrho) d\varrho. \tag{2.2}$$

Definition 2.6. *([19]) The Caputo-type generalized fractional derivative ${}_c^\rho D_{\kappa^+}^\zeta$ is given by*

$${}_c^\rho D_{\kappa^+}^\zeta g(\vartheta) = \left({}^\rho D_{\kappa^+}^\zeta \left[g(\varrho) - \sum_{k=0}^{n-1} \frac{g^{(k)}(\kappa)}{k!} (\varrho - \kappa)^k \right] \right) (\vartheta). \tag{2.3}$$

Lemma 2.1. *([19]) Let $\zeta, \rho \in \mathbb{R}^+$, and $g \in AC^{n-1}(\Theta, \Xi)$, then*

$$({}^{\rho}\mathcal{J}^{\zeta}_{\kappa^+}\,{}^{\rho}_{c}D^{\zeta}_{\kappa^+}g)(\vartheta) = g(\vartheta) - \sum_{k=0}^{n-1} c_k \left(\frac{\vartheta^{\rho} - \kappa^{\rho}}{\rho}\right)^k, \tag{2.4}$$

for some $c_k \in \mathbb{R}$, $n = [\zeta] + 1$.

Definition 2.7. *([12]) Let Ξ be a Banach space and Φ_{Ξ} the bounded subsets of Ξ, and B_1 the unit ball of Ξ. The De Blasi measure of weak noncompactness is the map $\mu : \Phi_{\Xi} \to [0, \infty)$ defined by*

$$\mu(B) = \inf\{\epsilon > 0 : \text{ there exists a weakly compact subset } \Phi \quad \text{of} \Xi : X \subset \epsilon B_1 + \Phi\}.$$

The next result follows directly from the Hahn-Banach theorem.

Proposition 2.2. *If Ξ is a normed space and $x_0 \in \Xi \backslash \{0\}$, then there exists $\varphi \in \Xi^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|\varphi\|$.*

The Blasi measure of weak noncompactness satisfies the following properties.

Lemma 2.2. *([12]) Let A and B bounded sets. Then we have*

(1) $\mu(B) = 0 \Leftrightarrow \overline{B}$ *is compact (B is weakly relatively compact)*
(2) $\mu(cov(B)) = \mu(B)$
(3) $\mu(B) = \zeta(\overline{B}^{\omega})$, *($\overline{B}^{\omega}$ denote the weak closure of B)*
(4) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$
(5) $\mu(A + B) \leq \mu(A) + \mu(B)$, *where* $A + B = \{x + y : x \in A, \quad y \in B\}$
(6) $\mu(\upsilon B) = |\upsilon|\mu(B)$; $\upsilon \in \mathbb{R}$, *where* $\upsilon B = \{\upsilon x : x \in B\}$
(7) $\mu(A \cup B) = max\{\mu(A), \mu(B)\}$
(8) $\mu(B + x_0) = \mu(B)$ *for any* $x_0 \in \Xi$.

Lemma 2.3. *([13]) Let $V \subset C(\Theta, \Xi)$ is a bounded and equicontinuous set, then*

(i) *the function $\vartheta \longmapsto \mu(V(\vartheta))$ is continuous on Θ, and*

$$\mu_C(V) = \max_{\vartheta \in \Theta} \mu(V(\vartheta)),$$

(ii)

$$\mu\left(\int_a^{\bar{\kappa}} y(\varrho)d\varrho : y \in V\right) = \int_a^{\bar{\kappa}} \mu(V(\varrho))d\varrho,$$

where

$$V(\vartheta) = \{y(\vartheta) : y \in V\}, \ \vartheta \in \Theta,$$

and μ_C is the De Blasi measure of weak noncompactness defined on the bounded sets of $C(\Theta)$.

Theorem 2.1. *([14]) Let D be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space $C(\Theta)$ such that $0 \in D$. Suppose $N : D \longrightarrow D$ is weakly-sequentially continuous. If the implication*

$$V = \overline{co}(N(V) \cup \{(0,0)\}) \Longrightarrow V \ \ \textit{is} \ \ \textit{relatively} \ \ \textit{weakly} \ \ \textit{compact}, \tag{2.5}$$

holds for every subset $V \subset D$, then the operator N has a fixed point.

3. Main results

Lemma 3.1. *Let $1 < \zeta \leq 2$, $\delta \in C([\kappa - \lambda_1, \kappa], \Xi)$ with $\delta(\kappa) = 0$, $\tilde{\delta} \in C([\bar{\kappa}, \bar{\kappa} + \lambda_2], \Xi)$ with $\tilde{\delta}(\bar{\kappa}) = 0$ and $\bar{\Psi} : \Theta \to \Xi$ be a continuous function. Then, the problem*

$${}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{p}(\vartheta) = \bar{\Psi}(\vartheta), \ \textit{for a.e} \ \vartheta \in \Theta := [\kappa, \bar{\kappa}], \ 1 < \zeta \leq 2, \tag{3.1}$$

$$\mathfrak{p}(\vartheta) = \delta(\vartheta), \ \vartheta \in [\kappa - \lambda_1, \kappa], \ \lambda_1 > 0 \tag{3.2}$$

$$\mathfrak{p}(\vartheta) = \tilde{\delta}(\vartheta), \ \vartheta \in [\bar{\kappa}, \bar{\kappa} + \lambda_2], \ \lambda_2 > 0, \tag{3.3}$$

has the following unique solution

$$\mathfrak{p}(\vartheta) = \begin{cases} \delta(\vartheta), \ \textit{if} \quad \vartheta \in [\kappa - \lambda_1, \kappa], \\ \\ -\displaystyle\int_a^T G(\vartheta, \varrho)\bar{\Psi}(\varrho)d\varrho, \ \textit{if} \quad \vartheta \in \Theta \\ \\ \tilde{\delta}(\vartheta), \ \textit{if} \quad \vartheta \in [\bar{\kappa}, \bar{\kappa} + \lambda_2], \end{cases} \tag{3.4}$$

where

(3.5)

$$G(\vartheta,\varrho)=\frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\begin{cases}\dfrac{(\vartheta^\rho-\kappa^\rho)(\bar{\kappa}^\rho-\varrho^\rho)^{\zeta-1}\varrho^{\rho-1}}{(\bar{\kappa}^\rho-\kappa^\rho)}-\varrho^{\rho-1}(\vartheta^\rho-\varrho^\rho)^{\zeta-1}, & \kappa\le\varrho\le\vartheta\le\bar{\kappa},\\ \\ \dfrac{(\vartheta^\rho-\kappa^\rho)(\bar{\kappa}^\rho-\varrho^\rho)^{\zeta-1}\varrho^{\rho-1}}{(\bar{\kappa}^\rho-\kappa^\rho)}, & \kappa\le\vartheta\le\varrho\le\bar{\kappa}.\end{cases}$$

Here $G(\vartheta,\varrho)$ is called the Green function of (3.1)-(3.3).

Proof. *From* (2.4), *we have*

$$\mathfrak{p}(\vartheta)=c_0+c_1\left(\frac{\vartheta^\rho-\kappa^\rho}{\rho}\right)+^\rho\mathcal{J}^{\zeta}_{\kappa^+}\bar{\Psi}(\varrho),\quad c_0,c_1\in\mathbb{R},\tag{3.6}$$

therefore

$$\mathfrak{p}(\kappa)=c_0=0,$$

$$\mathfrak{p}(\bar{\kappa})=c_1\left(\frac{\bar{\kappa}^\rho-\kappa^\rho}{\rho}\right)+\frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_a^{\bar{\kappa}}(\bar{\kappa}^\rho-\varrho^\rho)^{\zeta-1}\varrho^{\rho-1}\bar{\Psi}(\varrho)d\varrho,$$

and

$$c_1=-\frac{\rho^{2-\zeta}}{(\bar{\kappa}^\rho-\kappa^\rho)\Gamma(\zeta)}\int_a^{\bar{\kappa}}(\bar{\kappa}^\rho-\varrho^\rho)^{\zeta-1}\varrho^{\rho-1}\bar{\Psi}(\varrho)d\varrho.$$

Substitute the value of c_0 and c_1 into equation (3.6), we get equation (3.4).

$$\mathfrak{p}(\vartheta)=\begin{cases}\delta(\vartheta), & if\quad \vartheta\in[\kappa-\lambda_1,\kappa],\\ \\ -\displaystyle\int_a^T G(\vartheta,\varrho)\bar{\Psi}(\varrho)d\varrho, & if\quad \vartheta\in\Theta\\ \\ \tilde{\delta}(\vartheta), & if\quad \vartheta\in[\bar{\kappa},\bar{\kappa}+\lambda_2].\end{cases}$$

Lemma 3.2. *Let* $\Psi_j : \Theta \times C[-\lambda_1, \lambda_2] \times C[-\lambda_1, \lambda_2] \times \Xi^2 \longrightarrow \Xi$, $j = 1, 2$, *be continuous functions.* $(\mathfrak{p}, \mathfrak{q}) \in \Phi^2$ *is solution of* $(1.1) - (1.2)$ *if and only if* $(\mathfrak{p}, \mathfrak{q})$ *verifies the following coupled system:*

$$\mathfrak{p}(\vartheta) = \begin{cases} \delta_1(\vartheta), \ \text{if} \quad \vartheta \in [\kappa - \lambda_1, \kappa], \\ \\ -\displaystyle\int_a^T G(\vartheta, \varrho)\bar{\Psi}_1(\varrho)d\varrho, \ \text{if} \quad \vartheta \in \Theta \\ \\ \tilde{\delta}_1(\vartheta), \ \text{if} \quad \vartheta \in [\bar{\kappa}, \bar{\kappa} + \lambda_2], \end{cases}$$

$$\mathfrak{q}(\vartheta) = \begin{cases} \delta_2(\vartheta), \ \text{if} \quad \vartheta \in [\kappa - \lambda_1, \kappa], \\ \\ -\displaystyle\int_a^T G(\vartheta, \varrho)\bar{\Psi}_2(\varrho)d\varrho, \ \text{if} \quad \vartheta \in \Theta \\ \\ \tilde{\delta}_2(\vartheta), \ \text{if} \quad \vartheta \in [\bar{\kappa}, \bar{\kappa} + \lambda_2], \end{cases}$$

where $\bar{\Psi}_j \in C(\Theta)$ *verifies the system:*

$$\begin{cases} \bar{\Psi}_1(\vartheta) = \Psi_1(\vartheta, \mathfrak{p}^\vartheta, \mathfrak{q}^\vartheta, \bar{\Psi}_1(\vartheta), \bar{\Psi}_2(\vartheta)), \\ \bar{\Psi}_2(\vartheta) = \Psi_2(\vartheta, \mathfrak{p}^\vartheta, \mathfrak{q}^\vartheta, \bar{\Psi}_1(\vartheta), \bar{\Psi}_2(\vartheta)). \end{cases}$$

The hypotheses:

(H_1): The functions $(\mathfrak{p}, \mathfrak{q}, \bar{\mathfrak{p}}, \bar{\mathfrak{q}}) \longrightarrow \Psi_j(\vartheta, \mathfrak{p}, \mathfrak{q}, \bar{\mathfrak{p}}, \bar{\mathfrak{q}})$, $j = 1, 2$, are weakly sequentially continuous for $a.e.$ $\vartheta \in \Theta$.

(H_2): For all $\mathfrak{p}, \mathfrak{q} \in C([-\lambda_1, \lambda_2])$, $\bar{\mathfrak{p}}, \bar{\mathfrak{q}} \in \Xi$ the functions $\vartheta \longrightarrow \Psi_j(\vartheta, \mathfrak{p}, \mathfrak{q}, \bar{\mathfrak{p}}, \bar{\mathfrak{q}})$, $j = 1, 2$, are Pettis integrable.

(H_3): There exist $p_j, q_j \in C([\kappa, \bar{\kappa}], \mathbb{R}_+)$, $j = 1, 2$, such that, for all $\varphi \in \Xi^*$,

$$|\varphi(\Psi_j(\vartheta, \mathfrak{p}, \mathfrak{q}, \bar{\mathfrak{p}}, \bar{\mathfrak{q}}))| \leq \frac{p_j(\vartheta)\|\mathfrak{p}\|_{[-\lambda_1, \lambda_2]} + q_j(\vartheta)\|\mathfrak{q}\|_{[-\lambda_1, \lambda_2]}}{1 + \|\varphi\| + \|\mathfrak{p}\|_{[-\lambda_1, \lambda_2]} + \|\mathfrak{q}\|_{[-\lambda_1, \lambda_2]} + \|\bar{\mathfrak{p}}\|_\Xi + \|\bar{\mathfrak{q}}\|_\Xi}$$

for a.e. $\vartheta \in \Theta$, and each $\mathfrak{p}, \mathfrak{q} \in C([-\lambda_1, \lambda_2])$ and $\bar{\mathfrak{p}}, \bar{\mathfrak{q}} \in \Xi$.

(H_4): For each bounded measurable sets $B_j \subset C[-\lambda_1, \lambda_2], j = 1, 2,$ and each $\vartheta \in \Theta$, we have

$$\mu(\Psi_1(\vartheta, B_1, B_2, {}^{\rho}_{c}D^{\zeta}_{\kappa^+}(B_1), {}^{\rho}_{c}D^{\zeta}_{\kappa^+}(B_2)), 0) \leq p_1(\vartheta) \sup_{\varrho\in[-\lambda_1,\lambda_2]} \mu(B_1(\varrho)) + q_1(\vartheta) \sup_{\varrho\in[-\lambda_1,\lambda_2]} \mu(B_2(\varrho))$$

and

$$\mu(0, \Psi_2(\vartheta, B_1, B_2, {}^{\rho}_{c}D^{\zeta}_{\kappa^+}(B_1), {}^{\rho}_{c}D^{\zeta}_{\kappa^+}(B_2))) \leq p_2(\vartheta) \sup_{\varrho\in[-\lambda_1,\lambda_2]} \mu(B_1(\varrho)) + q_2(\vartheta) \sup_{\varrho\in[-\lambda_1,\lambda_2]} \mu(B_2(\varrho)),$$

where

$${}^{\rho}_{c}D^{\zeta}_{\kappa^+}(B_j) = \{{}^{\rho}_{c}D^{\zeta}_{\kappa^+}(\mathfrak{p}) : \mathfrak{p} \in B_j\}, j = 1, 2.$$

Set

$$p_j^* = \sup_{\vartheta\in\Theta} p_j(\vartheta), \quad q_j^* = \sup_{\vartheta\in\Theta} q_j(\vartheta), \quad j = 1, 2$$

$$\widetilde{G} = \sup\left\{\int_a^{\bar{\kappa}} |G(\vartheta, \varrho)| d\varrho, \vartheta \in \Theta\right\}.$$

Theorem 3.1. *Suppose that (H_1) - (H_4) hold. If*

$$\widetilde{G}(p_1^* + q_1^* + p_2^* + q_2^*) < 1, \tag{3.7}$$

then (1.1)-(1.2) has at least one weak solution defined on Θ.

Proof. Let $\mathfrak{S} : \Phi \times \Phi \longmapsto \Phi \times \Phi$ be the operator given by

(3.8)

$$\mathfrak{S}(\mathfrak{p}, \mathfrak{q})(\vartheta) = (\mathfrak{S}_1(\mathfrak{p}, \mathfrak{q}), \mathfrak{S}_2(\mathfrak{p}, \mathfrak{q})) = \begin{cases} (\delta_1(\vartheta), \delta_2(\vartheta)), \text{ if } \vartheta \in [\kappa - \lambda_1, \kappa], \\ -\left(\int_a^T G(\vartheta, \varrho)\bar{\Psi}_1(\varrho) d\varrho, \int_a^T G(\vartheta, \varrho)\bar{\Psi}_2(\varrho) d\varrho\right), \ \vartheta \in \Theta \\ (\tilde{\delta}_1(\vartheta), \tilde{\delta}_2(\vartheta)), \text{ if } \vartheta \in [\bar{\kappa}, \bar{\kappa} + \lambda_2]. \end{cases}$$

First, notice that the hypotheses imply that, for each $\bar{\Psi}_j \in C(\Theta),\ j = 1, 2,$ the function $\vartheta \longrightarrow G(\vartheta, \varrho)\bar{\Psi}_j(\vartheta)$ are Pettis integrable over Θ. Define

$$D = \left\{ (\mathfrak{p}, \mathfrak{q}) \in \Phi \times \Phi : \begin{array}{c} \|(\mathfrak{p}, \mathfrak{q})\|_{\overline{\Phi}} \leq \varpi, \\ \|\mathfrak{p}(\vartheta_2) - \mathfrak{p}(\vartheta_1)\|_{\Xi} \leq (p_1^* + q_1^*) \int_\kappa^{\bar{\kappa}} |G(\vartheta_2, \varrho) - G(\vartheta_1, \varrho)| d\varrho, \\ \textit{and} \ \|\mathfrak{q}(\vartheta_2) - \mathfrak{q}(\vartheta_1)\|_{\Xi} \leq (p_2^* + q_2^*) \int_\kappa^{\bar{\kappa}} |G(\vartheta_2, \varrho) - G(\vartheta_1, \varrho)| d\varrho, \end{array} \right\}$$

where

$$\varpi \geq \max\left\{L_1+L_2, \|\delta_1\|_{[\kappa-\lambda_1,\kappa]}+\|\delta_2\|_{[\kappa-\lambda_1,\kappa]}, \|\tilde{\delta}_1\|_{[\bar{\kappa},\bar{\kappa}+\lambda_2]}+\|\tilde{\delta}_2\|_{[\bar{\kappa},\bar{\kappa}+\lambda_2]}\right\}. \tag{3.9}$$

Clearly, the subset D is closed, convex end equicontinuous. We shall show that the operator $\mathfrak{S}$ satisfies all the assumptions of Theorem 2.1. The proof will be given in several steps.

Step 1. *$\mathfrak{S}$ maps D into itself.*

Let $(\mathfrak{p},\mathfrak{q}) \in D,\ \vartheta \in \Theta$ and assume that $(\mathfrak{S}(\mathfrak{p},\mathfrak{q}))(\vartheta) \neq (0,0)$. Then there exists $\varphi \in \Xi^*$ such that $\|\mathfrak{S}_j(\mathfrak{p},\mathfrak{q})(\vartheta)\|_\Xi = \varphi\left(\mathfrak{S}_j(\mathfrak{p},\mathfrak{q})(\vartheta)\right)$. Thus, for any $j \in \{1,2\}$ we have

$$\|\mathfrak{S}_j(\mathfrak{p},\mathfrak{q})(\vartheta)\|_\Xi = \varphi\left(\int_a^T G(\vartheta,\varrho)\bar{\Psi}_j(\varrho)d\varrho\right),$$

where $\bar{\Psi}_j \in C(\Theta)$, with

$$\bar{\Psi}_j(\vartheta) = \Psi_j(\vartheta, \mathfrak{p}^\vartheta, \mathfrak{q}^\vartheta, \bar{\Psi}_1(\vartheta), \bar{\Psi}_2(\vartheta)).$$

If $\vartheta \in [\kappa-\lambda_1,\kappa]$, then

$$\|\mathfrak{S}(\mathfrak{p},\mathfrak{q})(\vartheta)\|_\Xi \leq \|\delta_1\|_{[\kappa-\lambda_1,\kappa]}+\|\delta_2\|_{[\kappa-\lambda_1,\kappa]} \leq \varpi,$$

and if $\vartheta \in [\bar{\kappa},\bar{\kappa}+\lambda_2]$, then

$$\|\mathfrak{S}(\mathfrak{p},\mathfrak{q})(\vartheta)\|_\Xi \leq \|\tilde{\delta}_1\|_{[\bar{\kappa},\bar{\kappa}+\lambda_2]}+\|\tilde{\delta}_2\|_{[\bar{\kappa},\bar{\kappa}+\lambda_2]} \leq \varpi.$$

For each $\vartheta \in \Theta$, we obtain

$$\|(\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta)\|_\Xi \leq \int_\kappa^{\bar{\kappa}} |G(\vartheta,\varrho)||\varphi(\bar{\Psi}_j(\varrho))|d\varrho, j=1,2.$$

By (H_3), we get

$$|\varphi(\bar{\Psi}_j(\vartheta))| \leq p_j^* + q_j^*.$$

Therefore

$$\begin{aligned}\|(\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta)\|_\Xi &\leq (p_j^*+q_j^*)\int_\kappa^{\bar{\kappa}} |G(\vartheta,\varrho)|d\varrho\\ &\leq (p_j^*+q_j^*)\widetilde{G} = L_j.\end{aligned}$$

Thus, for each $\vartheta \in [\kappa-\lambda_1,\bar{\kappa}+\lambda_2]$, we have

$$\|\mathfrak{S}_j(\mathfrak{p},\mathfrak{q})(\vartheta)\|_\Xi \leq L_j,$$

which implies that

$$\|\mathfrak{S}_j(\mathfrak{p},\mathfrak{q})\|_\Phi \leq L_j.$$

Then we have

$$\begin{aligned}\|\mathfrak{S}(\mathfrak{p},\mathfrak{q})\|_{\overline{\Phi}} &\leq L_1 + L_2\\ &\leq \varpi.\end{aligned}$$

Next, Let $\vartheta_1, \vartheta_2 \in \Theta = [\kappa, \bar{\kappa}], \vartheta_1 < \vartheta_2,$ *and* $(\mathfrak{p},\mathfrak{q}) \in D$ be such that

$$(\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_2) - (\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_1) \neq 0.$$

Then there exists $\varphi \in \Xi^*$ such that

$$\|(\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_2) - (\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_1)\|_\Xi = \varphi((\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_2) - (\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_1)),$$

and $\|\varphi\| = 1$. Then, for any $j \in \{1, 2\}$, we get

$$\begin{aligned}\|(\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_2) - (\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_1)\|_\Xi &= \varphi((\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_2) - (\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_1))\\ &\leq \varphi\left(\int_\kappa^{\bar{\kappa}} |G(\vartheta_2,\varrho) - G(\vartheta_1,\varrho)|\bar{\Psi}_j(\varrho)\right)\end{aligned}$$

where $\bar{\Psi}_j \in C(\Theta)$, with

$$\bar{\Psi}_j(\vartheta) = \Psi_j(\vartheta, \mathfrak{p}^\vartheta, \mathfrak{q}^\vartheta, \bar{\Psi}_1(\vartheta), \bar{\Psi}_2(\vartheta)).$$

Thus, we have

$$\begin{aligned}\|(\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_2) - (\mathfrak{S}_j(\mathfrak{p},\mathfrak{q}))(\vartheta_1)\|_\Xi &\leq \int_\kappa^{\bar{\kappa}} |G(\vartheta_2,\varrho) - G(\vartheta_1,\varrho)||\varphi(\bar{\Psi}_j(\varrho))|d\varrho\\ &\leq (p_j^* + q_j^*)\int_\kappa^{\bar{\kappa}} |G(\vartheta_2,\varrho) - G(\vartheta_1,\varrho)|d\varrho.\end{aligned}$$

Consequently,

$$\mathfrak{S}(D) \subset D.$$

Step 2. $\mathfrak{S}$ is weakly sequentially continuous.

Let $\{(\mathfrak{p}_n, \mathfrak{q}_n)\}_n$ be a sequence in $D \times D$, and let $(\mathfrak{p}_n(\vartheta), \mathfrak{q}_n(\vartheta)) \longrightarrow (\mathfrak{p}(\vartheta), \mathfrak{q}(\vartheta))$ in $(\Xi, \omega) \times (\Xi, \omega)$ for each $\vartheta \in [\kappa - \lambda_1, \bar{\kappa} + \lambda_2]$. Fix $\vartheta \in [\kappa - \lambda_1, \bar{\kappa} + \lambda_2]$. Since for any $j \in 1, 2$, the function $\Psi_j(\vartheta, \mathfrak{p}_n^\vartheta, \mathfrak{q}_n^\vartheta, {}_c^\rho D_{\kappa^+}^\zeta \mathfrak{p}_n(\vartheta), {}_c^\rho D_{\kappa^+}^\zeta \mathfrak{q}_n(\vartheta))$ satisfies assumption (H_1), we have that it converges weakly uniformly to $\Psi_j(\vartheta,\mathfrak{p}^{\vartheta},\mathfrak{q}^{\vartheta},{}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{p}(\vartheta),{}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{q}(\vartheta))$. Hence the Lebesgue dominated convergence theorem for Pettis integral implies that $(\mathfrak{S}(\mathfrak{p}_n,\mathfrak{q}_n))(\vartheta)$ converges weakly uniformly to $(\mathfrak{S}(\mathfrak{p},\mathfrak{q}))(\vartheta)$ in (Ξ,ω). We do it for each $\vartheta\in\Theta$, so $\mathfrak{S}(\mathfrak{p}_n,\mathfrak{q}_n)\longrightarrow\mathfrak{S}(\mathfrak{p},\mathfrak{q})$. Then $\mathfrak{S}:D\longrightarrow D$ is weakly sequentially continuous.

Step 3. Now let V be a subset of D such that $V=conv(\mathfrak{S}(V)\cup\{(0,0)\})$. Obviously

$$V(\vartheta)\subset conv(\mathfrak{S}(V)(\vartheta)\cup\{(0,0)\}).$$

Since V is bounded and equicontinuous, the function $\vartheta\longmapsto\mathfrak{q}(\vartheta)=\mu(V(\vartheta))$ is continuous on $[\kappa-\lambda_1,\bar{\kappa}+\lambda_2]$. By $(H_1)-(H_3)$, Lemma 2.3, and the properties of measure μ, for each $\vartheta\in\Theta$, we have

$$\begin{aligned}
\mathfrak{q}(\vartheta) &\leq \mu(\mathfrak{S}(V)(\vartheta)\cup\{(0,0)\})\\
&\leq \mu((\mathfrak{S}V)(\vartheta))\\
&\leq \mu\left(\{((\mathfrak{S}_1\mathfrak{p})(\vartheta),(\mathfrak{S}_2\mathfrak{q})(\vartheta)):(\mathfrak{p},\mathfrak{q})\in V\}\right)\\
&\leq \int_{\kappa}^{\bar{\kappa}}|G(\vartheta,\varrho)|\mu(\{(\Psi_1(\vartheta,\mathfrak{p}^{\vartheta},\mathfrak{q}^{\vartheta},{}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{p}(\vartheta),{}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{q}(\vartheta)),0)\})d\varrho\\
&+ \int_{\kappa}^{\bar{\kappa}}|G(\vartheta,\varrho)|\mu(\{(0,\Psi_2(\vartheta,\mathfrak{p}^{\vartheta},\mathfrak{q}^{\vartheta},{}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{p}(\vartheta),{}^{\rho}_{c}D^{\zeta}_{\kappa^+}\mathfrak{q}(\vartheta)))\})d\varrho\\
&\leq \int_{\kappa}^{\bar{\kappa}}|G(\vartheta,\varrho)|(p_1(\varrho)\mu(\{(\mathfrak{p}(\varrho),0);(\mathfrak{p},0)\in V\})\\
&+ q_1(\varrho)\mu(\{(\mathfrak{q}(\varrho),0);(\mathfrak{q},0)\in V\})d\varrho\\
&+ \int_{\kappa}^{\bar{\kappa}}|G(\vartheta,\varrho)|(p_2(\varrho)\mu(\{(0,\mathfrak{p}(\varrho));(0,\mathfrak{p})\in V\})\\
&+ q_2(\varrho)\mu(\{(0,\mathfrak{q}(\varrho));(0,\mathfrak{q})\in V\}))d\varrho\\
&\leq \int_{\kappa}^{\bar{\kappa}}|G(\vartheta,\varrho)|(p_1(\varrho)+q_1(\varrho)+p_2(\varrho)+q_2(\varrho))\mu(V(\varrho))d\varrho\\
&\leq \widetilde{G}(p_1^*+q_1^*+p_2^*+q_2^*)\|\mathfrak{q}\|_c.
\end{aligned}$$

Thus

$$\|\mathfrak{q}\|_c\leq\widetilde{G}(p_1^*+q_1^*+p_2^*+q_2^*)\|\mathfrak{q}\|_c.$$

From (3.7), we get $\|\mathfrak{q}\|_c=0$, that is $\mu(V(\vartheta))=0$ for each $\vartheta\in\Theta$.
For $\vartheta\in[\kappa-\lambda_1,\kappa]$, we have

$$\begin{aligned}
\mathfrak{q}(\vartheta) &= \mu((\delta_1(\vartheta),\delta_2(\vartheta)))\\
&= 0.
\end{aligned}$$

Also for $\vartheta \in [\bar{\kappa}, \bar{\kappa} + \lambda_2]$ we have

$$\begin{aligned} \mathfrak{q}(\vartheta) &= \mu(\tilde{\delta}_1(\vartheta), \tilde{\delta}_2(\vartheta)) \\ &= 0, \end{aligned}$$

then $V(\vartheta)$ is relatively compact in Ξ. In view of Ascoli-Arzela theorem V is weakly relatively compact in $\overline{\Phi}$. Applying Theorem 2.1, we conclude that $\mathfrak{G}$ has a fixed point which is a solution of the problem (1.1) – (1.2).

4. An Example

Let

$$\Xi = l^1 = \left\{ \mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n, \dots), \sum_{k=1}^{\infty} |\mathfrak{p}_n| < \infty \right\},$$

be the Banach space with the norm

$$\|\mathfrak{p}\|_{\Xi} = \sum_{k=1}^{\infty} |\mathfrak{p}_n|.$$

Consider the following fractional differential equation:

$$(4.1) \qquad \begin{cases} (\mathfrak{p}(\vartheta), \mathfrak{q}(\vartheta)) = (\frac{1}{2}\vartheta, \vartheta^2 + \vartheta), \qquad \vartheta \in [-1, 0], \\ \\ {}^1_cD^{\frac{3}{2}}_{0^+}\mathfrak{p}_n(\vartheta) = \Psi(\vartheta,\ \mathfrak{p}^{\vartheta}_n,\ \mathfrak{q}^{\vartheta}_n,\ {}^1_cD^{\frac{3}{2}}_{0^+}\mathfrak{p}_n(\vartheta),\ {}^1_cD^{\frac{3}{2}}_{0^+}\mathfrak{p}_n(\vartheta)), \qquad \vartheta \in \Theta = [0, 1] \\ \\ {}^1_cD^{\frac{3}{2}}_{0^+}\mathfrak{q}_n(\vartheta) = g(\vartheta,\ \mathfrak{p}^{\vartheta}_n,\ \mathfrak{q}^{\vartheta}_n,\ {}^1_cD^{\frac{3}{2}}_{0^+}\mathfrak{p}_n(\vartheta),\ {}^1_cD^{\frac{3}{2}}_{0^+}\mathfrak{p}_n(\vartheta)), \qquad \vartheta \in \Theta = [0, 1] \\ \\ (\mathfrak{p}(\vartheta), \mathfrak{q}(\vartheta)) = (\vartheta - 1, \vartheta^2 - \vartheta), \qquad \vartheta \in [1, 2], \end{cases}$$

here $\bar{\kappa} = 1, \quad \kappa = 0, \quad \zeta = \dfrac{3}{2}, \quad \rho = 1.$

Set

$$\mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n, \dots), \qquad \Psi = (\Psi_1, \Psi_2, \dots, \Psi_n, \dots)$$

$$\Psi(\vartheta, \mathfrak{p}^{\vartheta}, \mathfrak{q}^{\vartheta}, {}^1_cD^{\frac{3}{2}}_{0^+}\mathfrak{p}(\vartheta), {}^1_cD^{\frac{3}{2}}_{0^+}\mathfrak{q}(\vartheta))$$

$$= \frac{sin(\vartheta)(\|\mathfrak{p}^{\vartheta}\|_{C([-1,1])} + \|\mathfrak{q}^{\vartheta}\|_{C([-1,1])})}{8(\vartheta+1)\left(1 + \|\mathfrak{p}^{\vartheta}\|_{C([-1,1])} + \|\mathfrak{q}^{\vartheta}\|_{C([-1,1])} + \|{}^3_cD^{\frac{3}{2}}_{0^+}\mathfrak{p}\|_{\Xi} + \|{}^3_cD^{\frac{3}{2}}_{2^+}\mathfrak{q}\|_{\Xi}\right)}.$$

and

$$g(\vartheta, \mathfrak{p}^{\vartheta}, \mathfrak{q}^{\vartheta}, {}^1_cD^{\frac{3}{2}}_{0^+}\mathfrak{p}(\vartheta), {}^1_cD^{\frac{3}{2}}_{0^+}\mathfrak{q}(\vartheta))$$

$$= \frac{cos(\vartheta)(\|\mathfrak{p}^{\vartheta}\|_{C([-1,1])} + \|\mathfrak{q}^{\vartheta}\|_{C([-1,1])})}{8(\vartheta^2+1)\left(1 + \|\mathfrak{p}^{\vartheta}\|_{C([-1,1])} + \|\mathfrak{q}^{\vartheta}\|_{C([-1,1])} + \|{}_c^1 D_{0^+}^{\frac{3}{2}}\mathfrak{p}\|_{\Xi} + \|{}_c^1 D_{0^+}^{\frac{3}{2}}\mathfrak{q}\|_{\Xi}\right)}.$$

For each $y \in \Xi$ and $\vartheta \in [0,1]$, we have

$$\|\Psi(\vartheta, \mathfrak{p}^{\vartheta}, \mathfrak{q}^{\vartheta}, {}_c^1 D_{0^+}^{\frac{3}{2}}\mathfrak{p}(\vartheta), {}_c^1 D_{0^+}^{\frac{3}{2}}\mathfrak{q}(\vartheta))\|_{\Xi} \leq \frac{sin(\vartheta)}{8(\vartheta+1)}$$

and

$$\|g(\vartheta, \mathfrak{p}^{\vartheta}, \mathfrak{q}^{\vartheta}, {}_c^1 D_{0^+}^{\frac{3}{2}}\mathfrak{p}(\vartheta), {}_c^1 D_{0^+}^{\frac{3}{2}}\mathfrak{q}(\vartheta))\|_{\Xi} \leq \frac{cos(\vartheta)}{8(\vartheta^2+1)}.$$

Hence (H_2) is satisfied with $P_j^* = q_j^* = \frac{1}{8}, j = 1, 2$.

For each $\vartheta \in \Theta$ we have

$$\begin{aligned}
\int_a^{\bar{\kappa}} |G(\vartheta, \varrho)| d\varrho &\leq \frac{1}{\Gamma(\zeta)}\left(\frac{\vartheta^{\rho}-\kappa^{\rho}}{\bar{\kappa}^{\rho}-\kappa^{\rho}}\right)\int_a^{\bar{\kappa}}\left|\left(\frac{\bar{\kappa}^{\rho}-\varrho^{\rho}}{\rho}\right)^{\zeta-1}\varrho^{\rho-1}\right| d\varrho \\
&+ \frac{1}{\Gamma(\zeta)}\int_a^{\vartheta}\left|\left(\frac{\vartheta^{\rho}-\varrho^{\rho}}{\rho}\right)^{\zeta-1}\varrho^{\rho-1}\right| d\varrho \\
&\leq \frac{2}{\Gamma(\zeta+1)}\left(\frac{\bar{\kappa}^{\rho}-\kappa^{\rho}}{\rho}\right)^{\zeta}.
\end{aligned}$$

Therefore

$$\widetilde{G} \leq \frac{2}{\Gamma(\zeta+1)}\left(\frac{\bar{\kappa}^{\rho}-\kappa^{\rho}}{\rho}\right)^{\zeta}.$$

Condition (3.7) holds, indeed,

$$\begin{aligned}
\widetilde{G}(p_1^* + q_1^* + p_2^* + q_2^*) &\leq \frac{1}{\Gamma(\frac{3}{2}+1)} \\
&\approx 0.7522527778 \\
&< 1.
\end{aligned}$$

as all the assumptions of Theorem 3.1 are met. Then, problem (4.1) has at least one solution.

Conclusion

We have provided some sufficient conditions guaranteeing the existence of weak solutions for some coupled systems of fractional differential equations. Mönch's fixed point theorem associated with the technique of measure of weak noncompactness were used. An example is included to show the applicability of our outcomes. It is interesting, in a forthcoming paper, to consider the set-valued analogue problem.

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