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STABLE RANGE CONDITIONS

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Abstract. In this paper, we discuss stable range conditions (stable range one, unit 1-stable range and weakly unit 1-stable range) for a semilocal ring. In particular, we prove that a semilocal ring satisfies weakly unit 1-stable range if and only if it is $\sum$-clean ring. We also show that none of the properties are local properties.

1. Introduction

Throughout the paper, R denotes the commutative ring with unity unless otherwise stated. The group of unit elements, the set of idempotent elements, the set of zero-divisors, the Jacobson radical and the nil radical are denoted by $U(R)$, $Id(R)$, $Z(R)$, $J(R)$ and $N(R)$ respectively. An associative ring R with unity satisfies stable range one provided that whenever $aR + bR = R$ with $a, b \in R$, there exists $y \in R$ such that $a + by \in U(R)$. This condition extensively studied in [2, 5, 6, 9, 11, 12]. We provide several results for specific rings satisfying stable range one. In particular, we give an elementary proof for a commutative semilocal ring (a ring having finite number of maximal ideals) satisfies stable range one condition. A characterization for Noetherian ring satisfying stable range one is also given. All these results have been given in section 2.

Following Xiao and Tong [14], an associative ring R with unity satisfies weakly unit 1-stable range provided that whenever $aR + bR = R$ with $a, b \in R$, there exists $w \in W(R)$ such that $a + bw \in U(R)$ where $W(R) = \{w \in R : \exists\, n \in \mathbb{N}\,\text{such that}\, w =$

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$e+u_1+u_2+\cdots+u_n$, where e is a central idempotent of R and $u_1, u_2, \ldots, u_n \in U(R)\}$. They proposed several characterizations and many other properties of rings satisfying weakly unit 1-stable range [14]. One of the important stable range condition is unit 1-stable range which is given as follows: an associative ring R with unity satisfies unit 1-stable range provided that whenever $aR+bR = R$ with $a,\ b \in R$, there exists $y \in U(R)$ such that $a + by \in U(R)$. Chen has established various results on rings satisfying unit 1-stable range, see [5, 6, 7]. In [8], Chen discussed the necessary and sufficient condition for a semilocal ring satisfying unit 1-stable range. Motivated from this result, we discuss how a semilocal ring behaves with respect to the property weakly unit 1-stable range. In section 3, several other results present for rings satisfying weakly unit 1-stable range and unit 1-stable range. We also prove none of the stable range conditions is a local property.

2. Stable Range One

In this section, we prove some of standard results by using element-wise technique. For convenience, we recall the definition of a ring R satisfies stable range one.

Definition 2.1. An associative ring R with unity satisfies stable range one provided that whenever $aR+bR = R$ with $a,\ b \in R$, there exists $y \in R$ such that $a+by \in U(R)$.

In the following theorem, we prove that every semilocal ring satisfies stable range one, which is a consequence of the result a ring R satisfies stable range one if and only if so does $R/J(R)$ [13] .

Theorem 2.1. *Every semilocal ring satisfies stable range one.*

Proof. Let R be a semilocal ring and $\{m_i\}_{i\in I}$ be maximal ideals of R where $I = \{1, 2, \ldots, n\}$. Let $aR + bR = R$ for $a, b \in R$. Then to prove the claim we have to find an element $y \in R$ such that $a + by \in U(R)$. We do this in the following three cases:
Case 1: If a is a unit, then we can take $y = 0$ so that $a + by \in U(R)$.
Case 2: If b is a unit, then we can take $y = -b^{-1}a + 1$ so that $a + by \in U(R)$.
Case 3: If a and b both are non-unit elements. Then $a \in \cap_{i\in S} m_i$ for some $S \subseteq I$ and $b \in \cap_{i\in S'} m_i - (\cup_{i\in S} m_i)$ where $S' \subseteq I$ and $S \cap S' = \phi$. Now there are two subcases:
Subcase 1: If $S \cup S' = I$, then $a + by \in U(R)$ for any $y \in U(R)$. For, if $a + by \in m_i$

for some $i \in S$, then $by \in m_i$ and thus $b \in m_i$, a contradiction. And if $a + by \in m_i$ for some $i \in S'$, then $a \in m_i$, again a contradiction.

Subcase 2: If $S \cup S' \neq I$ and assume $S'' = I - (S \cup S')$, then we take $y \in (\cap_{i \in S''} m_i) - (\cup_{j \in S \cup S'} m_j)$. This implies $a + by \in U(R)$. For, if $a + by \in m_i$ for some $i \in S$, then $by \in m_i$ for some $i \in S$ and thus $b \in m_i$ for some $i \in S$, a contradiction ; if $a + by \in m_i$ for some $i \in S'$, then $a \in m_i$ for some $i \in S'$, again a contradiction; if $a + by \in m_i$ for some $i \in S''$, then $a \in m_i$ for some $i \in S''$, a contradiction.

Existence of y: Suppose that $\cap_{i \in S''} m_i \subseteq \cup_{i \in S \cup S'} m_i$. Then we have $\cap_{i \in S''} m_i \subseteq m_i$ for some $i \in S \cup S'$. This implies $m_j \subseteq m_i$ for some $j \in S''$. Therefore $m_j = m_i$ for some $i \neq j$. Hence $\cap_{i \in S''} m_i \nsubseteq \cup_{i \in S \cup S'} m_i$. □

The following corollary follows immediately.

Corollary 2.1. *Every local ring satisfies stable range one.*

The next proposition provides a criteria for a principal ideal domain(PID) satisfying stable range one.

Proposition 2.1. *Let R be a PID. Then R satisfies stable range one if and only if for any pair $a, b \in R$, we have a is associate to $d - by$, where $y \in R$ and d is a greatest common divisor of a and b.*

Proof. Let R satisfy stable range condition and $a, b \in R$. Then $aR + bR = dR$ where d is a greatest common divisor of a and b. Then there exist $u \in U(R)$ and $y \in R$ such that $au + by = d$. This implies $a = u^{-1}(d - by)$. Hence a and $d - by$ are associates.
Conversely, suppose for any pair $a, b \in R$; we have a is associate to $d - by$ for some $y \in R$ and d is a greatest common divisor of a and b. Then we have to show that R satisfies stable range one. Let $aR + bR = R$. Then $aa' + bb' = 1$ for some a' and b' in R. Clearly the greatest common divisor of a' and b is 1. Therefore by assumption $a' = u(1 - by)$ for some $u \in U(R)$ and $y \in R$. Hence $a + b(-uy' + b')u^{-1} \in U(R)$. □

The prime ideals play an important role in studying rings such as: if all prime ideals of a ring R are finitely generated, then R is a Noetherian ring and if all prime ideals of a ring R are principal ideals, then R is principal ideal ring. In both statements, we reduce the conditions from all ideals to prime ideals. It is well known that prime ideals in PID are generated by irreducible elements. Therefore the natural question arises that if for any irreducibles a and b in a PID R, there exists $0 \neq y \in R$ with $a + by \in U(R)$, then does R satisfy stable range one? The answer is no. In PID, the stable range one condition for irreducible elements is not sufficient for PID to satisfy stable range one.

Example 2.1. Consider $R = \mathbb{C}[x]$. Then clearly only irreducible elements of R are of one degree polynomials. Now, we will show that $(ax+b) = (cx+d)$ if and only if $b/a = d/c$. If $b/a = d/c$, then $(ax + b) = (a(x + b/a)) = a(x + d/c) = (cx + d)$. Conversely, suppose $(ax + b) = (cx + d)$. Then $ax + b = e(cx + d)$ for some $e \in \mathbb{C}$. This implies $a = ec$ and $b = ed$. Thus $b = da/c$. Clearly, stable range condition is satisfied for irreducible polynomials but it is not satisfied for all elements.

The following theorem gives a characterization for a Noetherian ring satisfying stable range one.

Theorem 2.2. *Let R be Noetherian ring. Then R satisfies stable range one if and only if R/P satisfies stable range one for all prime ideals P.*

Proof. Let R satisfy stable range one condition. Let $\overline{a}\overline{R} + \overline{b}\overline{R} = \overline{R}$ where $\overline{R} = R/P$. This implies $ac + bd = 1 + r$ where $r \in P; c, d \in R$. Then there exists $u \in U(R),\ y \in R$ such that $au + by = 1 + r$ or equivalently $a + byu^{-1} = (1 + r)u^{-1}$. This implies $\overline{a + byu^{-1}} \in U(\overline{R})$. Conversely, there are two cases: (i) If $J(R) = 0$, then $N(R) = 0$. Since R is a Noetherian ring, $N(R) = \cap_{i=1}^{n} P_i$. Thus R is a subdirect product of R/P_i and the rings R/P_i satisfy stable range one condition. Hence by [5, Theorem 1.1.13] R satisfies stable range one. (ii) If $J(R) \neq 0$, then consider a ring $R' = R/J(R)$. Clearly, $J(R') = 0$ and R' is Noetherian so by case (i) R' satisfies stable range one. Hence R satisfies stable range one. □

For a ring R which is not a Noetherian, R does not satisfy stable range one if R/P satisfies stable range one for all the prime ideals P. For example, consider $R = \mathbb{Z}$. Clearly, R does not satisfy stable range one while the ring R/P for any prime ideal P of R is of the form $\mathbb{Z}_p$ where p is a prime number satisfies stable range one.

For the definition of clean rings and pm-rings, we refer the paper [1]. A ring R is called clean ring if every element of R is the sum of a unit element and an idempotent element, while a ring R is called pm-ring if each prime ideal is contained in a unique maximal ideal of R. We end this section by proving the proposition which says that a commutative Noetherian clean ring satisfies stable range one condition.

Proposition 2.2. *Let R be a Noetherian ring. Then R is clean implies R satisfies stable range one.*

Proof. Let R be a clean ring. Then by [1, Corollary 4], it is a pm-ring. Thus R is a Noetherian pm-ring. Therefore R has finite number of maximal ideals. Hence by Theorem 2.1, R satisfies stable range one. □

The converse need not be true. For $R = \mathbb{Z}_{(2)\cup(3)}$ which is Noetherian and satisfies stable range one but not clean, see [3].

3. Results On Weakly Unit 1-Stable Range And Unit 1-Stable Range

In this section, we prove some new results on weakly unit 1-stable range condition and unit 1-stable range condition. For the definition of a ring satisfying weakly unit 1-stable range and unit 1-stable range we follow the papers [14] and [10] respectively. Every local ring is a clean ring and satisfies stable range one. Consequently, every local ring satisfies weakly unit 1-stable range. The natural question is that if R is a semilocal then whether it satisfies weakly unit 1-stable range or not. The answer to this question is that in general, semilocal rings do not satisfy weakly unit 1-stable range. For example, consider a ring $R = (\mathbb{Z}_2[x])_{(x)\cup(1-x)}$. Obviously, R is semilocal ring but it does not satisfy weakly unit 1-stable range, see [4, Section 5]. In the following theorem we provide a condition for a semilocal ring to satisfy weakly unit 1-stable range.

Theorem 3.1. *Let R be a semilocal ring with dimension zero. Then R satisfies weakly unit 1-stable range.*

Proof. If $J(R) = 0$, then $R \cong \prod_{i=i}^{n} R/m_i$ where $(m_i)_{i=1}^{n}$ are maximal ideals of R. Any element of $\prod R/m_i$ can be written as a sum of a unit and an idempotent. For $(a_1, a_2, ..., a_n) \in \prod R/m_i$, if $a_i \neq 0$ for all i, then $(a_1, a_2, ..., a_n)$ is a unit. If $a_i = 0$ for some set $S \subseteq \{1, 2, \cdots, n\}$, then $(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n) - (c_1, c_2, ..., c_n)$ where $b_i = 0$ for $i \notin S, b_i = 1$ for $i \in S$ and $c_i = 1$ for $i \in S, c_i = -a_i$ for $i \notin S$. Thus R satisfies weakly unit 1-stable range as $\prod_{i=1}^{n} R/m_i$ satisfies stable range one. If $J(R) \neq 0$, then $J(R) = N(R)$ and $J(R/J(R)) = 0$ as the ring is zero dimensional. Thus $R/J(R)$ satisfies weakly unit 1-stable range and idempotents can be lifted modulo $N(R)$. Hence R satisfies weakly unit 1-stable range[14, Corollary 5.2]. □

Remark 1. *The example of a ring which is given just before Theorem 3.1 is of dimension one, while the ring* $\mathbb{Z}_6[[x]]$ *is also of dimension one but it satisfies weakly unit 1-stable range. Therefore for a semilocal ring, dimesion zero is only a sufficient condition to satisfy weakly unit 1-stable range.*

Following [14], a ring R is called a $\sum$-clean ring if every element of R is a sum of an idempotent and a finite number of units. The next theorem discusses the necessary and sufficient condition for a semilocal ring which satisfies weakly unit 1-stable range in terms of a $\sum$-clean ring.

Theorem 3.2. *Let R be a semilocal ring. Then R satisfies weakly unit 1-stable range iff R is a $\sum$-clean ring.*

Proof. Let R be weakly unit 1-stable range and let $a \in R$. Assume $aR + uR = R$, where u is a unit element. Then by assumption, there exists $w \in W(R)$ such that $a + uw \in U(R)$. This gives $a = v - u(u_1 + u_2 + \cdots + u_n + e)$ where $u_i, u, v \in U(R)$ and $e \in Id(R)$. Then $a = v - uu_1 + uu_2 + \cdots + uu_n - ue$. We know that $ue = u' + e'$ where $u' \in U(R), e' \in Id(R)$; for, take $u' = ue - (1 - e)$ and $e' = 1 - e$, then $v' = u^{-1}e - (1 - e)$ is the inverse of u' and $e'e' = e'$. Thus $a = v - uu_1 + uu_2 + \cdots + uu_n - u' - e'$. Clearly, $-e'$ is a periodic element and every periodic element is clean [3, Theorem 2.7]. Thus a is a $\sum$-clean element and hence R is a $\sum$-clean ring. Conversely, let R be a $\sum$-clean. Since R is a semilocal ring, Theorem 2.1 concludes that R satisfies stable range one condition. Hence R satisfies the weakly unit 1-stable range condition. □

Remark 2. **(i)** *By Theorem 3.2, we can get examples for $\sum$-clean ring. It is clear that all zero dimensional semilocal rings and semilocal rings for which*

$J(R) = 0$ are $\sum$-clean rings.

(ii) *In Theorem 3.2, the assumption R to be a semilocal ring cannot be let down. For, $\mathbb{Z}$ is not semilocal but it is $\sum$-clean ring which does not satisfy weakly unit 1-stable range.*

Next we announce the following theorem without proof.

Theorem 3.3. [8, Theorem 2.3.3] *Let R be a semilocal ring. Then the following statements are equivalent:*

(1) *R has unit 1-stable range.*
(2) *There exists $\alpha, \beta \in U(R)$ such that $\alpha + \beta = 1$.*
(3) *R has no homomorphic image $\mathbb{Z}_2$.*

From Theorem 3.3 and the definition of rings satisfying almost unit 1-stable range [4], we can conclude the following Corollary:

Corollary 3.1. *Let R be a semilocal ring of dimension zero. Then R satisfies unit 1-stable range if and only if R is almost unit 1-stable range.*

Proof. Clearly, unit 1-stable range condition implies almost unit 1-stable range condition. Conversely, let R satisfy almost unit 1-stable range. Suppose that for some maximal ideal m_i of R we have $R/m_i \cong \mathbb{Z}_2$. Then $R/J(R)$ does not satisfy almost unit 1-stable range. This implies that R does not satisfy almost unit 1-stable range. Hence $R/m_i \ncong \mathbb{Z}_2$ for all i. □

Remark 3. *For any ring R with $R = Z(R) \cup U(R)$, the almost unit 1-stable range condition is same as the unit 1-stable range condition. All finite rings have the property $R = Z(R) \cup U(R)$, therefore almost unit 1-stable range and unit 1-stable range are equivalent for finite rings.*

In the next theorem, we characterize the local rings that satisfy unit 1-stable range condition. This can be concluded from Theorem 3.3. Here we present an elementary proof.

Theorem 3.4. *If* (R,m) *is a local ring. Then* R *satisfies unit 1-stable range iff* $R/m \ncong \mathbb{Z}_2$.

Proof. Let $\overline{a}\overline{R} + \overline{b}\overline{R} = \overline{R}$ where $\overline{R} = R/m$. This gives $ac + bd = 1 + x$ where $x \in m$. Then by [6, Theorem 2.2], there exist $u, v \in U(R)$ such that $au + bv = 1 + x$. This implies $\overline{a} + \overline{b}\overline{v}\overline{u^{-1}} = \overline{u^{-1}}$ in $\overline{R}$. Thus $\overline{R}$ satisfies unit 1-stable range and therefore $\overline{R} \ncong \mathbb{Z}_2$. Conversely, let $aR + bR = R$ where $a, b \in R$. There are some cases:
Case 1: If $a \notin m$ and $b \in m$, then $a + bu \in U(R)$ where $u \in U(R)$.
Case 2: If $a \in m$ and $b \notin m$, then $a + bu \in U(R)$ where $u \in U(R)$.
Case 3: If both $a, b \notin m$:
subcase 1: if $1 - a \notin m$, then take $y = b^{-1}(1 - a)$ such that $a + by = 1 \in U(R)$ and we are done in this subcase.
subcase 2: if $1 - a \in m$ then agian two subcases arise:
subsubcase 1: if $2 \notin m$, then take $y = b^{-1}(2 - a) \in U(R)$ such that $a + by = 2 \in U(R)$.
subsubcase 2: if $2 \in m$ and since $R/m \ncong \mathbb{Z}_2$, $c \notin m$ such that $\overline{c} \neq \overline{0}, \overline{1}$, then take $y = b^{-1}(1 - c)$ such that $a + by = a + 1 - c \in U(R)$ as $a + 1 = 2 - (1 - a) \in m$. Hence R satisfies unit 1-stable range. ☐

One can not let down the condition of dimension zero in the Corollary 3.1. An example for this is given below:

Example 3.1. *Let* $R = \mathbb{Z}_2[[x]]$. *Then* R *is a local domain of dimesion one with maximal ideal* $m = (x)$. *Clearly,* $R/m \cong \mathbb{Z}_2$. *Thus by Theorem 3.4,* R *does not satisfy unit 1-stable range condition while* [4, Corollary 3.5] *shows that* R *satisfies almost unit 1-stable range condition.*

The following example shows that the unit 1-stable range is not a local property.

Example 3.2. Take $R = \mathbb{Z}_3[x]$. Then R does not satisfy unit 1-stable range because for $xR + 2R = R$, we cannot find $g \in U(R)$ such that $x + 2g \in U(R)$. But localization of R at each prime ideal $(p(x))$ satisfies unit 1-stable range. For, assume that $R_{(p(x))}/(p(x))_{(p(x))} \cong \mathbb{Z}_2$. This implies there exists ϕ from $R_{(p(x))}$ onto $\mathbb{Z}_2$ such that $ker\phi = (p(x))_{(p(x))}$. This gives $\phi(f(x)/g(x)) = 0$ when $f(x) \in (p(x))$ and

$\phi(f(x)/g(x)) = 1$ when $f(x) \notin (p(x))$. Now, consider $(f_1(x)p(x){+}1)/g(x), (f_2(x)p(x){+}1)/g(x) \in R_{(p(x))}$. Then $\phi(((f_1(x) + f_2(x))p(x) + 2)/g(x)) = 1$ and $\phi((f_1(x)p(x) + 1)/g(x)) + \phi((f_1(x)p(x) + 1)/g(x)) = 1 + 1 = 0$, which is a contradiction. Hence $R_{(p(x))}/(p(x))_{(p(x))} \ncong \mathbb{Z}_2$.

The ring of integers does not satisfy weakly unit 1-stable range while localization of this ring at each prime ideal of it satisfies weakly unit 1-stable range. This shows weakly unit 1-stable range is not a local property.

Theorem 3.5. *Let R be a principal ideal ring(PIR). Then R satisfies unit 1-stable range if and only if $\frac{R}{P}$ satisfies unit 1-stable range for all prime ideals P belonging to zero ideal.*

Proof. Since R is a PIR so $R = \prod_{i=1}^{n} R/q_i$ where R/q_i either a PID or a special PIR and q_i are primary ideals belonging to zero ideal [15, Theorem 33, pg 245]. Then the proof is trivial. □

The similar results is true for weakly unit 1-stable range.

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References

[1] D. D. Anderson, V. P. Camillo, Commutative rings whose elements are a sum of a unit and idempotent, *Comm. Algebra* **30**(2002), 3327–3336

[2] P. Ara, Strongly π-regular rings have stable range one, *Proc. Amer. Math. Soc.* **124**(1996), 293–3298

[3] N. Arora, S. Kundu, Semiclean rings and rings of continuous function, *J. Commut. Algebra* **6**(2014), 1–6

[4] N. Bisht, Stable range one on non-zero divisor, *Proc. Indian Acad. Sci. (Math. Sci.)* **129**(2019), pp 13

[5] H. Chen, Rings with stable range conditions, *Comm. Algebra* **26**(1998),3653–3668

[6] H. Chen, On stable range conditions, *Comm. Algebra* **28**(2000), 3913–3914

[7] H. Chen, Units, idempotents and stable range conditions, *Comm. Algebra* **29**(2001), 703–717

[8] H. Chen, *R*ings related to stable range condition, Series in Algebra Vol 11, World Scientific Publishing Co Pte Ltd (2011)

[9] H. Y. Chen, F. A. Li, Rings with many unit-regular elements, *Chinese Ann. Math. Ser. A* **46**(2000), 115–121

[10] K. R. Goodearl, P. Menal, Stable range one for Rings with Many Units, *J. Pure Appl. Alg.* **54**(1988), 261–287

[11] Jr. E. Graham Evans, Krull-Schmidt and cancellation over local rings, *Pacific J. Math.* **46**(1973), 115–121

[12] Jr. R. B. Warfield, Cancellation of modules and groups and stable range of endomorphism rings, Pacific J. Math. **91**(1980), 457–485

[13] L. N. Vaserstein, Bass's first stable range condition, *J. Pure Appl. Alg.* **34**(1984), 319–330

[14] G. Xiao, W. Tong, n-clean rings and weakly unit stable range rings, *Comm. Algebra* **33**(2005), 1501–1517

[15] O. Zariski, P. Samuel, *C*ommutative algebra Vol 1 Graduate Texts in Mathematics. Springer-Verlag New York-Heidelberg-Berlin(1975)

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