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NOTES ON GENERALIZATIONS OF HOPFIAN AND CO-HOPFIAN MODULES

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Abstract. A module M is called semi co-Hopfian (resp. semi Hopfian) if any injective (resp. surjective) endomorphism of M has a direct summand image (resp. kernel). We show that if M is semi Hopfian strongly co-Hopfian or semi co-Hopfian strongly Hopfian module, then $End_R(M)$ is strongly π-regular ring. As a consequence we obtain a version of Hopkins-Levitzki Theorem extend to semi Hopfian module and to semi co-Hopfian module. The semi Hopficity and semi co-Hopficity of modules over truncated polynomial rings are considered.

1. Introduction

Throughout this paper, R denotes an associative ring with identity and modules M are unitary left R-modules. The study of modules by properties of their endomorphisms has long been of interest. In 1986, Hiremath, [9], introduced the notion of Hopfian modules and rings. A bit later, in 1992, Varadarajan, [14], introduced the notion of co-Hopfian modules and rings. In 2001, Haghany and Vedadi, [8], and in 2002, Ghorbani and Haghany, [7], respectively, introduced and investigated the weakly co-Hopfian (respectively generalized Hopfian) modules (i.e., every injective endomorphism has an essential image) (respectively every surjective endomorphism has a small kernel). In 2007, Hmaimou, Kaidi and Sánchez Campos, [10], introduced and investigated the Generalized Fitting modules. In 2008, Aydogdu and Ozcan, [3], introduced the semi co-Hopfian and semi Hopfian modules. A module M is called

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semi co-Hopfian (resp. semi Hopfian) if any injective (resp. surjective) endomorphism of M has a direct summand image (resp. kernel). Such modules and others generalizations were introduced and studied by many authors, (for more information about this and others related topics, see, for instance, [6], [7], [9], [10], [13], [14], [15]).In Section 2, We show that if M is semi Hopfian strongly co-Hopfian or semi co-Hopfian strongly Hopfian module, then $End_R(M)$ is strongly π-regular (Theorem 2.4). As a consequence we obtain that if M is semi Hopfian strongly co-Hopfian or semi co-Hopfian strongly Hopfian module, then it is Fitting module (Corollary 2.5). And also we obtain a version of Hopkins-Levitzki Theorem extend to semi Hopfian module and to semi co-Hopfian module i.e., for a semi Hopfian (respectively semi co-Hopfian) module, M, if M is strongly co-Hopfian (respectively strongly Hopfian) then M is strongly Hopfian (respectively strongly co-Hopfian) (Corollary 2.6). It is clear that every Hopfian module is semi Hopfian, but the converse is not true (see Example 2.7). Then we prove that if M is semi Hopfian and co-Hopfian, then M is Hopfian, and if M is semi co-Hopfian and Hopfian, then M is co-Hopfian (Theorem 2.9). Varadarajan [14] showed that the left R-module M is Hopfian if and only if the left R[x]-module M[x] is Hopfian if and only if the left $R[x]/(x^{n+1})$-module $M[x]/(x^{n+1})$ is Hopfian, where n is a non-negative integer and x is a commuting indeterminate over R. However, for any R-module $M \neq 0$, the $R[x]$-module $M[x]$ is never co-Hopfian. In fact, the map "multiplication by x" is injective and non surjective. We are motivated to prove that, if $M[x]/(x^{n+1})$ is semi Hopfian (respectively, semi co-Hopfian) $R[x]/(x^{n+1})$-module then M is semi Hopfian (respectively, semi co-Hopfian) R-module, (Theorem 2.15) and (Theorem 2.16).

Also we prove that if M is semi Hopfian (respectively semi co-Hopfian) module, then Hopfian and generalized Hopfian (respectively co-Hopfian and weakly co-Hopfian) are coincide, (Proposition 2.20) (respectively (Proposition 2.22)). Let R be a ring and M an R-module. We recall the following definitions and facts:

Definition 1.1. (1) M is called Hopfian if every surjective endomorphism of M is an automorphism. [9]

(2) M is called co-Hopfian if every injective endomorphism of M is an automorphism. [14]

Definition 1.2. [3] A module M is called semi Hopfian if any surjective endomorphism of M has a direct summand kernel, i.e., any surjective endomorphism of M splits.

Definition 1.3. [3] A module M is called semi co-Hopfian if any injective endomorphism of M has a direct summand image, i.e., any injective endomorphism of M splits.

Definition 1.4. [2] An R-module M is said to be Fitting module if for any endomorphism f of M, there exists a positive integer $n \geq 1$ such: $M = Kerf^n \oplus Imf^n$.

Definition 1.5. A ring R is called Dedekind finite ring if $ba = 1$ whenever $ab = 1$. Equivalently, R is Dedekind finite ring if whenever a is left or right invertible, then a is invertible.

Clearly $ab = 1$ implies that ba is non-zero idempotent, so R is a Dedekind finite ring if and only if R is not isomorphic to any proper left or right ideal direct summand.

Definition 1.6. Let R be any unital ring and M be a unital R-module. M is called Dedekind finite module if its ring of endomorphisms $End_R(M)$ is a Dedekind finite ring.

Consequently, M is Dedekind finite module if and only if M is not isomorphic to any proper direct summand of itself.

Remark 1.7. *The following facts are well known:*

1. *Every Noetherian R-module M (i.e., M has ACC on submodules), is Hopfian* [1].
2. *Every Artinian R-module M (i.e., M has DCC on submodules), is co-Hopfian)* [1].
3. *The additive group $\mathbb{Q}$ of rational numbers is a non-Noetherian non-Artinian $\mathbb{Z}$-module, which is Hopfian and co-Hopfian* [10].
4. *A ring R is left Hopfian if and only if R is Dedekind finite, if and only if R is right Hopfian.*
5. *Every commutative ring is Hopfian.*

(6) Every Artinian and Noetherian R-module is Fitting. [1]

(7) Every Fitting R-module is Hopfian and co-Hopfian. [1]

(8) If R is a commutative ring, then every finitely generated R-module is Hopfian [15, Proposition 1.2].

2. Main Results

Definition 2.1. [10] Let M be an R-module.

(1) M is called strongly Hopfian if for every endomorphism f of M the ascending chain $Kerf \subseteq Kerf^2 \subseteq ... \subseteq Kerf^n \subseteq ...$ stabilizes.

(2) M is called strongly co-Hopfian if for every endomorphism f of M the descending chain $Imf \supseteq Imf^2 \supseteq ... \supseteq Imf^n \supseteq ...$ stabilizes.

Remark 2.2. *The left and right strongly π-regular rings have been introduced by Kaplansky* [11], *Azumaya proved in 1954 that a ring R is strongly π-regular if for every $a \in R$ there exist $m \in \mathbb{N}$ and $c \in R$ satisfying $ac = ca$ and $a^m = ca^{m+1}$* [4]. *Dischinger proved in 1976 that the strongly π-regularity is left-right symmetric* [5].

Example 2.3. *By* [10, Remark 2.16(3)], *the ring $R = \prod_{n\geq 1} \mathbb{Z}/\nvDash^{\ltimes}\mathbb{Z}$ is Hopfian (every commutative ring is Hopfian) but not strongly Hopfian. Since every Hopfian ring is semi Hopfian, the ring $\prod_{n\geq 1} \mathbb{Z}/\nvDash^{\ltimes}\mathbb{Z}$ is semi Hopfian but not strongly Hopfian.*

Theorem 2.4. *Let M be an R-module. Then we have:*

(1) If M is semi Hopfian strongly co-Hopfian, then $End_R(M)$ is strongly π-regular.

(2) If M is semi co-Hopfian strongly Hopfian, then $End_R(M)$ is strongly π-regular.

Proof. (1) Assume that M is a strongly co-Hopfian and semi Hopfian module and let f be an endomorphism of M. By [10, Proposition 2.6], *there exists an integer $n \geq 1$ such that $Imf^n = Imf^{n+1}$.*

Let $g : M \to Imf^n = Imf^{n+1}$, $g(x) = f^{n+1}(x)$, and $h : M \to Imf^n = Imf^{n+1}$, $h(x) = f^n(x)$, for every $x \in M$.

Since $g(x) = f^{n+1}(x)$ and $h(x) = f^n(x)$ are surjective, there exists α, $\beta \in End_R(M)$, such that: $f^{n+1}\alpha = 1$ and $f^n\beta = 1$, by the definition of semi Hopfian module. Then

β is an injective endomorphism and $f^{n+1}\alpha = f^n\beta$. Now as M is strongly co-Hopfian then it is co-Hopfian, so β is an automorphism and $f^{n+1}\alpha\beta^{-1} = f^n$. Therefore $f^{n+1}\gamma = f^n$ where $\gamma = \alpha\beta^{-1} \in End_R(M)$.

And finally by Dischinger theorem, [5, Remark 2.2(7)], *$End_R(M)$ is strongly π-regular.*

(2) Assume that M is a strongly Hopfian and semi co-Hopfian module and let f be an endomorphism of M. By [10, Proposition 2.5], *there exists an integer $n \geq 1$ such that $Ker f^n = Ker f^{n+1}$.*

Let $g : Im f^n \to M$, $g(f^n(x)) = f^{n+1}(x)$, and $h : Im f^n \to M$, the natural inclusion, $h(f^n(x)) = f^n(x)$, for every $x \in M$. If $f^{n+1}(x) = 0$, we have $x \in Ker f^{n+1} = Ker f^n$ then $f^n(x) = 0$, so g is injective.

Since $g(f^n(x)) = f^{n+1}(x)$ and $h(f^n(x)) = f^n(x)$ are injective, there exists α, β $\in End_R(M)$, such that: $\alpha f^{n+1} = 1$ and $\beta f^n = 1$, by the definition of semi co-Hopfian module. Then β is a surjective endomorphism and $\alpha f^{n+1} = \beta f^n$. Now as M is strongly Hopfian then it is Hopfian, so β is an automorphism and $\beta^{-1}\alpha f^{n+1} = f^n$. Therefore $\gamma f^{n+1} = f^n$ where $\gamma = \beta^{-1}\alpha \in End_R(M)$.

And finally by Dischinger theorem, [5, Remark 2.2(7)], *$End_R(M)$ is strongly π-regular.* □

Corollary 2.5. *Every semi Hopfian strongly co-Hopfian or semi co-Hopfian strongly Hopfian module is a Fitting module.*

Proof. This follows from Theorem 2.4 and [2, Proposition 2.3]. □

Now we obtain a version of Hopkins-Levitzki Theorem extend to semi Hopfian module and to semi co-Hopfian module

Corollary 2.6. *Let M be an R-module. Then we have:*

1. *(1) If M is semi Hopfian strongly co-Hopfian, then M is strongly Hopfian.*
2. *(2) If M is semi co-Hopfian strongly Hopfian, then M is strongly co-Hopfian.*

Proof. (1) By Theorem 2.4, $End(M)$ is strongly π-regular. Then M is a Fitting module by [2], *and finally M is strongly Hopfian by* [10, Proposition 2.7(2)].

(2) By Theorem 2.4, End(M) is strongly π-regular. Then M is a Fitting module by [2], *and finally M is strongly co-Hopfian by* [10, Proposition 2.7(2)]. □

It is clear that every Hopfian module is semi Hopfian, but the converse is not true.

Example 2.7. *By* [9, Theorem 16(ii)], *a vector space V over a field F is Hopfian if and only if it is finite dimensional. Thus an infinite-dimensional vector space over a field is semi Hopfian, but it is not Hopfian.*

Proposition 2.8. *Let M be a semi Hopfian R-module. If M is indecomposable, then it is Hopfian.*

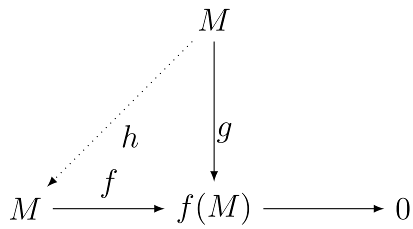
Proof. *Let $f : M \to M$ be a surjective endomorphism. Since M is semi Hopfian, $Ker f$ is a direct summand of M. Now as M is an indecomposable, then M can not be written as direct sum of its nonzero submodules. Therefore $Ker f = 0$. This shows that f is an automorphism, and hence M becomes Hopfian.* □

Theorem 2.9.
1. *Let M be a semi Hopfian R-module. If M is co-Hopfian, then it is Hopfian.*
2. *Let M be a semi co-Hopfian R-module. If M is Hopfian, then it is co-Hopfian.*

Proof. *(1) Let $f : M \to M$ be a surjective endomorphism. Since M is a semi Hopfian R-module, f splits, and hence there exists an endomorphism $g : M \to M$, such that $fg = 1$. This implies that g is an injective endomorphism. Now since M is co-Hopfian, g is an automorphism. Therefore f is an automorphism and M becomes a Hopfian R-module.*

(2) Let $f : M \to M$ be an injective endomorphism. Since M is a semi co-Hopfian R-module, f splits, and hence there exists an endomorphism $g : M \to M$, such that $gf = 1$. This implies that g is a surjective endomorphism. Now since M is Hopfian, g is an automorphism. Therefore f is an automorphism and M becomes a co-Hopfian R-module. □

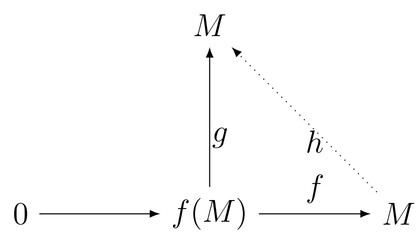
Definition 2.10. A module M is called quasi principally projective if every endomorphism f of M and every homomorphism g from M to $f(M)$, there exists an endomorphism h of M such that $fh = g$.



Hence every quasi principally projective module is semi Hopfian, by [12, Proposition 3.2]. So we have the following corollary.

Corollary 2.11. *Let M be a quasi principally projective R-module. If M is co-Hopfian, then it is Hopfian.*

Definition 2.12. A module M is called quasi principally injective if every endomorphism f of M and every homomorphism g from $f(M)$ to M, there exists an endomorphism h of M such that $hf = g$.



Hence any quasi principally injective module is semi co-Hopfian, by [12, Proposition 3.1]. So we have the following corollary.

Corollary 2.13. *Let M be a quasi principally injective R-module. If M is Hopfian, then it is co-Hopfian.*

Now we see an analogue to Hilbert's basis Theorem for semi Hopfian and for semi co-Hopfian Module.

Let M be an R-module. We will briefly recall the definitions of the modules $M[x]$ and $M[x]/(x^{n+1})$ from [13]. The elements of $M[x]$ are formal sums of the form $a_0+a_1x+...+a_kx^k$ with k an integer greater than or equal to 0 and $a_i \in M$. We denote this sum by $\sum_{i=1}^{k} a_i x^i$ (a_0x^0 is to be understood as the element $a_0 \in M$). Addition is defined by adding the corresponding coefficients. The $R[x]$-module structure is given by

$$(\sum_{i=0}^{k} \lambda_i x^i).(\sum_{j=0}^{z} a_j x^j) = \sum_{\mu=0}^{k+z} c_\mu x^\mu,$$

where $c_\mu = \sum_{i+j=\mu} \lambda_i a_j$, for any $\lambda_i \in R$, $a_j \in M$.

Any nonzero element β of $M[x]$ can be written uniquely as $(\sum_{i=k}^{l} m_i x^i)$ with $l \geq k \geq 0$, $m_i \in M$, $m_k \neq 0$ and $m_l \neq 0$. In this case, we refer to k as the order of β, l as the degree of β, m_k as the initial coefficient of β, and m_l as the leading coefficient of β.

Let n be any non-negative integer and

$$I_{n+1} = \{0\} \cup \{\beta; 0 \neq \beta \in R[x], \text{ order of } \beta \geq n+1\}.$$

Then I_{n+1} is a two-sided ideal of $R[x]$. The quotient ring $R[x]/I_{n+1}$ will be called the truncated polynomial ring, truncated at degree $n+1$. Since R has an identity element, I_{n+1} is the ideal generated by x^{n+1}. Even when R does not have an identity element, we will "symbolically" denote the ring $R[x]/I_{n+1}$ by $R[x]/(x^{n+1})$. Any element of $R[x]/(x^{n+1})$ can be uniquely written as $(\sum_{i=0}^{n} \lambda_i x^i)$ with $\lambda_i \in R$.

Let

$$D_{n+1} = \{0\} \cup \{\beta; 0 \neq \beta \in M[x], \text{ order of } \beta \geq n+1\}.$$

Then D_{n+1} is an $R[x]$-submodule of $M[x]$. Since $I_{n+1}M[x] \subset D_{n+1}$, we see that $R[x]/(x^{n+1})$ acts on $M[x]/D_{n+1}$. We denote the module $M[x]/D_{n+1}$ by $M[x]/(x^{n+1})$. Any nonzero element β of $M[x]/D_{n+1}$ can be written uniquely as $(\sum_{i=k}^{n} m_i x^i)$ with $n \geq k \geq 0$, $m_i \in M$, $m_k \neq 0$. In this case, we refer to k as the order of β, m_k as the initial coefficient of β. The action of $R[x]/(x^{n+1})$ on $M[x]/(x^{n+1})$ is given by

$$(\sum_{i=0}^{n} \lambda_i x^i).(\sum_{j=0}^{n} a_j x^j) = \sum_{\mu=0}^{n} c_\mu X^\mu,$$

where $c_\mu = \sum_{i+j=\mu} \lambda_i a_j$, for any $\lambda_i \in R$, $a_j \in M$.

The $R[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$-module $M[x_1, ..., x_k]/(X_1^{n_1+1}, ..., x_k^{n_k+1})$ is defined similarly.

Lemma 2.14. *Let M be an R-module and N be a submodule of M. If $N[x]/(x^{n+1})$ is a direct summand of $M[x]/(x^{n+1})$, then N is a direct summand of M.*

Proof. Assume that $N[x]/(x^{n+1})$ is a direct summand of $M[x]/(x^{n+1})$, then $M[x]/(x^{n+1}) = N[x]/(x^{n+1}) \oplus L$, for some submodule L of $M[x]/(x^{n+1})$. Let L' be the submodule of M which is generated by the constant polynomials of L. Let $m \in M$. Then $m \in M[x]/(x^{n+1})$ and so $m = g(x) + h(x)$ where $g(x) \in N[x]/(x^{n+1})$ and $h(x) \in L$. Since m is a constant polynomial in $M[x]/(x^{n+1})$, we have $m = g(0) + h(0)$ where $g(0) \in N$

and $h(0) \in L'$. *Hence* $M = N + L'$. *If* $x \in N \cap L'$ *then* $x \in N[x]/(x^{n+1}) \cap L = \{0\}$, *and finally* $M = N \oplus L'$. □

Theorem 2.15. *Let* M *be an* R*-module. If* $M[x]/(x^{n+1})$ *is semi Hopfian* $R[x]/(x^{n+1})$*-module, then* M *is semi Hopfian* R*-module.*

Proof. Let $f : M \to M$ be any surjective endomorphism in R-module. Then $\alpha : M[x]/(x^{n+1}) \to M[x]/(x^{n+1})$ defined by $\alpha(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} f(a_i) x^i$ is a surjective endomorphism in $R[x]/(x^{n+1})$-module. Since $M[x]/(x^{n+1})$ is semi Hopfian $R[x]/(x^{n+1})$-module, $Ker(\alpha) = (Ker f)[x]/(x^{n+1})$ is a direct summand of $M[x]/(x^{n+1})$. Then by Lemma 2.14, $Ker f$ is a direct summand of M, and finally M is semi Hopfian. □

Theorem 2.16. *Let* M *be an* R*-module. If* $M[x]/(x^{n+1})$ *is semi co-Hopfian* $R[x]/(x^{n+1})$*-module, then* M *is semi co-Hopfian* R*-module.*

Proof. Let $f : M \to M$ be any injective endomorphism in R-module. Then $\alpha : M[x]/(x^{n+1}) \to M[x]/(x^{n+1})$ defined by $\alpha(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} f(a_i) x^i$ is an injective endomorphism in $R[x]/(x^{n+1})$-module. Since $M[x]/(x^{n+1})$ is semi co-Hopfian $R[x]/(x^{n+1})$-module, $Im(\alpha) = (Im f)[x]/(x^{n+1})$ is a direct summand of $M[x]/(x^{n+1})$. Then by Lemma 2.14, $Im f$ is a direct summand of M, and finally M is semi co-Hopfian. □

Theorem 2.17. *Let* M *be an* R*-module. If* $M[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$ *is semi Hopfian (respectively, semi co-Hopfian)* $R[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$*-module, then* M *is semi Hopfian (respectively, semi co-Hopfian)* R*-module.*

Proof. Use induction and the
$(R[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1})$-module isomorphism
$(M[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1}) \simeq M[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$
and ring isomorphism
$(R[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1}) \simeq R[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$. □

A submodule K of an R-module M is said to be small in M, written $K \ll M$, if for every submodule $L \subseteq M$ with $K + L = M$ implies $L = M$.

Definition 2.18. [7] A module M is called generalized Hopfian if every surjective endomorphism of M has a small kernel.

It is clear that every Hopfian module is generalized Hopfian by [7, corollary 1.4], but the converse is not true.

Example 2.19. *(see [7, example 1.7]). Let $G = \mathbb{Z}_{p^\infty}$. Since in G every proper subgroup is small, we see that G is a generalized Hopfian Abelian group. However G is not Hopfian since the multiplication by p induces an epimorphism of G which is not an isomorphism.*

Proposition 2.20. *Let M be a semi Hopfian module. Then the following conditions are equivalent:*

(1) M is Hopfian.
(2) M is generalized Hopfian

Proof. *(1) $\Rightarrow$ (2) Evident.*
(2) $\Rightarrow$ (1) Let $f : M \to M$ be a surjective endomorphism. Since M is semi Hopfian, f splits, and hence there exists $g : M \to M$ such that $fg = 1$. Now as M is a generalized Hopfian, then by [7, Corollary 1.4], *M is Dedekind finite and hence $gf = 1$. Therefore f is an injective endomorphism. This shows that f is an automorphism, and hence M becomes Hopfian.* □

A submodule K of an R-module M is said to be essential in M, written $K \leq^e M$, if for every submodule $L \subseteq M$ with $K \cap L = 0$ implies $L = 0$.

Definition 2.21. [8] A module M is called weakly co-Hopfian if every injective endomorphism of M has an essential image.

Proposition 2.22. *Let M be a semi co-Hopfian module. Then the following conditions are equivalent:*

(1) M is co-Hopfian.
(2) M is weakly co-Hopfian

Proof. *(1) $\Rightarrow$ (2) Evident.*
(2) $\Rightarrow$ (1) Let $f : M \to M$ be an injective endomorphism. Since M is semi co-Hopfian, f splits, and hence there exists $g : M \to M$ such that $gf = 1$. Now as M is a weakly co-Hopfian, then by [8, Proposition 1.4], *M is Dedekind finite and*

hence $fg = 1$. Therefore f is a surjective endomorphism. This shows that f is an automorphism, and hence M becomes co-Hopfian. □

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