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SOME REFINEMENTS OF NUMERICAL RADIUS INEQUALITIES FOR HILBERT SPACE OPERATORS

EBRAHIM ALIZADEH [(1)] AND ALI FAROKHINIA [(2)]

Abstract. The main goal of this paper is to obtain some refinements of numerical radius inequalities for Hilbert space operators.

1. Introduction and Preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. For $T \in \mathcal{B}(\mathcal{H})$, let $\omega(T)$ and $\|T\|$ denote the numerical radius and the usual operator norm of T, respectively. Recall that $\omega(T) = \sup_{\|x\|=1} |\langle Tx, x \rangle|$. It is well-known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the operator norm $\|\cdot\|$. In fact, for every $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|. \tag{1.1}$$

Also, it is a basic fact that $\omega(\cdot)$ satisfies the power inequality

$$\omega(T^n) \leq \omega^n(T)$$

for all $n = 1, 2, \ldots$.

In [4], Kittaneh gave the following estimate of the numerical radius which refines the second inequality in (1.1):

For every $T \in \mathcal{B}(\mathcal{H})$,

$$\omega(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}\left\|T^2\right\|^{\frac{1}{2}}. \tag{1.2}$$

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The following estimate of the numerical radius has been given in [6]:

$$\omega^2(T) \le \frac{1}{2}\|T^*T + TT^*\|. \tag{1.3}$$

The inequality (1.3) also refines the inequality (1.1). This can be seen by using the fact that

$$\|T^*T + TT^*\| \le \|T\|^2 + \|T^2\|.$$

For other properties of the numerical radius and related inequalities, the reader may consult [8, 9, 10, 11, 13]. In this article, we give several refinements of numerical radius inequalities. Our results mainly extend and improve the inequalities in [4, 12].

2. Main Results

In the sequel the following lemmas will be needed.

Lemma 2.1. [5] *Let A be an operator in $\mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors.*

(a) If $0 \le \alpha \le 1$, then $|\langle Tx, y\rangle|^2 \le \langle |T|^{2\alpha} x, x\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle$.

(b) If f, g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)\, g(t) = t, (t \ge 0)$, then $|\langle Tx, y\rangle| \le \|f(|T|)\, x\| \, \|g(|T^*|)\, y\|$.

Lemma 2.2. [5] *Let A be a positive operator in $\mathcal{B}(\mathcal{H})$ and let $x \in \mathcal{H}$ be any unit vector. Then*

(a) $\langle A^r x, x\rangle \le \langle Ax, x\rangle^r$ for $0 < r < 1$.

(b) $\langle Ax, x\rangle^r \le \langle A^r x, x\rangle$ for $r \ge 1$.

Lemma 2.3. *For $a, b \ge 0, 0 \le \alpha \le 1, r \ge 1$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have $ab \le \frac{a^p}{p} + \frac{b^q}{q} \le \left(\frac{a^{pr}}{p} + \frac{b^{qr}}{q}\right)^{\frac{1}{r}}$.*

Minculete, in [7, Theorem 2.1], obtained an improvement of the Young inequality as follows:

Lemma 2.4. *Let $a, b > 0$ and $\nu \in (0, 1)$. Then*

$$a^\nu b^{1-\nu} \left(\frac{a+b}{2\sqrt{ab}}\right)^{2\mu} \le \nu a + (1-\nu)\, b \tag{2.1}$$

where $\mu = \min\{\nu, 1-\nu\}$.

Notice that, if $0 < m \leq a, b \leq M$, *then* $\frac{m+M}{\sqrt{mM}} \leq \frac{a+b}{\sqrt{ab}}$. *Based on this fact, from the inequality* (2.1) *we have*

$$a^{\nu} b^{1-\nu} \left(\frac{m+M}{2\sqrt{mM}} \right)^{2\mu} \leq \nu a + (1-\nu) b \tag{2.2}$$

where $\mu = \min \{\nu, 1-\nu\}$.

We recall the following refinement of the inequality (1.1) obtained by Dragomir in [3]:

$$\omega^2 (A) \leq \frac{1}{2} \left(\|A\|^2 + \omega \left(A^2\right) \right). \tag{2.3}$$

In addition to this, we have the following related inequality:

Theorem 2.1. *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* f, g *be non-negative continuous functions on* $[0, \infty)$ *satisfying* $f(t) g(t) = t$, $(t \geq 0)$, *and*

$$\sqrt[r]{m} \leq f^2 (|A^2|) \leq \sqrt[r]{M}, \quad \sqrt[r]{m} \leq g^2 \left(\left|(A^2)^*\right| \right) \leq \sqrt[r]{M}.$$

Then

$$\omega^r \left(A^2\right) \leq \left(\frac{m+M}{2\sqrt{mM}} \right)^{-1} \left\| \frac{f^{2r} (|A^2|) + g^{2r} \left(\left|(A^2)^*\right| \right)}{2} \right\|. \tag{2.4}$$

Proof. Let $x \in \mathcal{H}$ be a unit vector and $r \geq 1$. We have

$$\begin{aligned}
&\left|\left\langle A^2 x, x \right\rangle\right|^r \\
&\leq \left\langle f^2 \left(\left|A^2\right|\right) x, x \right\rangle^{\frac{r}{2}} \left\langle g^2 \left(\left|\left(A^2\right)^*\right|\right) x, x \right\rangle^{\frac{r}{2}} \\
&\qquad \text{(by Lemma 2.1 (b))} \\
&\leq \left(\frac{m+M}{2\sqrt{mM}} \right)^{-1} \left(\frac{\left\langle f^2 (|A^2|) x, x \right\rangle^r + \left\langle g^2 \left(\left|(A^2)^*\right|\right) x, x \right\rangle^r}{2} \right) \\
&\qquad \text{(by (2.2))} \\
&\leq \left(\frac{m+M}{2\sqrt{mM}} \right)^{-1} \left(\frac{\left\langle f^{2r} (|A^2|) x, x \right\rangle + \left\langle g^{2r} \left(\left|(A^2)^*\right|\right) x, x \right\rangle}{2} \right) \\
&\qquad \text{(by Lemma 2.2 (b))} \\
&= \left(\frac{m+M}{2\sqrt{mM}} \right)^{-1} \left\langle \left(\frac{f^{2r} (|A^2|) + g^{2r} \left(\left|(A^2)^*\right|\right)}{2} \right) x, x \right\rangle.
\end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we deduce the desired result (2.4).

$\square$

Remark 1. *If we take* $f(t) = t^{1-\alpha}, g(t) = t^{\alpha}, 0 \le \alpha \le 1$, *in Theorem 2.1, we get*

$$\omega^r\left(A^2\right) \le \left(\frac{m+M}{2\sqrt{mM}}\right)^{-1} \left\| \left|A^2\right|^{2r(1-\alpha)} + \left|\left(A^2\right)^*\right|^{2r\alpha} \right\|$$

whenever

$$\sqrt[r]{m} \le \left|A^2\right|^{2(1-\alpha)} \le \sqrt[r]{M}, \quad \sqrt[r]{m} \le \left|\left(A^2\right)^*\right|^{2\alpha} \le \sqrt[r]{M}.$$

Remark 2. *It follows from* [12, Proposition 2.5] *that*

$$\omega\left(A^2\right) \le \frac{1}{2}\left(\|A\|^2 + \left\| \frac{f^2\left(\left|A^2\right|\right) + g^2\left(\left|\left(A^2\right)^*\right|\right)}{2} \right\|\right). \tag{2.5}$$

Letting $r = 1$ *in* (2.4). *Therefore, from the inequality* (2.3),

$$\omega^2(A) \le \frac{1}{2}\left(\|A\|^2 + \left(\frac{m+M}{2\sqrt{mM}}\right)^{-1} \left\| \frac{f^2\left(\left|A^2\right|\right) + g^2\left(\left|\left(A^2\right)^*\right|\right)}{2} \right\|\right).$$

It is worth to mention that the above inequality is sharper than the inequality (2.5), *since* $\left(\frac{m+M}{2\sqrt{mM}}\right)^{-1} \le 1$.

The following result for several operators holds:

Theorem 2.2. *Let* $A, B, X \in \mathcal{B}(\mathcal{H})$ *be such that* A, B *are positive, and*

$$m \le A^r \le M, \quad m \le B^r \le M.$$

Then

$$\omega^r\left(A^{\alpha} X B^{1-\alpha}\right) \le \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu} \|X\|^r \left\|\alpha A^r + (1-\alpha) B^r\right\| \tag{2.6}$$

for all $0 \le \alpha \le 1, r \ge 2$ *and* $\mu = \min\{\alpha, 1-\alpha\}$.

Proof. Let $x \in \mathcal{H}$ be a unit vector. We have

$$\begin{aligned} \left|\left\langle A^{\alpha} X B^{1-\alpha} x, x\right\rangle\right|^{r} &= \left|\left\langle X B^{1-\alpha} x, A^{\alpha} x\right\rangle\right|^{r} \\ &\leq \|X\|^{r}\left\|B^{1-\alpha} x\right\|^{r}\left\|A^{\alpha} x\right\|^{r} \\ &= \|X\|^{r}\left\langle B^{2(1-\alpha)} x, x\right\rangle^{\frac{r}{2}}\left\langle A^{2\alpha} x, x\right\rangle^{\frac{r}{2}} \\ &\leq \|X\|^{r}\left\langle A^{r} x, x\right\rangle^{\alpha}\left\langle B^{r} x, x\right\rangle^{1-\alpha} \quad (2.7) \\ &\quad \text{(by Lemma 2.2 (a))} \\ &\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|X\|^{r}\left\langle\left(\alpha A^{r}+(1-\alpha) B^{r}\right) x, x\right\rangle \\ &\quad \text{(by (2.2))} \end{aligned}$$

where $\mu=\min\{\alpha, 1-\alpha\}$.

Now taking the supremum over $x \in \mathcal{H}$ with $\|x\|=1$ in the above inequality produces (2.6). □

As a consequence of the above theorem, we have:

Corollary 2.1. *Suppose that the assumptions of Theorem 2.2 are satisfied. Then*

$$\omega^{r}\left(A^{\frac{1}{2}} X B^{\frac{1}{2}}\right) \leq\left(\frac{m+M}{2\sqrt{mM}}\right)^{-1}\|X\|^{r}\left\|\frac{A^{r}+B^{r}}{2}\right\|. \tag{2.8}$$

The following result may be stated as well.

Theorem 2.3. *Let all the assumptions of Theorem 2.2 be valid. Then*

$$\omega^{r}\left(\frac{A^{\alpha} X B^{1-\alpha}+A^{1-\alpha} X B^{\alpha}}{2}\right) \leq\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|X\|^{r}\left\|\frac{A^{r}+B^{r}}{2}\right\|.$$

Proof. By the inequality (2.7), we have

$$\left|\left\langle A^{\alpha} X B^{1-\alpha} x, x\right\rangle\right|^{r} \leq\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu}\|X\|^{r}\left\langle\left(\alpha|A|^{r}+(1-\alpha)|B|^{r}\right) x, x\right\rangle$$

for all $0 \leq \alpha \leq 1, r \geq 2$ and $\mu=\min\{\alpha, 1-\alpha\}$. Hence

$$\left|\left\langle\frac{A^{\alpha} X B^{1-\alpha}+A^{1-\alpha} X B^{\alpha}}{2} x, x\right\rangle\right|^{r} \tag{2.9}$$

$$\leq\left(\frac{\left|\left\langle A^{\alpha} X B^{1-\alpha} x, x\right\rangle\right|+\left|\left\langle A^{1-\alpha} X B^{\alpha} x, x\right\rangle\right|}{2}\right)^{r} \tag{2.10}$$

$$\leq \frac{\left|\left\langle A^{\alpha} X B^{1-\alpha} x, x\right\rangle\right|^{r}+\left|\left\langle A^{1-\alpha} X B^{\alpha} x, x\right\rangle\right|^{r}}{2} \tag{2.11}$$

(2.12)

$$\leq \left(\tfrac{m+M}{2\sqrt{mM}}\right)^{-2\mu} \tfrac{\|X\|^r}{2}$$

$$\times \left(\langle(\alpha A^r + (1-\alpha) B^r) x, x\rangle + \langle((1-\alpha) A^r + \alpha B^r) x, x\rangle\right)$$

$$= \left(\tfrac{m+M}{2\sqrt{mM}}\right)^{-2\mu} \|X\|^r \left\langle\left(\tfrac{A^r+B^r}{2}\right) x, x\right\rangle.$$

Therefore,

$$\omega^r\left(\frac{A^\alpha X B^{1-\alpha} + A^{1-\alpha} X B^\alpha}{2}\right) \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\mu} \|X\|^r \left\|\frac{A^r + B^r}{2}\right\|.$$

This completes the proof. □

The following result concerning the sums of two operators may be stated as well:

Theorem 2.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\omega^r(A+B) \leq \sqrt{\left\|\frac{1}{p}|A+B|^{2pr\alpha} + \frac{1}{q}|(A+B)^*|^{2qr(1-\alpha)}\right\|} \tag{2.13}$$

for $0 \leq \alpha \leq 1, r \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. For any unit vector $x \in \mathcal{H}$ we have

$$|\langle(A+B) x, x\rangle|^{2r}$$

$$\leq \langle|A+B|^{2\alpha} x, x\rangle^r \left\langle|(A+B)^*|^{2(1-\alpha)} x, x\right\rangle^r$$

(by Lemma 2.1 (a))

$$\leq \langle|A+B|^{2r\alpha} x, x\rangle \left\langle|(A+B)^*|^{2r(1-\alpha)} x, x\right\rangle$$

(by Lemma 2.2 (b))

$$\leq \frac{1}{p}\langle|A+B|^{2r\alpha} x, x\rangle^p + \frac{1}{q}\left\langle|(A+B)^*|^{2r(1-\alpha)} x, x\right\rangle^q$$

(by Lemma 2.3)

$$\leq \frac{1}{p}\langle|A+B|^{2pr\alpha} x, x\rangle + \frac{1}{q}\left\langle|(A+B)^*|^{2qr(1-\alpha)} x, x\right\rangle$$

(by Lemma 2.2 (b))

$$= \left\langle\frac{1}{p}|A+B|^{2pr\alpha} + \frac{1}{q}|(A+B)^*|^{2qr(1-\alpha)} x, x\right\rangle.$$

Now taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ in the above inequality produces (2.13). □

Remark 3. *If we take* $B = A$, $p = q = 2$, $r = 1$, *and* $\alpha = \frac{1}{2}$ *in Theorem 2.4, we get*

$$\omega(A) \leq \frac{1}{2}\sqrt{\left\|2\left(|A|^2 + |A^*|^2\right)\right\|},$$

which is equivalent to the following inequality

$$\omega^2(A) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|.$$

In fact, our inequality (2.13)*, can be considered as an extension of the inequality* (1.3)*.*

Remark 4. *Let* $A, B \in \mathcal{B}(\mathcal{H})$*, such that*

$$\sqrt[r]{m} \leq |A+B|^{2\alpha} \leq \sqrt[r]{M}, \quad \sqrt[r]{m} \leq |(A+B)^*|^{(2(1-\alpha))} \leq \sqrt[r]{M}.$$

Then by using the inequality (2.2)*, we deduce*

$$\omega^r(A+B) \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-1}\left\|\frac{|A+B|^{2r\alpha} + |(A+B)^*|^{2r(1-\alpha)}}{2}\right\|.$$

In particular,

$$\omega^r(A+B) \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-1}\left\|\frac{|A+B|^r + |(A+B)^*|^r}{2}\right\|.$$

The following result is of interest in itself.

Theorem 2.5. *Let* $A \in \mathcal{B}(\mathcal{H})$*. Then for any* $r \geq 1$*,*

$$w^{2r}(A) \leq \frac{1}{4}\left(\|A^*A + AA^*\|^r + \|A^*A - AA^*\|^r\right) + \frac{1}{2}w^r\left(A^2\right).$$

Proof. The celebrated Boas–Bellman inequality asserts that

$$\sum_{i=1}^{n}|\langle a, b_i\rangle|^2 \leq \|a\|^2\left(\max_{1\leq i\leq n}\|b_i\|^2 + \left(\sum_{1\leq i\neq j\leq n}|\langle b_i, b_j\rangle|^2\right)^{\frac{1}{2}}\right)$$

for any $a \in \mathcal{H}$ (see [1, 2]).

Evidently, the case $n = 2$ in the above reduces to

$$|\langle z, x\rangle|^2 + |\langle z, y\rangle|^2 \leq \|z\|^2\left(\max\left(\|x\|^2, \|y\|^2\right) + |\langle x, y\rangle|\right).$$

Choosing $x = Ax$, $y = A^*x$, and $z = x$ with $\|x\| = 1$, we infer that

$$\begin{aligned}
&|\langle x, Ax\rangle|^2 + |\langle x, A^*x\rangle|^2 \\
&\leq \max\left(\|Ax\|^2, \|A^*x\|^2\right) + |\langle Ax, A^*x\rangle| \qquad (2.14)\\
&= \frac{1}{2}\left(|\langle A^*A + AA^*x, x\rangle| + |\langle A^*A - AA^*x, x\rangle|\right) + \left|\left\langle A^2x, x\right\rangle\right|,
\end{aligned}$$

due to $\max(a,b) = \frac{|a+b|+|a-b|}{2}$.

Applying the arithmetic–geometric mean inequality to the left hand side of the above inequality, to get

$$\begin{aligned}
&|\langle A^*x, x\rangle|\,|\langle Ax, x\rangle| \\
&\leq \frac{1}{4}\left(|\langle A^*A + AA^*x, x\rangle| + |\langle A^*A - AA^*x, x\rangle|\right) + \frac{1}{2}\left|\left\langle A^2x, x\right\rangle\right|.
\end{aligned}$$

It follows from the classical Jensen inequality that

$$\begin{aligned}
&|\langle A^*x, x\rangle|\,|\langle Ax, x\rangle| \\
&\leq \left[\frac{1}{2}\left(\frac{|\langle A^*A + AA^*x, x\rangle| + |\langle A^*A - AA^*x, x\rangle|}{2}\right)^r + \frac{1}{2}\left|\left\langle A^2x, x\right\rangle\right|^r\right]^{\frac{1}{r}}
\end{aligned}$$

for any $r \geq 1$.

Therefore,

$$\begin{aligned}
&|\langle A^*x, x\rangle|^r|\langle Ax, x\rangle|^r \\
&\leq \frac{1}{2}\left(\frac{|\langle A^*A + AA^*x, x\rangle| + |\langle A^*A - AA^*x, x\rangle|}{2}\right)^r + \frac{1}{2}\left|\left\langle A^2x, x\right\rangle\right|^r \\
&\leq \frac{1}{4}\left(|\langle A^*A + AA^*x, x\rangle|^r + |\langle A^*A - AA^*x, x\rangle|^r\right) + \frac{1}{2}\left|\left\langle A^2x, x\right\rangle\right|^r.
\end{aligned}$$

Now, by taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, the desired inequality is obtained. □

Corollary 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$w^2(A) - \frac{1}{2}w\left(A^2\right) \leq \frac{1}{4}\left(\left\||A|^2 + |A^*|^2\right\| + \left\||A|^2 - |A^*|^2\right\|\right).$$

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References

[1] R. Bellman, *Almost orthogonal series*, Bull. Amer. Math. Soc. **50** (1944), 517-519.

[2] R. P. Boas, *A general moment problem*, Amer. J. Math. **63** (1941), 361-370.

[3] S. S. Dragomir, *Some inequalities for the norm and the numerical radius of linear operators in Hilbert Spaces*, Tamkang J. Math. **39**(1) (2008), 1-7.

[4] F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. **158** (2003), 11-17.

[5] F. Kittaneh, *Notes on some inequalities for Hilbert Space operators*, Publ. Res. Inst. Math. Sci. **24**(2) (1988), 283–293.

[6] F. Kittaneh, *Numerical radius inequalities for Hilbert space operators*, Studia Math. **168**(1) (2005), 73-80.

[7] N. Minculete. *A result about Young inequality and several applications*. Sci. Magna. **7** (2011), 61-68.

[8] H. R. Moradi, M. Sababheh, *More accurate numerical radius inequalities (II)*, Linear Multilinear Algebra. https://doi.org/10.1080/03081087.2019.1703886

[9] H. R. Moradi, M. E. Omidvar, S. S. Dragomir, M. S. Khan, *Sesquilinear version of numerical range and numerical radius*, Acta Univ. Sapientiae, Math. **9**(2) (2017), 324-335.

[10] M. E. Omidvar, H. R. Moradi, K. Shebrawi, *Sharpening some classical numerical radius inequalities*, Oper. Matrices. **12**(2) (2018), 407-416.

[11] M. Sababheh, H. R. Moradi, *More accurate numerical radius inequalities (I)*, Linear Multilinear Algebra. https://doi.org/10.1080/03081087.2019.1651815

[12] M. Sattari, M. S. Moslehian, T. Yamazaki, *Some generalized numerical radius inequalities for Hilbert space operators*. Linear Algebra Appl. **470** (2015), 216-227.

[13] S. Tafazoli, H. R. Moradi, S. Furuichi and P. Harikrishnan, *Further inequalities for the numerical radius of Hilbert space operators*. J. Math. Inequal., **13**(4) (2019), 955-967.

(1,2) Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, Iran

Email address: (1) alizadeh.ebrahim91@yahoo.com

Email address: (2) farokhinia@iaushiraz.ac.ir