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HÖLDER'S INEQUALITIES FOR A CLASS OF ANALYTIC FUNCTIONS CONNECTED WITH $q-$ CONFLUENT HYPERGEOMETRIC DISTRIBUTION

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Abstract. In this paper, we introduce new a class of analytic functions connected with $q-$confluent hypergeometric distribution. The results on modified Hadamard product, Hölder's inequalities some interesting convolution properties, closure properties under integral transforms, integral means and partial sums, are considered for functions belonging to these classes.

1. Introduction

In [30] Srivastava presented and motivated about brief expository overview of the classical q-analysis versus the so-called (p, q)-analysis with an obviously redundant additional parameter p. We also briefly consider several other families of such extensively and widely-investigated linear convolution operators as (for example) the Dziok–Srivastava, Srivastava–Wright and Srivastava–Attiya linear convolution operators, together with their extended and generalized versions. The theory of (p, q)-analysis has important role in many areas of mathematics and physics. Our usages here of the q-calculus and the fractional $q-$calculus in geometric function theory of complex analysis are believed to encourage and motivate significant further developments on these and other related topics (see Srivastava and Karlsson [31, pp. 350–351], also [1, 9, 10, 11, 33, 12, 28, 29]).

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Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent and normalized by $f(0) = 0 = f'(0) - 1$. We denote by $\mathcal{S}^*(\delta)$ and $\mathcal{C}(\delta)$ the subclasses of S consisting of all functions which are, respectively, starlike and convex of order δ $(0 \leq \delta < 1)$. Thus,

$$\mathcal{S}^*(\delta) = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \delta \quad (0 \leq \delta < 1;\ z \in \Delta) \right\}, \tag{1.2}$$

and

$$\mathcal{C}(\delta) = \left\{ f \in\ \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta \quad (0 \leq \delta < 1;\ z \in \Delta) \right\}. \tag{1.3}$$

The classes $\mathcal{S}^*(\delta)$ and $\mathcal{C}(\delta)$ were introduced by Robertson [22]. From (1.2) and (1.3) it follows that

$$f(z) \in \mathcal{C}(\delta) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\delta). \tag{1.4}$$

We note that:

$$\mathcal{S}^*(0) = \mathcal{S}^*,\ \mathcal{C}(0) = \mathcal{C}.$$

Definition 1.1. For $f, g \in \mathcal{A}$, we say that f *is subordinate to* g, written $f(z) \prec g(z)$, if there exists a *Schwar'z function* w, which is analytic in Δ, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$, such that $f(z) = g(w(z))$, $z \in \Delta$. Furthermore, if the function g is univalent in Δ, then we have the following equivalence (see [3, 15]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

Also denote by $\mathcal{T}$ the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ a_k \geq 0; \quad z \in \Delta. \tag{1.5}$$

Recently, Nishiwaki et al. [18] have studied some results of Hölder-type inequalities for a subclass of uniformly starlike functions. Now, we recall the generalization of the convolution due to Choi et al. [4]. For functions $f_i(z) \in \mathcal{A}$ are given by

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k \left(a_{k,i} \geq 0;\ i = 1, 2, ..., n \right), \tag{1.6}$$

we define

$$\mathcal{G}_n(z) = z - \sum_{k=2}^{\infty} \left(\prod_{i=1}^{n} a_{k,i} \right) z^k, \tag{1.7}$$

and

$$\mathcal{H}_n(z) = z - \sum_{k=2}^{\infty} \left(\prod_{i=1}^{n} (a_{k,i})^{p_i} \right) z^k \quad (p_i > 0), \tag{1.8}$$

where $\mathcal{G}_n(z)$ denotes the modified Hadamard product of $f_i(z)$ $(i = 1, 2, ..., n)$ are given by (1.6). Therefore, $\mathcal{H}_n(z)$ are the generalization modified Hadamard product. We note that:

(i) For $n = 2$, then $\mathcal{G}_2(z) = (f_1 * f_2)(z)$;

(ii) For $p_i = 1$, we have $\mathcal{H}_n(z) = \mathcal{G}_n(z)$.

Further for functions $f_i(z)$ $(i = 1, 2, ..., n)$ are given by (1.6), the familiar Hölder inequality assumes the following form (see [16, 17, 34])

$$\sum_{k=2}^{\infty} \left(\prod_{i=1}^{n} a_{k,i} \right) \leq \prod_{i=1}^{n} \left(\sum_{k=2}^{\infty} (a_{k,i})^{p_i} \right)^{\frac{1}{p_i}} \quad \left(p_i \geq 1;\ i = 1, 2, ..., n;\ \sum_{i=1}^{n} \frac{1}{p_i} \geq 1 \right). \tag{1.9}$$

The confluent hypergeometric function of the first kind is given by the power series

$$\begin{aligned} F(b; c; z) &= 1 + \frac{b}{c} z + \frac{b\ (b+1)}{c\ (c+1)} \frac{z^2}{2!} + ... \\ &= \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k\ (1)_k} z^k, \quad \left(b \in \mathbb{C},\ c \in \mathbb{C} \setminus \{0, -1, -2, ...\} \right), \end{aligned}$$

where $(b)_k$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} 1, & \text{if} \quad k = 0, \\ b(b+1)...(b+k-1), & \text{if} \quad k \in \mathbb{N} = \{1, 2, ...\}. \end{cases}$$

is convergent for all finite values of z (see [21]). It can be written otherwise

$$F(b; c; m) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k\ (1)_k} m^k, \quad \left(b \in \mathbb{C},\ c \in \mathbb{C} \setminus \{0, -1, -2, ...\} \right),$$

is convergent for $b, c, m > 0$.

Very recently, Porwal and Kumar [20] introduced the confluent hypergeometric distribution (CHD) whose probability mass function is (see [7, 5])

$$P(k)=\frac{(b)_k}{(c)_k\ \ k!F(b;c;m)}m^k,\ (b,c,m>0,\ k=0,1,2,...)\,.$$

Porwal [19] introduced a series $\mathcal{I}(b;c;m;z)$ whose coefficients are probabilities of confluent hypergeometric distribution

$$\mathcal{I}(b;c;m;z)=z+\sum_{k=2}^{\infty}\frac{(b)_{k-1}\,m^{k-1}}{(c)_{k-1}\ \ (k-1)!F(b;c;m)}z^k,\ (b,c,m>0)\,, \tag{1.10}$$

and defined a linear operator $\Omega(b;c;m)f:\mathcal{A}\to\mathcal{A}$ as follows:

$$\begin{aligned}\Omega(b;c;m)f(z) &= \mathcal{I}(b;c;m;z)*f(z)\\ &= z+\sum_{k=2}^{\infty}\frac{(b)_{k-1}\,m^{k-1}}{(c)_{k-1}\ \ (k-1)!F(b;c;m)}a_k\ z^k,\ (b,c,m>0)\,.\end{aligned}$$

Srivastava [30] (see also [32], [33, 12]) made use of various operators of q-calculus and fractional q-calculus and recalling the definition and notations. The q-shifted factorial is defined for $\lambda,q\in\mathbb{C}$ and $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ as follows

$$(\lambda;q)_k=\begin{cases}1 & \text{if}\quad k=0,\\ (1-\lambda)\,(1-\lambda q)\,...\left(1-\lambda q^{k-1}\right) & \text{if}\quad k\in\mathbb{N}.\end{cases}$$

By using the q-gamma function $\Gamma_q(z)$, we get

$$\left(q^{\lambda};q\right)_k=\frac{(1-q)^k\ \Gamma_q\left(\lambda+k\right)}{\Gamma_q\left(\lambda\right)},\qquad (k\in\mathbb{N}_0)\,,$$

where

$$\Gamma_q(z)=(1-q)^{1-z}\,\frac{(q;q)_\infty}{(q^z;q)_\infty},\qquad (|q|<1)$$

for details (see [8]). Also, we note that

$$(\lambda;q)_\infty=\prod_{k=0}^{\infty}\left(1-\lambda q^k\right),\qquad (|q|<1)\,,$$

and, the q-gamma function $\Gamma_q(z)$ is known

$$\Gamma_q(z+1)=[z]_q\ \Gamma_q(z),$$

where $[k]_q$ denotes the basic q-number $q \in (0,1)$ defined as follows

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & \text{if } k \in \mathbb{C}, \\ 1 + \sum_{j=1}^{k-1} q^j, & \text{if } k \in \mathbb{N}. \end{cases} \tag{1.11}$$

Using the definition formula (1.11) we have the next two products:

(i) For any non negative integer k, the *q-shifted factorial* is given by

$$[k]_q! := \begin{cases} 1, & \text{if } k = 0, \\ \prod_{n=1}^{k} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the *q-generalized Pochhammer symbol* is defined by

$$[r]_{q,k} := \begin{cases} 1, & \text{if } k = 0, \\ \prod_{n=r}^{r+k-1} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(z)$, that

$$\Gamma_q(z) \to \Gamma(z) \qquad \text{as } q \to 1^-.$$

Also, we observe that

$$\lim_{q\to 1^-} \left\{ \frac{(q^\lambda; q)_k}{(1-q)^k} \right\} = (\lambda)_k .$$

For $0 < q < 1$, the *q-derivative operator* [10] for $\mathcal{I}(b;c;m;z)$ is defined by

$$\begin{aligned} \mathfrak{D}_q \left(\Omega(b;c;m) f(z)\right) \quad : \quad &= \frac{\Omega(b;c;m)f(z) - \Omega(b;c;m)f(qz)}{z(1-q)} \\ &= 1 + \sum_{k=2}^{\infty} [k]_q \frac{(b)_{k-1}\, m^{k-1}}{(c)_{k-1}\ (k-1)! F(b;c;m)} a_k\ z^{k-1}, \ , \end{aligned}$$

where $b, c, m > 0,\ z \in \Delta$ and

$$[k]_q := \frac{1-q^k}{1-q} = 1 + \sum_{j=1}^{k-1} q^j, \qquad [0,q] := 0 \quad \textit{and} \quad [2]_q = q+1. \tag{1.12}$$

For $\lambda > -1$ and $0 < q < 1$, we defined the linear operator $\mathcal{I}^{\lambda,q}(b;c;m)f : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{I}^{\lambda,q}(b;c;m)f(z) * \mathcal{N}^{m}_{q,\lambda+1}(z) = z\, \mathfrak{D}_q \left(\Omega(b;c;m)f(z)\right),\ z \in \Delta,$$

where the function $\mathcal{N}^m_{q,\lambda+1}$ is given by

$$\mathcal{N}^m_{q,\lambda+1}(z) := z + \sum_{k=2}^{\infty} \frac{[\lambda+1]_{q,k-1}}{[k-1]_q!} z^k, \; z \in \Delta.$$

A simple computation shows that

$$\mathcal{I}^{\lambda,q}(b;c;m)f(z) := z + \sum_{k=2}^{\infty} \psi_k \; a_k \; z^k \quad (z \in \Delta), \tag{1.13}$$

where

$$\psi_k = \frac{(b)_{k-1} m^{k-1} [k]_q!}{(c)_{k-1} \; (k-1)! F(b;c;m) [\lambda+1]_{q,k-1}}, \; (b,c,m > 0, \; \lambda > -1, \; 0 < q < 1). \tag{1.14}$$

From the definition relation (1.13), we can easily verify that the next relations hold for all $f \in \mathcal{A}$:

(1.15)

(i) $[\lambda+1]_q \; \mathcal{I}^{\lambda,q}(b;c;m)f(z) = [\lambda]_q \mathcal{I}^{\lambda+1,q}(b;c;m)f(z) + q^{\lambda} \, z \, \mathfrak{D}_q \left(\mathcal{I}^{\lambda+1,q}(b;c;m)f(z)\right),$

(1.16)

(ii) $\mathcal{N}^{\lambda}(b;c;m)f(z) := \lim_{q \to 1^-} \mathcal{I}^{\lambda,q}(b;c;m)f(z) = z + \sum_{k=2}^{\infty} \frac{[k]_q (b)_{k-1} m^{k-1}}{(c)_{k-1} \; F(b;c;m)(\lambda+1)_{k-1}} a_k z^k,$

$z \in \Delta$.

Remark 1. *Putting $b = c$ in the operator $\mathcal{I}^{\lambda,q}(b;c;m)$, we obtain the q-analogue of Poisson operator $I^{\lambda,m}_q$ defined by El-Deeb et al.* [6] *as follows*

$$P^{\lambda,m}_q f(z) := z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} \cdot \frac{[k]_q!}{[\lambda+1]_{q,k-1}} a_k \; z^k, \; z \in \Delta. \tag{1.17}$$

By using the operator $\mathcal{I}^{\lambda,q}(b;c;m)$, we defined a subclass $\mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta,\alpha,\eta,A,B)$ of the class $\mathcal{T}$ as follows:

Definition 1.2. For $0 \le \rho \le 1$, $0 < \alpha \le 1$, $0 \le \beta < 1$, $-1 \le B < A \le 1$, $0 \le \eta \le 1$, $b,c,m > 0$, $\lambda > -1$, $0 < q < 1$, we let the class $\mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta,\alpha,\eta,A,B)$ denote the subclass of $\mathcal{T}$ consisting of functions of the form (1.5) and satisfying the condition:

$$\left| \frac{\mathcal{F}^{\lambda,q,\rho}_{b,c,m}(z) - 1}{\eta\,(B-A)\left(\mathcal{F}^{\lambda,q,\rho}_{b,c,m}(z) - \beta\right) - B\left(\mathcal{F}^{\lambda,q,\rho}_{b,c,m}(z) - 1\right)} \right| < \alpha, \tag{1.18}$$

where

(1.19)

$$\mathcal{F}_{b,c,m}^{\lambda,q,\rho}(z)=\frac{z\mathfrak{D}_q(\mathcal{I}^{\lambda,q}(b;c;m)f(z))+\rho z^2\mathfrak{D}_q[\mathfrak{D}_q(\mathcal{I}^{\lambda,q}(b;c;m)f(z))]}{(1-\rho)\mathcal{I}^{\lambda,q}(b;c;m)f(z)+\rho z\mathfrak{D}_q(\mathcal{I}^{\lambda,q}(b;c;m)f(z))},\qquad (0\leq\rho\leq 1).$$

Remark 2. *Taking different particular cases for the coefficients* b, c *and* q, *we obtain the next special cases for the class* $\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$:
(i) Putting $q\to 1^-$, *we obtain that* $\lim\limits_{q\to 1^-}\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)=:\mathcal{R}_{b,c,m}^{\lambda,\rho}(\beta,\alpha,\eta,A,B)$, *where* $\mathcal{R}_{b,c,m}^{\lambda,\rho}(\beta,\alpha,\eta,A,B)$ *represents the functions* $f\in\mathcal{A}$ *that satisfies* (1.5) *for* $\mathcal{I}^{\lambda,q}(b;c;m)$ *replaced with* $\mathcal{N}^{\lambda}(b;c;m)$;
(ii) Putting $b=c$, *we obtain the class* $\mathcal{W}_m^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$, *that represents the functions* $f\in\mathcal{A}$ *satisfies* (1.5) *for* $\mathcal{I}^{\lambda,q}(b;c;m)$ *replaced with* $P_q^{\lambda,m}$.

Inspired by the recent works in [6, 7, 5, 11, 30, 28, 31], in this paper based on convolution, we discuss some interesting properties of functions $f\in\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$ .Further we discuss certain closure properties under integral transformation.

2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that $0\leq\rho,\eta\leq 1;\ 0<\beta\leq 1;\ b,c,m>0;\ \lambda>-1;\ 0<q<1;\ 0<\alpha\leq 1;\ -1\leq A<B\leq 1$ and $z\in\Delta$, the powers are understood as principle values. In the following theorem we obtain necessary and sufficient conditions for functions $f\in\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$.

Theorem 2.1. *Let the function* f *be defined by* (1.5). *Then* $f\in\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$ *if and only if*

(2.1)
$$\sum_{k=2}^{\infty}[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta)]\psi_k a_k \leq\alpha(B-A)\eta(1-\beta),$$

where ψ_k *is given by* (1.14) *and we let*

(2.2)

$$\Phi(k,\beta,\alpha,\eta,A,B)=[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta)]\psi_k.$$

Proof. Assume that the inequality (2.2) holds true, we find from (1.5) and (1.18) that

$$\begin{aligned}
&\left|\mathcal{F}_{b,c,m}^{\lambda,q,\rho}(z)-1\right|-\alpha\left|\eta\left(B-A\right)\left(\mathcal{F}_{b,c,m}^{\lambda,q,\rho}(z)-\beta\right)-B\left(\mathcal{F}_{b,c,m}^{\lambda,q,\rho}(z)-1\right)\right| \\
= &\left|\sum_{k=2}^{\infty}(1+[k]_q\rho-\rho)\left([k]_q-1\right)\psi_k a_k z^k\right| \\
&-\Bigg|\alpha\left(B-A\right)\eta\left(1-\beta\right)z \\
&\quad+\sum_{k=2}^{\infty}\alpha[1+\rho([k]_q-1)]\left[\eta\left(B-A\right)\left(k-\beta\right)-B\left([k]_q-1\right)\right]\psi_k a_k z^k\Bigg| \\
&\le\sum_{k=2}^{\infty}(1+[k]_q\rho-\rho)\left([k]_q-1\right)\psi_k a_k z^{k-1}r^k-\alpha\left(B-A\right)|\eta|\left(1-\beta\right)r \\
&+\sum_{k=2}^{\infty}\alpha(1+[k]_q\rho-\rho)\left[\eta\left(B-A\right)\left([k]_q-\beta\right)-B\left([k]_q-1\right)\right]\psi_k a_k r^k \\
&\le\sum_{k=2}^{\infty}(1+[k]_q\rho-\rho)[(1-B\alpha)\left([k]_q-1\right)+\left(B-A\right)|\eta|\,\alpha\left([k]_q-\beta\right)]\psi_k a_k \\
&\qquad -\alpha\left(B-A\right)|\eta|\left(1-\beta\right)\le 0\ \ \left(z\in\Delta\right).
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z)\in\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$. Conversely, Let

$$\begin{aligned}
&\left|\frac{\mathcal{F}_{b,c,m}^{\lambda,q,\rho}(z)-1}{\eta\left(B-A\right)\left(\mathcal{F}_{b,c,m}^{\lambda,q,\rho}(z)-\beta\right)-B\left(\mathcal{F}_{b,c,m}^{\lambda,q,\rho}(z)-1\right)}\right|= \\
&\left|\frac{\sum\limits_{k=2}^{\infty}[1+\rho([k]_q-1)]\left([k]_q-1\right)\psi_k z^k}{\left(B-A\right)\eta\left(1-\beta\right)z-\sum\limits_{k=2}^{\infty}(1+[k]_q\rho-\rho)\left[\eta\left(B-A\right)\left([k]_q-\beta\right)-B\left([k]_q-1\right)\right]\psi_k a_k z^k}\right| \\
&<\alpha,\ \ \left(z\in\Delta\right).
\end{aligned}$$

Now since $\Re\left\{z\right\}\le|z|$ for all z, we have

$$\Re\left\{\frac{\sum\limits_{k=2}^{\infty}[1+\rho([k]_q-1)](k-1)\psi_k z^k}{(B-A)\eta(1-\beta)z-\sum\limits_{k=2}^{\infty}[1+\rho([k]_q-1)][\eta(B-A)([k]_q-\beta)-B([k]_q-1)]\psi_k a_k z^k}\right\}<\alpha. \tag{2.3}$$

Choose values of z on the real axis so that $f'(z)$ is real. Then upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we have

$$\sum_{k=2}^{\infty}[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta)]\psi_k a_k \leq \alpha(B-A)\eta(1-\beta).$$

This completes the proof of Theorem 2.1. □

Taking $b = c$ in Theorem 2.1, we get the following corollary.

Corollary 2.1. *Let the function f be defined by* (1.5)*.Then $f \in \mathcal{W}_m^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$ if and only if*

$$\sum_{k=2}^{\infty}[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta)]\frac{m^{k-1}[k]_q!}{(k-1)![\lambda+1]_{q,k-1}}e^{-m}a_k \leq \alpha(B-A)\eta(1-\beta).$$

Taking $\rho = 0$ in Theorem 2.1, we get the following example.

Example 2.1. *Let the function f be defined by* (1.5)*. Then $f \in \mathcal{H}_{b,c,m}^{\lambda,q,0}(\beta,\alpha,\eta,A,B)$ if and only if*

$$\sum_{k=2}^{\infty}[(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta)]\psi_k a_k \leq \alpha(B-A)\eta(1-\beta).$$

Taking $\rho = 1$ in Theorem 2.1, we get the following example.

Example 2.2. *Let the function f be defined by* (1.5)*. Then $f \in \mathcal{H}_{b,c,m}^{\lambda,q,1}(\beta,\alpha,\eta,A,B)$ if and only if*

$$\sum_{k=2}^{\infty}[k]_q[(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta)]\psi_k a_k \leq \alpha(B-A)\eta(1-\beta).$$

3. Convolution properties for functions in the class $\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$

In this section, using the techniques of Schild and Silverman [23], we obtain some convolution properties for functions $f(z) \in \mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$.

Theorem 3.1. *Let f_1 be given by (1.6) and $f_1 \in \mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta_1, \alpha, \eta, A, B)$ and the function f_2 be given by (1.6) and $f_2 \in \mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta_2, \alpha, \eta, A, B)$. If the sequence $\{C_n\}$ is non-decreasing then $f_1 * f_2 \in \mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta^*, \alpha, \eta, A, B)$, where*

$$\beta^* = 1 - \frac{\alpha\eta(B-A)(1-\beta_1)(1-\beta_2)[1-B\alpha+\eta\alpha(B-A)]}{(1+\rho)\sqrt{\Upsilon_2(\beta_1,\alpha,\eta,A,B)}\sqrt{\Upsilon_2(\beta_2,\alpha,\eta,A,B)}\psi_2-(\alpha\eta(B-A))^2(1-\beta_1)(1-\beta_2)}, \tag{3.1}$$

where

$$\Upsilon_2(\beta_1, \alpha, \eta, A, B) = [(1 - B\alpha)\, q + (B - A)\, \eta\alpha\, (q + 1 - \beta_1)], \tag{3.2}$$

$$\Upsilon_2(\beta_2, \alpha, \eta, A, B) = [(1 - B\alpha)\, q + (B - A)\, \eta\alpha\, (q + 1 - \beta_2)], \tag{3.3}$$

and ψ_2 is given by (1.14) for $k = 2$.

Proof. In view of Theorem 3.1, it's enough to show that

$$\sum_{k=2}^{\infty} \frac{[1+\rho([k]_q-1)][(1-B\alpha)(k-1)+(B-A)\eta\alpha(k-\beta^*)]\psi_k\ a_{k,1}a_{k,2}}{\alpha\eta(B-A)(1-\beta^*)} \leq 1 \tag{3.4}$$

where β^* is defined by (3.1). Since $f_1 \in \mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta_1, \alpha, \eta, A, B)$, we have

$$\sum_{k=2}^{\infty} \frac{[1 + \rho([k]_q - 1)]\Upsilon_k(\beta_1, \alpha, \eta, A, B)\psi_k\ a_{k,1}}{\alpha\eta\,(B - A)\,(1 - \beta_1)} \leq 1, \tag{3.5}$$

and $f_2 \in \mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta_2, \alpha, \eta, A, B)$, we have

$$\sum_{k=2}^{\infty} \frac{[1 + \rho([k]_q - 1)]\Upsilon_k(\beta_2, \alpha, \eta, A, B)\psi_k a_{k,2}}{\alpha\eta\,(B - A)\,(1 - \beta_2)} \leq 1, \tag{3.6}$$

such that

$$\begin{aligned} \Upsilon_k(\beta_1, \alpha, \eta, A, B) &= [(1 - B\alpha)\,([k]_q - 1) + (B - A)\,\eta\alpha\,([k]_q - \beta_1)] \quad \textit{and} \\ \Upsilon_k(\beta_2, \alpha, \eta, A, B) &= [(1 - B\alpha)\,([k]_q - 1) + (B - A)\,\eta\alpha\,([k]_q - \beta_2)]. \end{aligned}$$

On the other hand, under the hypothesis and by the Cauchy's-Schwarz inequality that

$$\sum_{k=2}^{\infty} \frac{[1 + \rho([k]_q - 1)]\sqrt{\Upsilon_k(\beta_1, \alpha, \eta, A, B)}\sqrt{\Upsilon_k(\beta_2, \alpha, \eta, A, B)}}{\alpha\eta\,(B - A)\,\sqrt{(1 - \beta_1)\,(1 - \beta_2)}}\psi_k\sqrt{a_{k,1}a_{k,2}} \leq 1, \tag{3.7}$$

from (3.5) and (3.6), it follows that

$$\sum_{k=2}^{\infty} \frac{[1 + \rho([k]_q - 1)]^2\ \Upsilon_k(\beta_1, \alpha, \eta, A, B)\ \Upsilon_k(\beta_2, \alpha, \eta, A, B)\ \psi_k^2 a_{k,1}a_{k,2}}{[\alpha\eta\,(B - A)]^2\,(1 - \beta_1)\,(1 - \beta_2)} \leq 1. \tag{3.8}$$

Now, we have to find largest β^* such that

$$\sum_{k=2}^{\infty} \frac{[1+\rho([k]_q-1)][(1-B\alpha)(k-1)+(B-A)\eta\alpha(k-\beta^*)]\psi_k\ a_{k,1}a_{k,2}}{\alpha\eta(B-A)(1-\beta^*)}$$
$$\leq \sum_{k=2}^{\infty} \frac{[1+\rho([k]_q-1)]\sqrt{\Upsilon_k(\beta_1,\alpha,\eta,A,B)}\sqrt{\Upsilon_k(\beta_2,\alpha,\eta,A,B)}}{\alpha\eta(B-A)\sqrt{(1-\beta_1)(1-\beta_2)}}\psi_k\sqrt{a_{k,1}a_{k,2}},$$

then, we have

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(1-\beta^*)\sqrt{\Upsilon_k(\beta_1,\alpha,\eta,A,B)}\sqrt{\Upsilon_k(\beta_2,\alpha,\eta,A,B)}}{\sqrt{(1-\beta_1)(1-\beta_2)}[(1-B\alpha)q+(B-A)\eta\alpha([k]_q-\beta^*)]}.$$

From (3.7), it is sufficient to find largest β^* such that

$$\frac{\alpha\eta(B-A)\sqrt{(1-\beta_1)(1-\beta_2)}}{[1+\rho([k]_q-1)]\sqrt{\Upsilon_k(\beta_1,\alpha,\eta,A,B)}\sqrt{\Upsilon_k(\beta_2,\alpha,\eta,A,B)}\psi_k}$$
$$\leq \frac{(1-\beta^*)\sqrt{\Upsilon_k(\beta_1,\alpha,\eta,A,B)}\sqrt{\Upsilon_k(\beta_2,\alpha,\eta,A,B)}}{\sqrt{(1-\beta_1)(1-\beta_2)}[(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta^*)]},$$

we have

$$\beta^* \leq 1-\frac{\alpha\eta(B-A)(1-\beta_1)(1-\beta_2)([k]_q-1)[1-B\alpha+\eta\alpha(B-A)]}{[1+\rho([k]_q-1)]\sqrt{\Upsilon_k(\beta_1,\alpha,\eta,A,B)}\sqrt{\Upsilon_k(\beta_2,\alpha,\eta,A,B)}\psi_k-(\alpha\eta(B-A))^2(1-\beta_1)(1-\beta_2)}.$$

Let

$$\Psi(k) = \frac{\alpha\eta(B-A)(1-\beta_1)(1-\beta_2)([k]_q-1)[1-B\alpha+\eta\alpha(B-A)]}{[1+\rho([k]_q-1)]\sqrt{\Upsilon_k(\beta_1,\alpha,\eta,A,B)}\sqrt{\Upsilon_k(\beta_2,\alpha,\eta,A,B)}\psi_k-(\alpha\eta(B-A))^2(1-\beta_1)(1-\beta_2)}$$

Since $\Psi(k)$ is non decreasing function of k $(k \geq 2)$, then we have $\beta^* \leq 1-\Psi(k)$. That is,

$$\beta^* \leq 1-\frac{\alpha\eta q(B-A)(1-\beta_1)(1-\beta_2)[1-B\alpha+\eta\alpha(B-A)]}{(1+q\rho)\sqrt{\Upsilon_2(\beta_1,\alpha,\eta,A,B)}\sqrt{\Upsilon_2(\beta_2,\alpha,\eta,A,B)}\psi_2-(\alpha\eta(B-A))^2(1-\beta_1)(1-\beta_2)}.$$

This completes the proof of Theorem 3.1.

□

Taking $b=c$ in Theorem 3.1, we get the following corollary.

Corollary 3.1. *Let the function* f_1 *be given by* (1.6) *and* $f_1 \in \mathcal{W}_m^{\lambda,q,\rho}(\beta_1,\alpha,\eta,A,B)$, f_2 *defined by* (1.6)*and* $f_2 \in \mathcal{W}_m^{\lambda,q,\rho}(\beta_2,\alpha,\eta,A,B)$. *If the sequence* $\{C_n\}$*is non-decreasing then* $f_1 * f_2 \in \mathcal{W}_m^{\lambda,q,\rho}(\beta^{**},\alpha,\eta,A,B)$, *where*

$$\beta^{**} = 1-\frac{\alpha\eta(B-A)(1-\beta_1)(1-\beta_2)[1-B\alpha+\eta\alpha(B-A)]}{\frac{(1+\rho)[2]_q!me^{-m}}{[\lambda+1]_q}\sqrt{\Upsilon_2(\beta_1,\alpha,\eta,A,B)}\sqrt{\Upsilon_2(\beta_2,\alpha,\eta,A,B)}-(\alpha\eta(B-A))^2(1-\beta_1)(1-\beta_2)},$$

where $\Upsilon_2(\beta_1,\alpha,\eta,A,B)$ *and* $\Upsilon_2(\beta_2,\alpha,\eta,A,B)$ *are given by* (3.2) *and* (3.3).

Theorem 3.2. *Let the function f_i $(i = 1, 2)$ defined by* (1.6) *be in the class $\mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta, \alpha, \eta, A, B)$. If the sequence $\{C_n\}$ is non-decreasing then the function*

$$h(z) = z - \sum_{k=2}^{\infty} \left(a^2_{k,1} + a^2_{k,2}\right) z^k \tag{3.9}$$

belongs to the class $\mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\zeta, \alpha, \eta, A, B)$, where

$$\zeta = 1 - \frac{2\eta\alpha\,(B - A)\,(1 - \beta)^2\,[1 - B\alpha + \eta\alpha\,(B - A)]}{(1 + \rho)[1 - B\alpha + \eta\alpha\,(B - A)\,(2 - \beta)]^2\psi_2 - 2\,[\eta\alpha\,(B - A)\,(1 - \beta)]^2}, \tag{3.10}$$

and ψ_2 is given by (1.14).

Proof. From Theorem 2.1, it is sufficient prove that

$$\sum_{k=2}^{\infty} \frac{[1 + \rho([k]_q - 1)][(1 - B\alpha)\,([k]_q - 1) + (B - A)\,\eta\alpha\,([k]_q - \zeta)]\psi_k}{\alpha\,(B - A)\,\eta\,(1 - \zeta)} \left(a^2_{k,1} + a^2_{k,2}\right) \le 1.$$

Since the functions f_i $(i = 1, 2)$ be in the class $\mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta, \alpha, \eta, A, B)$, we have

$$\sum_{k=2}^{\infty} \left\{\frac{\Phi(k, \beta, \alpha, \eta, A, B)}{\alpha\,(B - A)\,\eta\,(1 - \beta)}\right\}^2 a^2_{k,1} \le \sum_{k=2}^{\infty} \left\{\frac{\Phi(k, \beta, \alpha, \eta, A, B)}{\alpha\,(B - A)\,\eta\,(1 - \beta)} a_{k,1}\right\}^2 \le 1, \tag{3.11}$$

and

$$\sum_{k=2}^{\infty} \left\{\frac{\Phi(k, \beta, \alpha, \eta, A, B)}{\alpha\,(B - A)\,\eta\,(1 - \beta)}\right\}^2 a^2_{k,2} \le \sum_{k=2}^{\infty} \left\{\frac{\Phi(k, \beta, \alpha, \eta, A, B)}{\alpha\,(B - A)\,\eta\,(1 - \beta)} a_{k,2}\right\}^2 \le 1, \tag{3.12}$$

where $\Phi(k, \beta, \alpha, \eta, A, B)$ is as assumed in (2.2). It follows from (3.11) and (3.12) that

$$\frac{1}{2} \sum_{k=2}^{\infty} \left\{\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta)]\psi_k}{\alpha(B-A)\eta(1-\beta)}\right\}^2 \left(a^2_{k,1} + a^2_{k,2}\right) \le 1$$

Therefore, we need to find the largest ζ, such that

$$\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\zeta)]\psi_k}{\alpha(B-A)\eta(1-\zeta)} \le \frac{1}{2} \left\{\frac{\Phi(k,\beta,\alpha,\eta,A,B)}{\alpha(B-A)\eta(1-\beta)}\right\}^2 \quad (k \ge 2)$$

that is

$$\zeta = 1 - \frac{2\eta\alpha(B-A)(1-\beta)^2([k]_q-1)[1-B\alpha+\eta\alpha(B-A)]}{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta)]^2\psi_k-2[\eta\alpha(B-A)(1-\beta)]^2}$$

Let

$$\Psi(k) = \frac{2\eta\alpha(B-A)(1-\beta)^2([k]_q-1)[1-B\alpha+\eta\alpha(B-A)]}{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\beta)]^2\psi_k-2[\eta\alpha(B-A)(1-\beta)]^2}$$

Since $\Psi(k)$ is non decreasing function of k $(k \geq 2)$, then we have $\zeta \leq 1-\Psi(k)$. That is,

$$\zeta \leq 1-\frac{2q\eta\alpha\,(B-A)\,(1-\beta)^2\,[1-B\alpha+\eta\alpha\,(B-A)]}{(1+q\rho)[(1-B\alpha)q+\eta\alpha\,(B-A)\,(1+q-\beta)]^2\psi_2-2\,[\eta\alpha\,(B-A)\,(1-\beta)]^2}.$$

This completes the proof of Theorem 3.2.

□

Taking $b=c$ in Theorem 3.2, we get the following corollary.

Corollary 3.2. *Let f_i $(i=1,2)$ be given by* (1.6) *and $f_i \in \mathcal{W}_m^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$. If the sequence $\{C_n\}$ is non-decreasing then the function $h(z)$ defined by* (3.9) *belongs to the class $\mathcal{W}_m^{\lambda,q,\rho}(\delta,\alpha,\eta,A,B)$, where*

$$\delta = 1-\frac{2q\eta\alpha\,(B-A)\,(1-\beta)^2\,[1-B\alpha+\eta\alpha\,(B-A)]}{\frac{(1+q\rho)[2]_q!me^{-m}}{[\lambda+1]_q}[(1-B\alpha)q+\eta\alpha\,(B-A)\,(1+q-\beta)]^2-2\,[\eta\alpha\,(B-A)\,(1-\beta)]^2}.$$

4. Hölder's inequality

Theorem 4.1. *Let the functions f_i $(i=1,2,...,n)$ defined by* (1.6) *be in the class $\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta_i,\alpha,\eta,A,B)$. Then $\mathcal{H}_n(z)$ are in the class $\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\zeta,\alpha,\eta,A,B)$ such that*

$$\zeta \leq 1-\frac{[\eta\alpha(B-A)]^{r-1}[1-B\alpha+\eta\alpha(B-A)]\prod_{i=1}^{n}(1-\beta_i)^{p_i}}{\left[[(1+\rho)\psi_2]^{r-1}\prod_{i=1}^{n}[(1-B\alpha)+\eta\alpha(B-A)(2-\beta_i)]^{p_i}-[\eta\alpha(B-A)]^{r}\prod_{i=1}^{n}[(1-\beta_i)]^{p_i}\right]},$$

where

$$(r=\sum_{i=1}^{n}p_i \geq 1;\ p_i \geq \frac{1}{q_i};\ \sum_{i=1}^{n}\frac{1}{q_i} \geq 1;\ q_i>1;\ i=1,2...,n),$$

and where ψ_2 is given by (1.14).

Proof. Let $f_i(z) \in \mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta_i,\alpha,\eta,A,B)$, we have

$$\sum_{k=2}^{\infty}\frac{[1+\rho([k]_q-1)][(1-B\alpha)\,([k]_q-1)+\eta\alpha\,(B-A)\,([k]_q-\beta_i)]\psi_k}{\eta\alpha\,(B-A)\,(1-\beta_i)}a_{k,i} \leq 1, \tag{4.1}$$

which implies

$$\left(\sum_{k=2}^{\infty}\frac{[1+\rho([k]_q-1)][(1-B\alpha)\,([k]_q-1)+\eta\alpha\,(B-A)\,([k]_q-\beta_i)]\psi_k}{\eta\alpha\,(B-A)\,(1-\beta_i)}a_{k,i}\right)^{\frac{1}{q_i}} \leq 1, \tag{4.2}$$

with $q_i > 1$ and $\sum_{i=1}^{n} \frac{1}{q_i} \geq 1$.
From (4.2), we have

$$\prod_{i=1}^{n}\left(\sum_{k=2}^{\infty}\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\beta_i)]\psi_k}{\eta\alpha(B-A)(1-\beta_i)}a_{k,i}\right)^{\frac{1}{q_j}} \leq 1.$$

Applying Hölder's inequality (1.9), we find that

$$\sum_{k=2}^{\infty}\left[\prod_{i=1}^{n}\left(\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\beta_i)]\psi_k}{\eta\alpha(B-A)(1-\beta_i)}\right)^{\frac{1}{q^i}} a_{k,i}^{\frac{1}{q_i}}\right] \leq 1.$$

Thus, we have to determine the largest ζ such that

$$\sum_{k=2}^{\infty}\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\zeta)]\psi_k}{\eta\alpha(B-A)(1-\zeta)}\left(\prod_{i=1}^{n}a_{k,i}^{p_i}\right) \leq 1$$

that is, that

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\zeta)]\psi_k}{\eta\alpha(B-A)(1-\zeta)}\left(\prod_{i=1}^{n}a_{k,i}^{p_i}\right)\\ \leq\ &\sum_{k=2}^{\infty}\left[\prod_{i=1}^{n}\left(\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\beta_i)]\psi_k}{\eta\alpha(B-A)(1-\beta_i)}\right)^{\frac{1}{q^i}} a_{k,j}^{\frac{1}{q_j}}\right].\end{aligned}$$

Therefore, we need to find the largest ζ such that

$$\begin{aligned}&\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\zeta)]\psi_k}{\eta\alpha(B-A)(1-\zeta)}\left(\prod_{i=1}^{n}a_{k,i}^{p_i-\frac{1}{q_i}}\right)\\ \leq\ &\prod_{i=1}^{n}\left(\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\beta_i)]\psi_k}{\alpha(B-A)\eta(1-\beta_i)}\right)^{\frac{1}{q_i}},\end{aligned}$$

for $(k \geq 2)$, Since

$$\prod_{i=1}^{n}\left(\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\zeta)]\psi_k}{\eta\alpha(B-A)(1-\zeta)}\right)^{p_i-\frac{1}{q_i}} a_{k,i}^{p_i-\frac{1}{q_i}} \leq 1$$

$$(p_i - \frac{1}{q_i} \geq 0),$$

we see that,

$$\prod_{i=1}^{n}a_{k,i}^{p_i-\frac{1}{q_i}} \leq \frac{1}{\prod_{i=1}^{n}\left(\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\beta_i)]\psi_k}{\eta\alpha(B-A)(1-\beta_i)}\right)^{p_i-\frac{1}{q_i}}}.$$

This implies that

$$\frac{[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\zeta)]\psi_k}{\eta\alpha(B-A)(1-\zeta)}$$
$$\leq \frac{\prod_{i=1}^{n}[[1+\rho([k]_q-1)]((1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\beta_i))\psi_k]^{p_i}}{\prod_{i=1}^{n}[\eta\alpha(B-A)(1-\beta_i)]^{p_i}},$$

which is equivalent to

$$\zeta \leq 1-\frac{([k]_q-1)\Theta_i\ [1-B\alpha+\eta\alpha(B-A)]}{\eta\alpha(B-A)\left[\prod_{i=1}^{n}[[1+\rho([k]_q-1)]\psi_k]^{p_i-1}[(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\beta_i)]^{p_i}-\Theta_i\right]},$$

where $\Theta_i=\prod_{i=1}^{n}[\eta\alpha(B-A)(1-\beta_i)]^{p_i}$. Let

$$\Phi(k) \leq 1-\frac{([k]_q-1)\Theta_i\ [1-B\alpha+\eta\alpha(B-A)]}{\eta\alpha(B-A)\left[\prod_{i=1}^{n}[[1+\rho([k]_q-1)]\psi_k]^{p_i-1}[(1-B\alpha)([k]_q-1)+\eta\alpha(B-A)([k]_q-\beta_i)]^{p_i}-\Theta_i\right]},$$

which is an increasing function in k, hence we have

$$\zeta \leq \Phi(2)=1-\frac{q[\eta\alpha(B-A)]^{r-1}[1-B\alpha+\eta\alpha(B-A)]\prod_{i=1}^{n}(1-\beta_i)^{p_i}}{\left[[(1+q\rho)\psi_2]^{r-1}\prod_{i=1}^{n}[(1-B\alpha)+\eta\alpha(B-A)(1+q-\beta_i)]^{p_i}-[\eta\alpha(B-A)]^{r}\prod_{i=1}^{n}[(1-\beta_i)]^{p_i}\right]}.$$

This completes the proof of Theorem 4.1. □

Taking $b=c$ in Theorem 4.1, we get the following corollary.

Corollary 4.1. *Let the functions f_i $(i=1,2,...,n)$ defined by (1.6) be in the class $\mathcal{W}_m^{\lambda,q,\rho}(\beta_i,\alpha,\eta,A,B)$. Then $\mathcal{H}_n(z)$ are in the class $\mathcal{W}_m^{\lambda,q,\rho}(\gamma^*,\alpha,\eta,A,B)$ with*

$$\gamma^* \leq 1-\frac{q[\eta\alpha(B-A)]^{r-1}[1-B\alpha+\eta\alpha(B-A)]\prod_{i=1}^{n}(1-\beta_i)^{p_i}}{\left[\left[\frac{(1+\rho)[2]_q!me-m}{[\lambda+1]_q}\right]^{r-1}\prod_{i=1}^{n}[(1-B\alpha)+\eta\alpha(B-A)(1+q-\beta_i)]^{p_i}-[\eta\alpha(B-A)]^{r}\prod_{i=1}^{n}[(1-\beta_i)]^{p_i}\right]},$$

where

$$\left(r=\sum_{i=1}^{n}p_i\geq 1;\ p_i\geq\frac{1}{q_i};\ \sum_{i=1}^{n}\frac{1}{q_i}\geq 1;\ q_i>1;\ i=1,2...,n\right).$$

Taking $p_i=1$ in Theorem 4.1, we obtain of the following corollary:

Example 4.1. *Let the functions f_i $(i=1,2,...,m)$ defined by (1.6) be in the class $\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta_i,\alpha,\eta,A,B)$. Then $\mathcal{G}_m(z)$ are in the class $\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\gamma,\alpha,\eta,A,B)$ whith*

$$\zeta \leq 1-\frac{q[\eta\alpha(B-A)]^{m-1}[1-B\alpha+\eta\alpha(B-A)]\prod_{i=1}^{m}(1-\beta_i)}{\left[[(1+q\rho)\psi_2]^{m-1}\prod_{i=1}^{m}[(1-B\alpha)+\eta\alpha(B-A)(1+q-\beta_i)]-[\eta\alpha(B-A)]^{m}\prod_{i=1}^{m}(1-\beta_i)\right]}.$$

5. Closure properties under integral transforms

Murugusundaramoorthy et al. [17] defined the integral transform

$$\mathcal{I}_{\rho}(f)(z) = \int_0^1 \rho(t)\ \frac{f(tz)}{t} dt, \tag{5.1}$$

where ρ is a real valued and non-negative weight function normalized such that $\int_0^1 \rho(t)dt = 1$, for which $\mathcal{I}_{\rho}$ is known as Bernardi operator [2] and

$$\rho(t) = \frac{(\mu+1)^{\sigma}}{\Gamma(\sigma)} t^{\mu} \left(\log \frac{1}{t}\right)^{\sigma-1}, \ \mu > -1,\ \sigma \geq 0, \tag{5.2}$$

which gives the Komatu operator [13].

Theorem 5.1. *Let the function f be defined by* (1.5) *belongs to $\mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta,\alpha,\eta,A,B)$. Then $\mathcal{I}^{\mu}_{\sigma}(f)$ is in the class $\mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta,\alpha,\eta,A,B)$.*

Proof. From (5.1) and (5.2), we have

$$\begin{aligned}\mathcal{I}^{\mu}_{\sigma}(f)(z) &= \frac{(-1)^{\sigma-1}(\mu+1)^{\sigma}}{\Gamma(\sigma)} \int_0^1 t^{\mu} (\log t)^{\sigma-1} \left(z - \sum_{k=2}^{\infty} a_k z^k t^{k-1}\right) dt \\ &= z - \sum_{k=2}^{\infty} \left(\frac{\mu+1}{\mu+k}\right)^{\sigma} a_k z^k. \end{aligned}\tag{5.3}$$

We need to prove that $\mathcal{I}^{\mu}_{\sigma}(f) \in \mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta,\alpha,\eta,A,B)$, it's enough to show that

$$\sum_{k=2}^{\infty} \frac{\Phi(k,\beta,\alpha,\eta,A,B)}{\alpha(B-A)\eta(1-\beta)} \left(\frac{\mu+1}{\mu+k}\right)^{\sigma} a_k \leq 1 \tag{5.4}$$

where $\Phi(k,\beta,\alpha,\eta,A,B)$ is as assumed in (2.2). In view of Theorem 2.1. Since $f \in \mathcal{H}^{\lambda,q,\rho}_{b,c,m}(\beta,\alpha,\eta,A,B)$ if and only if

$$\sum_{k=2}^{\infty} \frac{\Phi(k,\beta,\alpha,\eta,A,B)}{\alpha(B-A)\eta(1-\beta)} a_k \leq 1. \tag{5.5}$$

Hence $\frac{\mu+1}{\mu+k} < 1$, then (5.4) holds and this completes the proof of Theorem 5.1. □

Taking $b = c$ in Theorem 5.1, we get the following corollary.

Corollary 5.1. *Let the function f be defined by (1.5) belongs to $\mathcal{W}_m^{\lambda,q,\rho}(\beta_i,\alpha,\eta,A,B)$. Then $\mathcal{I}_\sigma^\mu(f)$ is in the class $\mathcal{W}_m^{\lambda,q,\rho}(\beta_i,\alpha,\eta,A,B)$.*

Theorem 5.2. *Let the function f be defined by (1.5)and $f\in\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$. Then $\mathcal{I}_\sigma^\mu(f)$ is starlike of order η $(0\leq\eta<1)$ in $|z|<R_1$, where*

$$R_1=\inf_k\left\{\left(\frac{\mu+k}{\mu+1}\right)^\sigma\frac{(1-\eta)\,\Phi(k,\beta,\alpha,\eta,A,B)}{\alpha\,(k-\eta)\,(B-A)\,\eta\,(1-\beta)}\right\}^{\frac{1}{k-1}},\tag{5.6}$$

where $\Phi(k,\beta,\alpha,\eta,A,B)$ is given by (2.2).

Proof. It's sufficient to prove

$$\left|\frac{z\mathfrak{D}_q\left(\mathcal{I}_\sigma^\mu(f)(z)\right)}{\mathcal{I}_\sigma^\mu(f)(z)}-1\right|<1-\eta\quad\text{for}\quad|z|<R_1,$$

where R_1 is given by (5.6), such that

$$\left|\frac{z\mathfrak{D}_q\left(\mathcal{I}_\sigma^\mu(f)(z)\right)}{\mathcal{I}_\sigma^\mu(f)(z)}-1\right|\leq\frac{\sum\limits_{k=2}^{\infty}([k]_q-1)\left(\frac{\mu+1}{\mu+k}\right)^\sigma a_k\,|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}\left(\frac{\mu+1}{\mu+k}\right)^\sigma a_k\,|z|^{k-1}}.$$

Therefore

$$\left|\frac{z\mathfrak{D}_q\left(\mathcal{I}_\sigma^\mu(f)(z)\right)}{\mathcal{I}_\sigma^\mu(f)(z)}-1\right|<1-\eta$$

if

$$\sum_{k=2}^{\infty}\left(\frac{[k]_q-\eta}{1-\eta}\right)\left(\frac{\mu+1}{\mu+k}\right)^\sigma a_k\,|z|^{k-1}\leq 1.\tag{5.7}$$

Since $f\in\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$, then from (5.5) and (5.7), we have

$$\left(\frac{[k]_q-\eta}{1-\eta}\right)\left(\frac{\mu+1}{\mu+k}\right)^\sigma|z|^{k-1}\leq\frac{\Phi(k,\beta,\alpha,\eta,A,B)}{\alpha\,(B-A)\,\eta\,(1-\beta)}.$$

Thus

$$|z|\leq\left\{\left(\frac{\mu+k}{\mu+1}\right)^\sigma\frac{(1-\eta)\,\Phi(k,\beta,\alpha,\eta,A,B)}{\alpha\,([k]_q-\eta)\,(B-A)\,\eta\,(1-\beta)}\right\}^{\frac{1}{k-1}}.$$

This completes the proof of Theorem 5.2.

□

Taking $b=c$ in Theorem 5.2, we get the following corollary.

Corollary 5.2. *Let the function f be defined by* (1.5) *belongs to* $\mathcal{W}_m^{\lambda,q,\rho}(\beta_i,\alpha,\eta,A,B)$. *Then* $\mathcal{I}_\sigma^\mu(f)$ *is starlike of order* η^* $(0\leq\eta^*<1)$ *in* $|z|<R_1^*$, *where*

$$R_1^*=\inf_k\left\{\left(\frac{\mu+k}{\mu+1}\right)^\sigma\frac{(1-\eta)[1+\rho([k]_q-1)][k]_q!m^{k-1}e^{-m}[(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta)]}{\alpha([k]_q-\eta)(B-A)\eta(1-\beta)(k-1)![\lambda+1]_{q,k-1}}\right\}^{\frac{1}{k-1}}.$$

Theorem 5.3. *Let the function f be defined by* (1.5) *belongs to* $\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$. *Then* $\mathcal{I}_\sigma^\mu(f)$ *is convex of order* η $(0\leq\eta<1)$ *in* $|z|<R_2$, *where*

$$R_2=\inf_k\left\{\left(\frac{\mu+k}{\mu+1}\right)^\sigma\frac{(1-\eta)[1+\rho([k]_q-1)][(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta)]\psi_k}{\alpha k([k]_q-\eta)(B-A)\eta(1-\beta)}\right\}^{\frac{1}{k-1}}, \tag{5.8}$$

where ψ_k *is given by* (1.14).

Proof. The proof is similar to the proof of theorem 5.2, so it omitted. □

Taking $b=c$ in Theorem 5.3, we get the following corollary.

Corollary 5.3. *Let the function f be defined by* (1.5) *belongs to* $\mathcal{W}_m^{\lambda,q,\rho}(\beta_i,\alpha,\eta,A,B)$. *Then* $\mathcal{I}_\sigma^\mu(f)$ *is convex of order* η^* $(0\leq\eta^*<1)$ *in* $|z|<R_2^*$, *where*

$$R_2^*=\inf_k\left\{\left(\frac{\mu+k}{\mu+1}\right)^\sigma\frac{(1-\eta)[1+\rho([k]_q-\rho)][k]_q!m^{k-1}e^{-m}[(1-B\alpha)([k]_q-1)+(B-A)\eta\alpha([k]_q-\beta)]}{\alpha k([k]_q-\eta)(B-A)\eta(1-\beta)(k-1)![\lambda+1]_{q,k-1}}\right\}^{\frac{1}{k-1}}.$$

6. Integral Means and Partial Sums

In [24], Silverman found that the function $f_2(z)=z-\frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$. He applied this function to resolve his integral means inequality, conjectured in [26] and settled in [24], that

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^\vartheta d\theta\leq\int_0^{2\pi}\left|f_2(re^{i\theta})\right|^\vartheta d\theta,$$

for all $f\in\mathcal{T}$, $\vartheta>0$ and $0<r<1$. In [24], he also proved his conjecture for the subclasses $\mathcal{T}^*(\gamma)$ and $C(\gamma)$ of $\mathcal{T}$.

Now, we prove the Silverman's conjecture for the functions in the family $\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$.

Lemma 6.1. [14] *If the functions f and g are analytic in Δ with $g\prec f$, then for* $\vartheta>0$, *and* $0<r<1$,

$$\int_0^{2\pi}\left|g(re^{i\theta})\right|^\vartheta d\theta\leq\int_0^{2\pi}\left|f(re^{i\theta})\right|^\vartheta d\theta. \tag{6.1}$$

Applying Lemma 6.1, Theorem 2.1 , we prove the following result.

Theorem 6.1. *Suppose* $f \in \mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$, $\vartheta > 0$, $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$, $k \geq 0$ *and* $f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{\alpha\,(B-A)\,\eta\,(1-\beta)}{\Phi(2,\beta,\alpha,\eta,A,B)]\psi_k} z^2,$$

where

$$\Phi(k,\beta,\alpha,\eta,A,B) = [1+\rho([k]_q-1)][(1-B\alpha)\,([k]_q-1)+(B-A)\,\eta\alpha\,([k]_q-\beta)]\psi_k$$

is defined in 2.2. Then for $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} |f(z)|^{\vartheta}\, d\theta \leq \int_0^{2\pi} |f_2(z)|^{\vartheta}\, d\theta. \tag{6.2}$$

Proof. For $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$, (6.2) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} |a_k| z^{k-1} \right|^{\vartheta} d\theta \leq \int_0^{2\pi} \left| 1 - \frac{\alpha\,(B-A)\,\eta\,(1-\beta)}{\Phi(2,\beta,\alpha,\eta,A,B)} z \right|^{\vartheta} d\theta.$$

By Lemma 6.1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} |a_k| z^{k-1} \prec 1 - \frac{\alpha\,(B-A)\,\eta\,(1-\beta)}{\Phi(2,\beta,\alpha,\eta,A,B)} z.$$

Setting

$$1 - \sum_{k=2}^{\infty} |a_k| z^{k-1} = 1 - \frac{\alpha\,(B-A)\,\eta\,(1-\beta)}{\Phi(2,\beta,\alpha,\eta,A,B)} w(z), \tag{6.3}$$

and using (2.2), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{k=2}^{\infty} \frac{\Phi(k,\beta,\alpha,\eta,A,B)}{\alpha\,(B-A)\,\eta\,(1-\beta)} |a_k| z^{k-1} \right| \\ &\leq |z| \sum_{k=2}^{\infty} \frac{\Phi(k,\beta,\alpha,\eta,A,B)}{\alpha\,(B-A)\,\eta\,(1-\beta)} |a_k| \\ &\leq |z|. \end{aligned}$$

This completes the proof . □

For a function $f \in \mathcal{A}$ given by (1.1) Silverman [27] and Silvia [25] investigated the partial sums f_1 and f_m defined by

$$f_1(z) = z; \;\text{ and }\; f_\tau(z) = z + \sum_{k=2}^{\tau} a_k z^k, \;(\tau = 2, 3\ldots) \tag{6.4}$$

We consider in this section partial sums of functions in the class $\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta, \alpha, \eta, A, B)$, and obtain sharp lower bounds for the ratios of real part of f to $f_\tau(z)$ and f' to f'_τ.

Theorem 6.2. *Let a function f of the form (1.1) belong to the class $\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta, \alpha, \eta, A, B)$, and assume (6.4). Then*

$$Re\left\{\frac{f(z)}{f_\tau(z)}\right\} \geq 1 - \frac{1}{d_{\tau+1}}, \quad z \in \Delta, \;\tau \in \mathbb{N} \tag{6.5}$$

and

$$Re\left\{\frac{f_\tau(z)}{f(z)}\right\} \geq \frac{d_{\tau+1}}{1 + d_{\tau+1}}, \quad z \in \Delta, \;\tau \in \mathbb{N}, \tag{6.6}$$

where

$$d_k := \frac{\alpha\,(B - A)\,\eta\,(1 - \beta)}{\Phi(k, \beta, \alpha, \eta, A, B)}. \tag{6.7}$$

Proof. From (6.7), it is not difficult to verify that

$$d_{k+1} > d_k > 1, \quad k = 2, 3 \ldots. \tag{6.8}$$

Thus by Theorem 2.1 we have

$$\sum_{k=2}^{\tau} |a_k| + d_{\tau+1} \sum_{k=\tau+1}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} d_k |a_k| \leq 1. \tag{6.9}$$

Setting

$$g(z) = d_{\tau+1}\left\{\frac{f(z)}{f_\tau(z)} - \left(1 - \frac{1}{d_{\tau+1}}\right)\right\} = 1 + \frac{d_{\tau+1}\sum\limits_{k=\tau+1}^{\infty} a_k z^{k-1}}{1 + \sum\limits_{k=2}^{\tau} a_k z^{k-1}}, \tag{6.10}$$

it suffices to show that

$$\Re\, g(z) \geq 0, \quad z \in \Delta.$$

Applying (6.9), we find that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{d_{\tau+1}\sum\limits_{k=\tau+1}^{\infty}|a_k|}{2-2\sum\limits_{k=2}^{\tau}|a_k|-d_{\tau+1}\sum\limits_{k=\tau+1}^{\infty}|a_k|} \leq 1, \quad z\in\Delta,$$

which readily yields the assertion (6.5) of Theorem 6.2. In order to see that

$$f(z)=z+\frac{z^{\tau+1}}{d_{\tau+1}}, \quad z\in\Delta, \tag{6.11}$$

gives sharp the result, we observe that for $z=re^{i\pi/\tau}$ we have

$$\frac{f(z)}{f_\tau(z)}=1+\frac{z^\tau}{d_{\tau+1}} \overset{z\to1^-}{\longrightarrow} 1-\frac{1}{d_{\tau+1}}.$$

Similarly, if we take

$$\begin{aligned} h(z) &= (1+d_{\tau+1})\left\{\frac{f_\tau(z)}{f(z)}-\frac{d_{\tau+1}}{1+d_{\tau+1}}\right\} \\ &= 1-\frac{(1+d_{k+1})\sum\limits_{k=\tau+1}^{\infty}a_kz^{k-1}}{1+\sum\limits_{k=2}^{\infty}a_kz^{k-1}}, \quad z\in\Delta, \end{aligned}$$

and making use of (6.9), we can deduce that

$$\left|\frac{h(z)-1}{h(z)+1}\right| \leq \frac{(1+d_{\tau+1})\sum\limits_{k=\tau+1}^{\infty}|a_k|}{2-2\sum\limits_{k=2}^{\tau}|a_k|-(1-d_{\tau+1})\sum\limits_{k=\tau+1}^{\infty}|a_k|} \leq 1, \quad z\in\Delta,$$

which leads us immediately to the assertion (6.6) of Theorem 6.2. The bound in (6.6) is sharp for each $\tau\in\mathbb{N}$ with the extremal function f given by (6.11), and the proof is complete. □

Theorem 6.3. *Let a function f of the form (1.1)and $f\in\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta,\alpha,\eta,A,B)$, and assume* (2.2). *Then*

$$Re\left\{\frac{f'(z)}{f'_\tau(z)}\right\} \geq 1-\frac{\tau+1}{d_{\tau+1}} \tag{6.12}$$

and

$$Re\left\{\frac{f'_\tau(z)}{f'(z)}\right\} \geq \frac{d_{\tau+1}}{\tau+1+d_{\tau+1}}, \tag{6.13}$$

where d_k is defined by (6.7)

Proof. By setting

$$g(z) = d_{\tau+1}\left\{\frac{f'(z)}{f'_\tau(z)} - \left(1 - \frac{\tau+1}{d_{\tau+1}}\right)\right\}, \quad z \in \Delta,$$

and

$$h(z) = [(\tau+1) + d_{\tau+1}]\left\{\frac{f'_\tau(z)}{f'(z)} - \frac{d_{\tau+1}}{\tau+1+d_{\tau+1}}\right\}, \quad z \in \Delta,$$

the proof is analogous to that of Theorem 6.2, and we omit the details. □

Remark 3. *For the function f be defined by* (1.5) *belongs to* $\mathcal{W}_m^{\lambda,q,\rho}(\beta_i, \alpha, \eta, A, B)$ *one can easily prove integral means and partial sum results on lines similar to above theorems.*

Conclusions

In this paper, we used the concept of $q-$confluent hypergeometric distribution , and we defined and investigated a new subclass $\mathcal{H}_{b,c,m}^{\lambda,q,\rho}(\beta, \alpha, \eta, A, B)$ of analytic functions in the open unit disk Δ. We also derived several properties and characteristics of newly defined subclasses of analytic functions such as coefficients estimates,necessary and sufficient conditions, closure theorems, convolution properties and results on Hölder's inequalities, integral means and partial sums results. We have highlighted some consequences of our main results as corollaries.

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