

## PRIME AND SEMIPRIME RINGS INVOLVING MULTIPLICATIVE (GENERALIZED)-SKEW DERIVATIONS

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ABSTRACT. In this article, we will cover a concept called multiplicative (generalized) skew derivation on rings, and we will generalize some of the important results in the literature. After, we enrich this paper with examples which show that our used assumptions are essential.

### 1. INTRODUCTION

Throughout the present paper,  $\mathcal{R}$  denotes an associative ring with center  $Z(\mathcal{R})$ . Recall that  $\mathcal{R}$  is semiprime if for  $a \in \mathcal{R}$ ,  $a\mathcal{R}a = \{0\}$  implies  $a = 0$  and is prime if for  $a, b \in \mathcal{R}$ ,  $a\mathcal{R}b = 0$  implies either  $a = 0$  or  $b = 0$ . In a prime ring  $\mathcal{R}$ ,  $Q_r(\mathcal{R})$  ( $Q_s(\mathcal{R})$ ) denote the right (symmetric) Martindale ring of quotients, respectively. The center of  $Q_r(\mathcal{R})$  is denoted by  $C$  and is called the extended centroid of  $\mathcal{R}$ . It is clear that  $C$  is also the center of  $Q_s(\mathcal{R})$ . The ring  $\mathcal{R}_C = \mathcal{R}C$  is a prime ring and is called the central closure of  $\mathcal{R}$ . Hence, one can construct the ring  $Q_r(\mathcal{R}C)$ . For more details see [6]. An additive mapping  $d : \mathcal{R} \rightarrow \mathcal{R}$  is said to be a derivation on  $\mathcal{R}$  if  $d(xy) = d(x)y + xd(y)$  for all  $x, y \in \mathcal{R}$ . Derivations appeared in many papers as in [7] and [13]. Also, an additive mapping  $F : \mathcal{R} \rightarrow \mathcal{R}$  is called a generalized derivation with associated derivation  $d$  if  $F(xy) = F(x)y + xd(y)$  for all  $x, y \in \mathcal{R}$ . Recalling that generalized derivation was initially introduced by Brešar in [11]. It is straightforward to see that every derivation is a generalized derivation, but there exist generalized derivations which are not derivations. There has been a great deal of work concerning

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the commutativity of rings admitting derivations or generalized derivations satisfying certain polynomial conditions (see for references [1, 3, 4, 5, 8] where more references can be found).

The notion of derivation were extended to the notion of skew derivation as follows: An additive map  $D : \mathcal{R} \rightarrow \mathcal{R}$  is called skew derivations if  $D(xy) = D(x)y + \alpha(x)D(y)$  for all  $x, y \in \mathcal{R}$ , where  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  is an automorphism. The concept of skew derivations appeared in many papers as in [2]. Also the concepts of derivation, generalized derivation and skew derivation were extended to the concept of a generalized skew derivation as follows : An additive map  $F : \mathcal{R} \rightarrow \mathcal{R}$  is called a generalized skew derivation if  $F(xy) = F(x)y + \alpha(x)D(y)$  holds for all  $x, y \in \mathcal{R}$ , where  $D$  is a skew derivation and  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  is an automorphism.

A multiplicative derivation of  $\mathcal{R}$  is a mapping  $D : \mathcal{R} \rightarrow \mathcal{R}$  (not necessary additive) which satisfies  $D(xy) = D(x)y + xD(y)$  for all  $x, y \in \mathcal{R}$ . Then a multiplicative derivation will be a derivation when it is an additive map. The notion of multiplicative derivation was extended to the notion of multiplicative generalized derivation as follows: a mapping  $F : \mathcal{R} \rightarrow \mathcal{R}$  is called a multiplicative generalized derivation if there exists a derivation  $d$  of  $\mathcal{R}$  such that  $F(xy) = F(x)y + xd(y)$  for all  $x, y \in \mathcal{R}$ . Now if we take  $d$  as any map (not necessarily additive), then it is more sensible to call  $F$  a multiplicative (generalized) derivation.

In [12], the concept of multiplicative (generalized)-derivation was extended to the concept of multiplicative (generalized)-skew derivations over rings as follows: A mapping  $F : \mathcal{R} \rightarrow \mathcal{R}$  (not necessarily additive) is called a multiplicative (generalized)-skew derivation of  $\mathcal{R}$  if it satisfies  $F(xy) = F(x)y + \alpha(x)h(y) = F(x)\alpha(y) + xh(y)$  for all  $x, y \in \mathcal{R}$ , where  $h : \mathcal{R} \rightarrow \mathcal{R}$  is any map (not necessarily additive) and  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  is an automorphism of  $\mathcal{R}$ . Every multiplicative (generalized) derivation is a multiplicative (generalized) skew derivation but converse may not be true in general (see [[15], Example 1.1]. Hence, in this more general setting, the results obtained for multiplicative (generalized) skew derivation cover the results proved for multiplicative (generalized)derivation.

For any  $x, y \in \mathcal{R}$ , the symbol  $[x, y]$  will denote the commutator  $xy - yx$  and the symbol  $x \circ y$  will represent the anti commutator  $xy + yx$ . Now we denote for all  $x, y \in \mathcal{R}$ ,  $[x, y]_\alpha = x\alpha(y) - yx$  and  $(x \circ y)_\alpha = x\alpha(y) + yx$ . In particular, if  $\alpha = id_{\mathcal{R}}$  (the identity map on  $\mathcal{R}$ ), then  $[x, y]_{id_{\mathcal{R}}} = [x, y]$  and  $(x \circ y)_{id_{\mathcal{R}}} = x \circ y$  for all  $x, y \in \mathcal{R}$ . In [9], Dhara and Pradhan studied the cases  $a(F(xy) \pm xy) = 0$ ,  $a(F(xy) \pm yx) = 0$ ,  $a(F(x)F(y) \pm xy) = 0$ ,  $a(F(x)F(y) \pm yx) = 0$ ,  $a(d(x)F(y) \pm xy) \in Z(\mathcal{R})$ ,  $a(d(x)F(y) \pm yx) \in Z(\mathcal{R})$  and  $a(F(xy) \pm F(x)F(y)) = 0$  for all  $x, y \in I$ , where  $0 \neq a \in \mathcal{R}$ ,  $I$  is a nonzero ideal of  $\mathcal{R}$  and  $F$  is a multiplicative (generalized) derivation on the prime ring  $\mathcal{R}$  associated with the map  $d : \mathcal{R} \rightarrow \mathcal{R}$ .

In the present paper, we consider the above identities for multiplicative (generalized) skew derivation of  $\mathcal{R}$  and obtain results which generalize Theorems 3.1, 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7 of [9].

## 2. SOME PRELIMINARIES

We begin with the following known lemma. We shall use basic commutator identities without any specific mention.

For all  $x, y, t \in \mathcal{R}$ ,

$$[xy, t] = x[y, t] + [x, t]y \text{ and } [x, yt] = y[x, t] + [x, y]t.$$

**Lemma 2.1.** ([7], Lemma 2) *If  $\mathcal{R}$  is prime with a nonzero central ideal, then  $\mathcal{R}$  is commutative.*

**Lemma 2.2.** *Let  $\mathcal{R}$  be a prime ring,  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  be an automorphism and  $I$  be a nonzero ideal of  $\mathcal{R}$ . If  $a[\alpha(x), a]_\alpha = 0$  for all  $x \in I$ , where  $a \in \mathcal{R} \setminus \{0\}$ , then  $a \in Z(\mathcal{R})$ .*

*Proof.* Assume that  $a[\alpha(x), a]_\alpha = 0$  for all  $x \in I$ , where  $a \in \mathcal{R} \setminus \{0\}$ . Then  $a\alpha(x)\alpha(a) = aa\alpha(x)$  for all  $x \in I$ . Replacing  $x$  by  $xr$ , where  $r \in \mathcal{R}$ , in the last expression and use it to get  $a\alpha(x)\alpha(a)\alpha(r) = a\alpha(x)\alpha(r)\alpha(a)$  for all  $x \in I$ ,  $r \in \mathcal{R}$ . Thus  $a\alpha(x)[\alpha(a), \alpha(r)] = 0$  for all  $x \in I$ ,  $r \in \mathcal{R}$ . Taking  $r_1xr_2$  where  $r_1, r_2 \in \mathcal{R}$  and using the fact that  $\alpha$  is an automorphism, we obtain  $a\mathcal{R}\alpha(x)\mathcal{R}[\alpha(a), \alpha(r)] = \{0\}$  for all  $x \in I$ ,  $r \in \mathcal{R}$ . Since  $\mathcal{R}$  is prime,  $I \neq \{0\}$  and  $a \neq 0$ ,  $[\alpha(a), \alpha(r)] = 0$  for all

$r \in \mathcal{R}$ . But  $\alpha$  is an automorphism, and hence we have  $[a, r] = 0$  for all  $r \in \mathcal{R}$ , i.e.,  $a \in Z(\mathcal{R})$ .  $\square$

**Lemma 2.3.** *Let  $\mathcal{R}$  be a prime ring,  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  be an automorphism and  $I$  be a nonzero ideal of  $\mathcal{R}$ . If  $[\alpha(x), u]_\alpha = 0$  for all  $x \in I, u \in \mathcal{R}$ , then  $\mathcal{R}$  is commutative.*

*Proof.* Suppose that  $[\alpha(x), u]_\alpha = 0$  for all  $x \in I, u \in \mathcal{R}$ . Then  $\alpha(x)\alpha(u) = u\alpha(x)$  for all  $x \in I, u \in \mathcal{R}$ . Substituting  $xr$  for  $x$ , where  $r \in \mathcal{R}$ , we get  $\alpha(x)\alpha(r)\alpha(u) = \alpha(x)\alpha(u)\alpha(r)$  for all  $x \in I, r, u \in \mathcal{R}$ . Hence we have  $\alpha(x)[\alpha(u), \alpha(r)] = 0$  for all  $x \in I, r, u \in \mathcal{R}$ . Since  $\mathcal{R}$  is prime,  $[\alpha(u), \alpha(r)] = 0$  for all  $r, u \in \mathcal{R}$ . But since  $\alpha$  is an automorphism, we obtain  $[u, r] = 0$  for all  $r, u \in \mathcal{R}$ , i.e.,  $\mathcal{R}$  is commutative.  $\square$

**Lemma 2.4.** *Let  $\mathcal{R}$  be a prime ring and  $I$  be a nonzero ideal of  $\mathcal{R}$ . If  $d : \mathcal{R} \rightarrow \mathcal{R}$  is a skew derivation of  $\mathcal{R}$  associated with the automorphism  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  such that  $[d(x), x] = 0$  for all  $x \in I$ , then  $d = 0$  or  $\mathcal{R}$  is commutative.*

*Proof.* Suppose that  $[d(x), x] = 0$  for all  $x \in I$ . Linearizing the above relation, we obtain

$$(2.1) \quad [d(x), y] + [d(y), x] = 0 \quad \text{for all } x, y \in I.$$

Substituting  $yx$  for  $y$  in (2.1), we obtain

$$(2.2) \quad [d(x), yx] + [d(y)x + \alpha(y)d(x), x] = 0 \quad \text{for all } x, y \in I.$$

By (2.2), we obtain

$$(2.3) \quad [d(x), y]x + [d(y), x]x + [\alpha(y)d(x), x] = 0 \quad \text{for all } x, y \in I.$$

By hypothesis, (2.3) becomes  $[\alpha(y)d(x), x] = 0$  for all  $x, y \in I$  which can be rewritten as  $\alpha(y)d(x)x = x\alpha(y)d(x)$  for all  $x, y \in I$ . Replacing  $y$  by  $\alpha^{-1}(t)y$ , where  $t \in I$ , in the last relation and using it to get  $tx\alpha(y)d(x) = xt\alpha(y)d(x)$  for all  $x, y, t \in I$ . Hence  $[x, t]\alpha(y)d(x) = 0$  for all  $x, y, t \in I$ . Since  $\mathcal{R}$  is prime, we find that

$$(2.4) \quad [x, t] = 0 \quad \text{or} \quad d(x) = 0 \quad \text{for all } x, t \in I.$$

Suppose there exists  $x_0 \in I$  such that  $[x_0, t] = 0$  for all  $t \in I$ . Replace  $t$  by  $rt$ , where  $r \in \mathcal{R}$ , to obtain  $[x_0, r]t = 0$  for all  $t \in I, r \in \mathcal{R}$  which implies that  $[x_0, r]RI = \{0\}$  for all  $r \in \mathcal{R}$ . Since  $\mathcal{R}$  is prime,  $x_0 \in Z(\mathcal{R})$ . In this case, (2.3) becomes  $[d(x_0), y]x_0 = 0$

for all  $y \in I$ , and hence  $[d(x_0), y]\mathcal{R}x_0 = \{0\}$  for all  $y \in I$ . Using the primeness of  $\mathcal{R}$  again, we arrive at  $d(x_0)y = yd(x_0)$  for all  $y \in I$ . Thus (2.4) forces that  $d(x)y = yd(x)$  for  $x, y \in I$ . Replacing  $x$  by  $xt$ , we get

$$d(x)ty + \alpha(x)d(t)y = yd(x)t + y\alpha(x)d(t) \quad \text{for all } x, y, t \in I.$$

This implies

$$[d(x), y]t = [\alpha(x), y]d(t) \quad \text{for all } x, y, t \in I.$$

Taking  $tr$  in place of  $t$ , where  $r \in \mathcal{R}$ , we obtain  $[\alpha(x), y]\alpha(t)d(r) = 0$  for all  $x, y, t \in I$ ,  $r \in \mathcal{R}$ . By primeness of  $\mathcal{R}$ , we arrive at  $d = 0$  or  $[\alpha(x), y] = 0$  for all  $x, y \in I$ . But  $\alpha$  is an automorphism, we conclude that either  $d = 0$  or  $\mathcal{R}$  is commutative.  $\square$

**Lemma 2.5.** *Let  $\mathcal{R}$  be a prime ring,  $I$  be a nonzero ideal of  $\mathcal{R}$  and  $a \in \mathcal{R} \setminus \{0\}$ . Suppose that  $d : \mathcal{R} \rightarrow \mathcal{R}$  is a skew derivation of  $\mathcal{R}$  with associated automorphism  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  such that  $\alpha \neq id_{Z(\mathcal{R})}$ . If  $[ad(x), x] = 0$  for all  $x \in I$ , then  $d = 0$  or  $\mathcal{R}$  is commutative.*

*Proof.* Assume that  $[ad(x), x] = 0$  for all  $x \in I$ . Linearizing the above relation, we get

$$(2.5) \quad [ad(x), y] + [ad(y), x] = 0 \quad \text{for all } x, y \in I.$$

Substituting  $yx$  for  $y$  in (2.5), we obtain

$$[ad(x), yx] + [ad(y)x + a\alpha(y)d(x), x] = 0 \quad \text{for all } x, y \in I.$$

By hypothesis, we get

$$[ad(x), y]x + [ad(y), x]x + [a\alpha(y)d(x), x] = 0 \quad \text{for all } x, y \in I.$$

By (2.5), we obtain  $[a\alpha(y)d(x), x] = 0$  for all  $x, y \in I$ . Replacing  $y$  with  $\alpha^{-1}(a)y$ , we get

$$\begin{aligned} 0 &= [aa\alpha(y)d(x), x] \\ &= a[a\alpha(y)d(x), x] + [a, x]a\alpha(y)d(x) \\ &= [a, x]a\alpha(y)d(x) \quad \text{for all } x, y \in I. \end{aligned}$$

Since  $\mathcal{R}$  is prime and  $\alpha$  is an automorphism, we get

$$(2.6) \quad [a, x]a = 0 \text{ or } d(x) = 0 \text{ for all } x \in I.$$

Suppose there exists  $x_0 \in I$  such that  $d(x_0) = 0$ . Replacing  $y$  by  $yx_0$  in (2.5) and using it again, we arrive at

$$(2.7) \quad y[ad(x), x_0] + ad(y)[x_0, x] = 0 \text{ for all } x, y \in I.$$

Taking  $yz$  in place of  $y$ , where  $z \in Z(\mathcal{R})$ , in (2.7) and using it again, we obtain  $a\alpha(y)d(z)[x_0, x] = 0$  for all  $x, y \in I$ . Since  $\mathcal{R}$  is prime,  $a \in Z(\mathcal{R}) \setminus \{0\}$  and  $\alpha$  is an automorphism, we find that  $d(z)[x_0, x] = 0$  for all  $x \in I, z \in Z(\mathcal{R})$ . Putting  $xt$  in place of  $x$  in the last expression and developing it, we arrive at  $d(z)x[x_0, t] = 0$  for all  $x \in I, t \in \mathcal{R}, z \in Z(\mathcal{R})$  which implies that  $d(Z(\mathcal{R}))I[x_0, t] = \{0\}$  for all  $t \in \mathcal{R}$ . Since  $\mathcal{R}$  is prime, either  $d(Z(\mathcal{R})) = \{0\}$  or  $x_0 \in Z(\mathcal{R})$ . For  $z \in Z(\mathcal{R})$ , we have  $zt = tz$  for all  $t \in \mathcal{R}$  which implies that  $\alpha(z)d(t) = d(t)z$  for all  $t \in \mathcal{R}$ , and hence  $(\alpha(z) - z)d(t) = 0$  for all  $t \in \mathcal{R}$ . Using the primeness of  $\mathcal{R}$  and the fact that  $\alpha$  is an automorphism, we arrive at  $\alpha(z) = z$  for all  $z \in Z(\mathcal{R})$ . In all cases, we get  $\alpha = id_{Z(\mathcal{R})}$  or  $x_0 \in Z(\mathcal{R})$ . In this case, (2.6) becomes  $\alpha = id_{Z(\mathcal{R})}$  or  $[a, x]a = 0$  for all  $x \in I$ . Replacing  $x$  by  $rx$  in the second case, we arrive at  $[a, r]Ia = \{0\}$  for all  $r \in \mathcal{R}$  and using the primeness of  $\mathcal{R}$ , we find that  $a \in Z(\mathcal{R})$ . In this case our hypothesis together with primeness of  $\mathcal{R}$  force that  $a = 0$  or  $[d(x), x] = 0$  for all  $x \in I$ . Since  $a \neq 0$ , by Lemma 2.4, we conclude that  $d = 0$  or  $\mathcal{R}$  is commutative.  $\square$

### 3. MULTIPLICATIVE (GENERALIZED)-SKEW DERIVATIONS IN SEMIPRIME RINGS

The following theorem is a generalization of Theorem 3.1 in [9].

**Theorem 3.1.** *Let  $\mathcal{R}$  be a prime ring,  $I$  be a nonzero ideal of  $\mathcal{R}$  and  $a \in \mathcal{R} \setminus \{0\}$ . If  $\mathcal{R}$  admits a multiplicative (generalized)-skew derivation  $F : \mathcal{R} \rightarrow \mathcal{R}$  associated with a map  $d : \mathcal{R} \rightarrow \mathcal{R}$  and an automorphism  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  such that  $a(F(xy) \pm xy) = 0$  for all  $x, y \in I$ , then  $aF = \pm id_{\mathcal{R}}$ .*

*Proof.* Assume that

$$(3.1) \quad a(F(xy) \pm xy) = 0 \text{ for all } x, y \in I.$$

Replacing  $y$  by  $yz$ , where  $z \in I$ , in (3.1), we get

$$\begin{aligned}
 0 &= a(F(xyz) \pm xyz) \\
 &= a(F(xy)z + \alpha(xy)d(z) \pm xyz) \\
 &= a((F(xy) \pm xy)z + \alpha(xy)d(z)) \\
 &= a(F(xy) \pm xy)z + a\alpha(xy)d(z) \\
 &= a\alpha(xy)d(z) \quad \text{for all } x, y, z \in I.
 \end{aligned}$$

But  $\alpha$  is an automorphism,  $a \neq 0$  and  $\mathcal{R}$  is prime so we have  $d(z) = 0$  for all  $z \in I$ . Hence from (3.1), we get

$$\begin{aligned}
 0 &= a(F(xy) \pm xy) \\
 &= a(F(x)y \pm xy) \\
 &= a(F(x) \pm x)y \quad \text{for all } x, y \in I.
 \end{aligned}$$

By primeness of  $\mathcal{R}$ , we arrive at  $a(F(x) \pm x) = 0$  for all  $x \in I$ . Therefore, it gives  $0 = a(F(ux) \pm ux) = a(F(u) \pm u)x$  for all  $u \in \mathcal{R}$  and  $x \in I$ . Thus, we conclude that  $a(F(u) \pm u) = 0$  for all  $u \in \mathcal{R}$ .  $\square$

Note that if we put  $\alpha = id_{\mathcal{R}}$  in Theorem 3.1, we obtain [[9], Theorem 3.1].

The next theorem is a generalization of Theorem 3.2 of [9].

**Theorem 3.2.** *Let  $\mathcal{R}$  be a prime ring,  $I$  be a nonzero ideal of  $\mathcal{R}$  and  $a \in \mathcal{R} \setminus \{0\}$ . If  $\mathcal{R}$  admits a multiplicative (generalized)-skew derivation  $F : \mathcal{R} \rightarrow \mathcal{R}$  associated with a map  $d : \mathcal{R} \rightarrow \mathcal{R}$  and an automorphism  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  such that  $a(F(xy) \pm yx) = 0$  for all  $x, y \in I$ , then  $\mathcal{R}$  is commutative and  $F = \pm id_{\mathcal{R}}$ .*

*Proof.* Suppose that

$$(3.2) \quad a(F(xy) \pm yx) = 0 \quad \text{for all } x, y \in I.$$

Substitute  $yz$  for  $y$ , where  $z \in I$ , we get

$$\begin{aligned}
 0 &= a(F(xyz) \pm yzx) \\
 &= a(F(xy)z + \alpha(xy)d(z) \pm yzx) \\
 &= a(F(y) \pm yx)z \mp y[x, z] + \alpha(xy)d(z) \quad \text{for all } x, y, z \in I.
 \end{aligned}$$

Using (3.2), we obtain

$$(3.3) \quad a(\alpha(xy)d(z) \mp y[x, z]) = 0 \quad \text{for all } x, y, z \in I.$$

Replacing  $y$  by  $ay$  in (3.3), we find that

$$(3.4) \quad a(\alpha(xay)d(z) \mp ay[x, z]) = 0 \quad \text{for all } x, y, z \in I.$$

Left-multiplying (3.3) by  $a$ , we have

$$(3.5) \quad a(a\alpha(xy)d(z) \mp ay[x, z]) = 0 \quad \text{for all } x, y, z \in I.$$

Subtracting (3.5) from (3.4), we get

$$(3.6) \quad a[\alpha(x), a]_{\alpha} \alpha(y)d(z) = 0 \quad \text{for all } x, y, z \in I.$$

Since  $\alpha$  is an automorphism and  $\mathcal{R}$  is prime,  $a[\alpha(x), a]_{\alpha} = 0$  for all  $x \in I$  or  $d(z) = 0$  for all  $z \in I$ . Using the first case together with Lemma 2.3, we have  $a \in Z(\mathcal{R})$ . But the center of the prime ring does not contain a zero divisor, and hence from (3.3), we obtain

$$(3.7) \quad \alpha(xy)d(z) \mp y[x, z] = 0 \quad \text{for all } x, y, z \in I.$$

Putting  $uy$  in place of  $y$ , where  $u \in \mathcal{R}$ , in (3.7), we get

$$(3.8) \quad \alpha(xuy)d(z) \mp uy[x, z] = 0 \quad \text{for all } x, y, z \in I.$$

Left-multiplying (3.7) by  $u$ , we obtain

$$(3.9) \quad u\alpha(xy)d(z) \mp uy[x, z] = 0 \quad \text{for all } x, y, z \in I.$$

Subtracting (3.9) from (3.8), we get

$$(3.10) \quad [\alpha(x), u]_{\alpha} \alpha(y)d(z) = 0 \quad \text{for all } x, y, z \in I.$$

Again since  $\alpha$  is an automorphism and  $\mathcal{R}$  is prime, we get  $[\alpha(x), u]_{\alpha} = 0$  for all  $x \in I, u \in \mathcal{R}$  or  $d(z) = 0$  for all  $z \in I$ . If  $[\alpha(x), u]_{\alpha} = 0$  for all  $x \in I, u \in \mathcal{R}$ , then  $\mathcal{R}$  is commutative by Lemma 2.4. In this case, the fact that  $a \neq 0$  with Theorem 3.1 force that  $F = \pm id_{\mathcal{R}}$ . If  $d(I) = \{0\}$ , then by (3.3), we have  $ay[x, z] = 0$  for all  $x, y, z \in I$ . Using the primeness of  $\mathcal{R}$  with  $a \neq 0$ , we get  $[x, z] = 0$  for all  $x, z \in I$  which forces that  $\mathcal{R}$  is commutative.  $\square$



Notice that if we put  $\alpha = id_{\mathcal{R}}$  in the previous theorem, we obtain [[9], Theorem 3.2].

The following theorem is a generalization of Theorem 3.3 of [9].

**Theorem 3.3.** *Let  $\mathcal{R}$  be a prime ring,  $I$  be a nonzero ideal of  $\mathcal{R}$  and  $a \in \mathcal{R} \setminus \{0\}$ . If  $\mathcal{R}$  admits a multiplicative (generalized)-skew derivation  $F : \mathcal{R} \rightarrow \mathcal{R}$  associated with a map  $d : \mathcal{R} \rightarrow \mathcal{R}$  and an automorphism  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  such that  $a(F(x)F(y) \pm xy) = 0$  for all  $x, y \in I$ , then  $F(xy) = F(x)y$  for all  $x, y \in \mathcal{R}$  and  $[F(y), y] = 0$  for all  $y \in I$ .*

*Proof.* Suppose that

$$(3.11) \quad a(F(x)F(y) \pm xy) = 0 \quad \text{for all } x, y \in I.$$

Substitute  $yt$  for  $y$ , where  $t \in I$ , in (3.11) and use it to get

$$\begin{aligned} 0 &= a(F(x)F(yt) \pm xyt) \\ &= a(F(x)(F(y)t + \alpha(y)d(t)) \pm xyt) \\ &= a((F(x)F(y) \pm xy)t + F(x)\alpha(y)d(t)) \\ &= aF(x)\alpha(y)d(t) \quad \text{for all } x, y \in I, t \in I. \end{aligned}$$

But since  $\alpha$  is an automorphism on  $\mathcal{R}$  and  $\mathcal{R}$  is prime, we get either  $aF(I) = \{0\}$  or  $d(\mathcal{R}) = \{0\}$ . Using the first case with (3.11), we obtain  $axy = 0$  for all  $x, y \in I$ . This implies that  $a = 0$  which is a contradiction and thus  $d(\mathcal{R}) = \{0\}$ . Hence, we have  $F(xy) = F(x)y$  for all  $x, y \in \mathcal{R}$ . Again substitute  $xy$  for  $x$  in (3.11), to get

$$(3.12) \quad a(F(xy)F(y) \pm xy^2) = 0 \quad \text{for all } x, y \in I.$$

Equivalently,

$$(3.13) \quad a(F(x)yF(y) \pm xy^2) = 0 \quad \text{for all } x, y \in I.$$

Right-Multiplying (3.11) by  $y$  and subtracting from (3.13), we get  $aF(x)[F(y), y] = 0$  for all  $x, y \in I$ . Again replace  $x$  with  $xr$  in the last relation to get  $aF(x)r[F(y), y] = 0$  for all  $x, y \in I, r \in \mathcal{R}$ . By primeness of  $\mathcal{R}$ , we have either  $aF(x) = 0$  for all  $x \in I$ , or  $[F(y), y] = 0$  for all  $y \in I$ . But as above  $aF(I) = \{0\}$  leads to a contradiction,  $a = 0$ , and hence we obtain that  $[F(y), y] = 0$  for all  $y \in I$ .  $\square$

If we put  $\alpha = id_{\mathcal{R}}$  in the previous theorem, we obtain [[9], Theorem 3.3].

The next theorem is a generalization of Theorem 3.4 of [9].

**Theorem 3.4.** *Let  $\mathcal{R}$  be a prime ring,  $I$  be a nonzero ideal of  $\mathcal{R}$  and  $a \in \mathcal{R} \setminus \{0\}$ . If  $\mathcal{R}$  admits a multiplicative (generalized)-skew derivation  $F : \mathcal{R} \rightarrow \mathcal{R}$  associated with a map  $d : \mathcal{R} \rightarrow \mathcal{R}$  and an automorphism  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  such that  $a(F(x)F(y) \pm yx) = 0$  for all  $x, y \in I$ , then  $\mathcal{R}$  is commutative and  $F(xy) = F(x)y$  for all  $x, y \in \mathcal{R}$ .*

*Proof.* Assume that

$$(3.14) \quad a(F(x)F(y) \pm yx) = 0 \quad \text{for all } x, y \in I.$$

Substituting  $yx$  for  $y$  in (3.14) and using it, we obtain

$$\begin{aligned} 0 &= a(F(x)F(yx) \pm yx^2) \\ &= a(F(x)(F(y)x + \alpha(y)d(x)) \pm yx^2) \\ &= a((F(x)F(y) \pm yx)x + F(x)\alpha(y)d(x)) \\ &= aF(x)\alpha(y)d(x) \quad \text{for all } x, y \in I. \end{aligned}$$

Using the primeness of  $\mathcal{R}$  with the fact that  $\alpha$  is an automorphism, we find that  $aF(I) = \{0\}$  or  $d(x) = 0$  for all  $x \in I$ . But as above  $aF(I) = \{0\}$  leads to a contradiction, and hence  $d(x) = 0$  for all  $x \in I$ . By definition of  $F$ , we have  $F(xy) = F(x)\alpha(y) + xd(y)$  for all  $x, y \in \mathcal{R}$ . If we put  $x = y = 0$ , then  $F(0) = 0$ . Also for  $y = 0$  we get  $F(0) = 0 = xd(0)$  for all  $x \in I$ . Using the primeness of  $\mathcal{R}$ , we arrive at  $d(0) = 0$ . In all cases, we have  $d(I) = \{0\}$  and  $F(xy) = F(x)y$  for all  $x, y \in I$ . Replacing  $y$  by  $yt$ , where  $t \in I$ , in (3.14), we get

$$(3.15) \quad a(F(x)F(y)t \pm ytx) = 0 \quad \text{for all } x, y, t \in I.$$

Right-multiplying (3.14) by  $t$  and then subtracting from (3.15), we get that  $ay[x, t] = 0$  for all  $x, y, t \in I$ . But since  $\mathcal{R}$  is prime and  $a \neq 0$ , we have  $[x, t] = 0$  for all  $x, t \in I$ , and hence  $\mathcal{R}$  is commutative. Since  $a \in \mathcal{R} \setminus \{0\}$ , (3.14) becomes  $F(x)F(y) \pm xy = 0$

for all  $x, y \in I$ . Substitute  $yr$  for  $y$  where  $r \in \mathcal{R}$  in the last relation, we obtain

$$\begin{aligned} 0 &= F(x)F(yr) \pm yrx \\ &= F(x)(F(y)r + \alpha(y)d(r)) \pm xy r \\ &= (F(x)F(y) \pm yx)r + F(x)\alpha(y)d(r) \\ &= F(x)\alpha(y)d(r) \quad \text{for all } x, y \in I, r \in \mathcal{R}. \end{aligned}$$

Since  $\alpha$  is an automorphism and  $\mathcal{R}$  is prime,  $F(x) = 0$  for all  $x \in I$  or  $d(r) = 0$  for all  $r \in \mathcal{R}$ . If  $F(I) = \{0\}$ , and hence from (3.14), we get  $axy = 0$  for all  $x, y \in I$ . This implies that  $a = 0$  which is a contradiction. Hence  $d(\mathcal{R}) = \{0\}$ , and  $F(xy) = F(x)y$  for all  $x, y \in \mathcal{R}$ .  $\square$

Notice that if we put  $\alpha = id_{\mathcal{R}}$  in Theorem 3.4, we obtain [[9], Theorem 3.4].

The following theorem is a generalization of Theorem 3.5 of [9], just take  $\alpha = id_{\mathcal{R}}$ .

**Theorem 3.5.** *Let  $\mathcal{R}$  be a semiprime ring,  $I$  be a nonzero ideal of  $\mathcal{R}$  and  $a \in \mathcal{R} \setminus \{0\}$ . If  $\mathcal{R}$  admits a multiplicative (generalized)-skew derivation  $F : \mathcal{R} \rightarrow \mathcal{R}$  associated with a map  $d : \mathcal{R} \rightarrow \mathcal{R}$  and an automorphism  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  such that  $a(d(x)F(y) \pm xy) \in Z(\mathcal{R})$  for all  $x, y \in I$ , then  $[ad(x), x] = 0$  for all  $x \in I$ . Moreover, if  $\mathcal{R}$  is a prime ring,  $d$  is a skew derivation of  $\mathcal{R}$  and  $\alpha \neq id_{Z(\mathcal{R})}$ , then  $\mathcal{R}$  is commutative.*

*Proof.* Suppose that

$$(3.16) \quad a(d(x)F(y) \pm xy) \in Z(\mathcal{R}) \quad \text{for all } x, y \in I.$$

Substitute  $yx$  in place of  $y$  in (3.16), we obtain

$$(3.17) \quad a(d(x)(F(y)x + \alpha(y)d(x)) \pm xyx) \in Z(\mathcal{R}) \quad \text{for all } x, y \in I.$$

This yields that

$$(3.18) \quad a(d(x)F(y) \pm xy)x + ad(x)\alpha(y)d(x) \in Z(\mathcal{R}) \quad \text{for all } x, y \in I.$$

But since  $a(d(x)F(y) \pm xy) \in Z(\mathcal{R})$  for all  $x, y \in I$ , we have  $ad(x)\alpha(y)d(x) \in Z(\mathcal{R})$  for all  $x, y \in I$ . Thus,  $[ad(x)\alpha(y)d(x), x] = 0$  for all  $x, y \in I$  which implies that

$$(3.19) \quad ad(x)\alpha(y)d(x)x = xad(x)\alpha(y)d(x) \quad \text{for all } x, y \in I.$$

Substituting  $y\alpha^{-1}(ad(x))z$  for  $y$ , where  $z \in I$ , in (3.19), we get

$$(3.20) \quad ad(x)\alpha(y)ad(x)\alpha(z)d(x)x = xad(x)\alpha(y)ad(x)\alpha(z)d(x) \quad \text{for all } x, y, z \in I.$$

Again by using (3.19), we get

$$ad(x)\alpha(y)xad(x)\alpha(z)d(x) = ad(x)\alpha(y)ad(x)x\alpha(z)d(x) \quad \text{for all } x, y, z \in I,$$

and hence

$$ad(x)\alpha(y)[ad(x), x]\alpha(z)d(x) = 0 \quad \text{for all } x, y, z \in I.$$

This implies that

$$[ad(x), x]\alpha(y)[ad(x), x]\alpha(z)[ad(x), x] = 0 \quad \text{for all } x, y, z \in I.$$

Hence, we find that  $([ad(x), x]\alpha(I))^3 = \{0\}$ . Since  $\mathcal{R}$  is semiprime,  $[ad(x), x]\alpha(I) = \{0\}$ . Moreover, if  $\mathcal{R}$  is prime, then  $d = 0$  or  $\mathcal{R}$  is commutative by Lemma 2.5. If  $d = 0$  then our hypothesis become  $axy = 0$  for all  $x, y \in I$ . Hence  $[axy, t] = 0$  for all  $x, y \in I$  and  $t \in \mathcal{R}$ . Thus,  $0 = [axy, t] = axy[r, t]$  for all  $x, y \in I$  and  $r, t \in \mathcal{R}$ . But since  $\mathcal{R}$  is prime and  $a \neq 0$ , we find that  $[r, t] = 0$  for all  $r, t \in \mathcal{R}$ , and  $\mathcal{R}$  is commutative.  $\square$

The next theorem is a generalization of Theorem 3.6 of [9].

**Theorem 3.6.** *Let  $\mathcal{R}$  be a semiprime ring,  $I$  be a nonzero ideal of  $\mathcal{R}$  and  $a \in \mathcal{R} \setminus \{0\}$ . If  $\mathcal{R}$  admits a multiplicative (generalized)-skew derivation  $F : \mathcal{R} \rightarrow \mathcal{R}$  associated with a map  $d : \mathcal{R} \rightarrow \mathcal{R}$  and an automorphism  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  such that  $a(d(x)F(y) \pm yx) \in Z(\mathcal{R})$  for all  $x, y \in I$ , then  $[ad(x), x] = 0$  for all  $x \in I$ . Moreover, if  $\mathcal{R}$  is a prime ring and  $d$  is a skew derivation of  $\mathcal{R}$ , then  $\mathcal{R}$  is commutative.*

*Proof.* Let

$$(3.21) \quad a(d(x)F(y) \pm yx) \in Z(\mathcal{R}) \quad \text{for all } x, y \in I.$$

Substituting  $yx$  in place of  $y$  in (3.21), we get

$$(3.22) \quad a(d(x)(F(y)x + \alpha(y)d(x)) \pm yx^2) \in Z(\mathcal{R}) \quad \text{for all } x, y \in I.$$

This is equivalent to

$$(3.23) \quad a(d(x)(F(y) \pm y)x + ad(x)\alpha(y)d(x)) \in Z(\mathcal{R}) \quad \text{for all } x, y \in I.$$

But since  $a(d(x)F(y) \pm yx) \in Z(\mathcal{R})$  for all  $x, y \in I$ , the above yields that  $ad(x)\alpha(y)d(x) \in Z(\mathcal{R})$  for all  $x, y \in I$ . Thus  $[ad(x)\alpha(y)d(x), x] = 0$  for all  $x, y \in I$ , which is the same as above. Hence, using the similar arguments as used in the proof of Theorem 3.5, we can get that  $[ad(x), x] = 0$  for all  $x \in I$ . Moreover, if  $\mathcal{R}$  is prime and  $d$  is a skew derivation of  $\mathcal{R}$ , then by Lemma 2.5, we conclude that  $d = 0$  or  $\mathcal{R}$  is commutative. Again using the same arguments as used in the proof of Theorem 3.5, we can show that  $d = 0$  implies that  $\mathcal{R}$  is commutative. Hence the proof of this theorem is complete.  $\square$

If we put  $\alpha = id_{\mathcal{R}}$  in Theorem 3.6, we obtain [[9], Theorem 3.6].

The following theorem is a generalization of Theorem 3.7 of [9].

**Theorem 3.7.** *Let  $\mathcal{R}$  be a prime ring,  $I$  be a nonzero ideal of  $\mathcal{R}$  and  $a \in \mathcal{R} \setminus \{0\}$ . If  $\mathcal{R}$  admits a multiplicative (generalized)-skew derivation  $F : \mathcal{R} \rightarrow \mathcal{R}$  associated with a map  $d : \mathcal{R} \rightarrow \mathcal{R}$  and an automorphism  $\alpha : \mathcal{R} \rightarrow \mathcal{R}$  such that  $a(F(xy) \pm F(x)F(y)) = 0$  for all  $x, y \in I$ , then one of the following holds*

- (i)  $d(\mathcal{R}) = \{0\}$  and  $aF(\mathcal{R}) = \{0\}$ ,
- (ii)  $d(\mathcal{R}) = \{0\}$  and  $F(r) = \mp r$  for all  $r \in \mathcal{R}$ .

*Proof.* Assume that

$$(3.24) \quad a(F(xy) \pm F(x)F(y)) = 0 \quad \text{for all } x, y \in I.$$

Replacing  $y$  by  $yz$ , where  $z \in I$ , in (3.24), we get

$$(3.25) \quad a(F(xy)z + \alpha(xy)d(z) \pm F(x)(F(y)z + \alpha(y)d(z))) = 0 \quad \text{for all } x, y, z \in I.$$

Then

$$(3.26) \quad a(F(xy) + \pm F(x)F(y))z + a(\alpha(xy)d(z) \pm F(x)\alpha(y)d(z)) = 0 \quad \text{for all } x, y, z \in I.$$

By (3.25) and (3.26), we obtain

$$(3.27) \quad a(\alpha(x) \pm F(x))\alpha(y)d(z) = 0 \quad \text{for all } x, y, z \in I.$$

Substitute  $xu$  for  $x$ , where  $u \in I$ , in (3.27), we get

$$(3.28) \quad a(\alpha(xu) \pm (F(x)\alpha(u) + xd(u)))\alpha(y)d(z) = 0 \quad \text{for all } x, y, z \in I.$$

Replacing  $y$  by  $uy$  in (3.27), we obtain

$$(3.29) \quad a(\alpha(x) \pm F(x))\alpha(uy)d(z) = 0 \quad \text{for all } x, y, z \in I.$$

From (3.28) and (3.29), we get

$$(3.30) \quad axd(u)\alpha(y)d(z) = 0 \quad \text{for all } x, y, z \in I.$$

Hence we find that  $aId(I)\alpha(I)d(I) = \{0\}$ . Now since  $\mathcal{R}$  is prime, we get either  $a = 0$  or  $d(I) = \{0\}$ . But  $a \neq 0$ , it is clear that  $d(I) = \{0\}$ . Then  $F(xy) = F(x)y + \alpha(x)d(y) = F(x)y$  for all  $x, y \in I$ . Hence, by (3.24) we have  $aF(x)(y \pm F(y)) = 0$  for all  $x, y \in I$ . Now for  $x = xz$  where  $z \in I$  in the last relation, we obtain  $aF(x)z(y \pm F(y)) = 0$  for all  $x, y, z \in I$ . Again by the primeness of  $\mathcal{R}$ , we get  $aF(x) = 0$  for all  $x \in I$  or  $F(y) = \mp y$  for all  $y \in I$ . First, if  $aF(I) = \{0\}$ , then we get  $\{0\} = aF(\mathcal{R}I) = aF(\mathcal{R})I$ , since  $d(I) = \{0\}$ . This implies that  $aF(\mathcal{R}) = \{0\}$ , and hence

$$\begin{aligned} \{0\} &= aF(\mathcal{R}^2) \\ &= a(F(\mathcal{R})\mathcal{R} + \alpha(\mathcal{R})d(\mathcal{R})) \\ &= a\alpha(\mathcal{R})d(\mathcal{R}) \\ &= a\mathcal{R}d(\mathcal{R}). \end{aligned}$$

But  $\mathcal{R}$  is prime and  $a \neq 0$ , we find that  $d(\mathcal{R}) = \{0\}$  and hence we get assertion (i).

Second, if  $F(y) = \mp y$  for all  $y \in I$ , then for all  $r \in \mathcal{R}$ ,  $y \in I$ , we have

$$\begin{aligned} F(ry) \pm ry &= F(r)y + \pm ry \\ &= (F(r) \pm r)y \\ &= 0. \end{aligned}$$

Again since  $\mathcal{R}$  is a prime ring, we get  $F(r) = \mp r$  for all  $r \in \mathcal{R}$ . Then we have

$$\begin{aligned} \mp rs &= F(rs) \\ &= F(r)s + \alpha(r)d(s) \\ &= \mp rs + \alpha(r)d(s) \quad \text{for all } r, s \in \mathcal{R}. \end{aligned}$$

Hence  $\alpha(r)d(s) = 0$  for all  $r, s \in \mathcal{R}$ , but since  $\mathcal{R}$  is prime and  $\alpha$  is an automorphism, we arrive at  $d(\mathcal{R}) = \{0\}$ .  $\square$

Now we introduce an example which shows that the hypothesis of primeness in the hypotheses of various theorems can not be omitted.

**Example 3.1.** Let us define  $\mathcal{R}, I$  and  $d, \alpha, F : \mathcal{R} \rightarrow \mathcal{R}$  by:

$$\mathcal{R} = \left\{ \left( \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ z & 0 & 0 \end{pmatrix} \mid x, y \in \mathbb{Z} \right) \right\}, \quad I = \left\{ \left( \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in \mathbb{Z} \right) \right\},$$

$$d \left( \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ z & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ x^2 & 0 & y^2 \\ 0 & 0 & 0 \end{pmatrix}, \quad F \left( \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ z & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix} \text{ and}$$

$$\alpha = id_{\mathcal{R}}.$$

It is clear that  $\mathcal{R}$  is not prime and  $I$  is a nonzero ideal of  $\mathcal{R}$ . Moreover,  $F$  is a multiplicative (generalized)-skew derivation of  $\mathcal{R}$  associated with the map  $d$  and the automorphism  $\alpha$ . Let  $q$  be a non zero matrix in  $\mathcal{R}$ , we can easily see that  $q(F(r) \pm r) \neq 0$  for some  $r \in \mathcal{R}$  and  $\mathcal{R}$  is not commutative. Also, one may notice that  $F(xy) \neq F(x)y$  for some  $x, y \in \mathcal{R}$ ,  $d(\mathcal{R}) \neq \{0\}$ ,  $[F(x), x] \neq 0$  for some  $x \in I$  and  $F(x) \neq \mp x$  for some  $x \in \mathcal{R}$ . Hence the primeness hypotheses in Theorems 3.1, 3.2, 3.3, 3.4 and 3.7 can not be omitted.

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