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ON 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLDS ADMITTING $*$-RICCI SOLITONS

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Abstract. The object of the present paper is to characterize 3-dimensional trans-Sasakian manifolds admitting $*$-Ricci solitons. First, 3-dimensional trans-Sasakian manifolds admitting $*$-Ricci solitons satisfying the conditions $R(\xi,.)\cdot S$, $S(\xi,.)\cdot R=0$ and $Q\cdot R=0$ are studied. Further, 3-dimensional manifolds admitting $*$-Ricci solitons satisfying certain conditions on the projective curvature tensor are considered and obtained several interesting results. Finally, the existence of $*$-Ricci solitons in a 3-dimensional trans-Sasakian manifold has been proved by a concrete example.

1. Introduction

In the Gray-Hervella classification of almost Hermitian manifolds [7], there appears a class, W_4 of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [14] if the product manifold $M\times R$ belongs to the class W_4. The class $C_6\oplus C_5$ [13] coincides with the class of trans-Sasakian structures of type (α,β). We note that trans-Sasakian structures of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are cosymplectic [1], β-Kenmotsu [10] and α-Sasakian [10], respectively. Therefore, trans-Sasakian manifolds generalize a large class of almost contact manifolds.

In 1982, Hamilton [9] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a

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Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively.
The notion of $*$-Ricci tensor on almost Hermitian manifolds was introduced by Tachibana [17]. Later, Hamada [8] studied $*$-Ricci flat real hypersurfaces of complex space forms and Blair [2] defined $*$-Ricci tensor in contact metric manifolds given by

$$S^*(X, Y) = g(Q^*X, Y) = \text{Trace}\,\{\phi \circ R(X, \phi Y)\} \tag{1.2}$$

for any vector fields X, Y on M, where Q^* is the (1,1) $*$-Ricci operator and S^* is a tensor field of type $(0, 2)$.
A Riemannian metric g on M is called a $*$-Ricci soliton, if

$$(\pounds_V g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) = 0 \tag{1.3}$$

for all vector fields X, Y on M and λ is a constant. If a trans-Sasakian manifold satisfies (1.3), then we say that the manifold admits a $*$-Ricci soliton. Recently, the $*$-Ricci solitons have been studied by various authors in several ways to a different extent such as [6, 12, 15, 18], and many others.

2. **Preliminaries**

Let M be an almost contact metric manifold [1] with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and

g is a compatible Riemannian metric such that

$$\phi^2X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(\phi X) = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y),\ g(X, \xi) = \eta(X) \tag{2.3}$$

for all vector fields $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on M. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.4}$$

for any $X, Y \in \chi(M)$.

An almost contact metric structure (ϕ, ξ, η, g) on M is called a trans-Sasakian structure [14], if $(M \times R, J, G)$ belongs to the class W_4 [7], where J is the almost complex structure on $M \times R$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ for all vector fields X on M and a smooth functions f on $M \times R$. This may be expressed by the condition

$$(\nabla_X\phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.5}$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β). From Eq.(2.5) it follows that

$$\nabla_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.6}$$

$$(\nabla_X\eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \tag{2.7}$$

where ∇ is the Levi Civita connection of g. In a 3-dimensional trans-Sasakian manifold, we have [5]

$$\begin{aligned} R(X, Y)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) \\ &\quad +2\alpha\beta((\eta(Y)\phi X - \eta(X)\phi Y) \\ &\quad +(Y\alpha)\phi X - (X\alpha)\phi Y \\ &\quad +(Y\beta)\phi^2X - (X\beta)\phi^2Y, \end{aligned} \tag{2.8}$$

$$R(\xi, X)Y = (\alpha^2 - \beta^2)(g(X, Y)\xi - \eta(Y)X) + 2\alpha\beta(g(\phi X, Y)\xi - \eta(Y)\phi X) + (Y\alpha)\phi X + g(\phi Y, X)(grad\alpha) + (Y\beta)(X - \eta(X)\xi) - g(\phi X, \phi Y)(grad\beta), \tag{2.9}$$

$$2\alpha\beta + \xi\beta = 0, \tag{2.10}$$

$$S(X, \xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - X\beta - (\phi X)\alpha, \tag{2.11}$$

where R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature of the manifold M. From [5] we know that for a 3-dimensional trans-Sasakian manifold

$$\phi(grad\alpha) = (n - 1)grad\beta. \tag{2.12}$$

Using Eq.(2.10) and Eq.(2.12),for constants α and β, we have

$$R(\xi, X)Y = (\alpha^2 - \beta^2)(g(X, Y)\xi - \eta(Y)X), \tag{2.13}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X), \tag{2.14}$$

$$R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y), \tag{2.15}$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \beta^2)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)), \tag{2.16}$$

$$S(X, \xi) = 2(\alpha^2 - \beta^2)\eta(X), \tag{2.17}$$

$$(\pounds_\xi g)(X, Y) = 2\beta(g(X, Y) - \eta(X)\eta(Y)) \tag{2.18}$$

for all $X, Y, Z \in \chi(M)$. Throughout in the paper, we are using the fact that $\alpha = \beta =$constant. We define endomorphisms $R(X, Y)$ and $X \wedge_A Y$ of $\chi(M)$ by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{2.19}$$

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \tag{2.20}$$

respectively, where A is the symmetric $(0, 2)-$tensor.

Definition 2.1. A 3-dimensional trans-Sasakian manifold is said to be an η-Einstein manifold if the Ricci tensor S is of the form [1]

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are smooth functions on the manifold. If $b = 0$, then the manifold is said to be an Einstein manifold.

Lemma 2.1. *In a 3-dimensional trans-Sasakian manifold, the following identity holds:*

$$\begin{aligned}\bar{R}(X,Y,\phi Z,\phi W) &= R(X,Y,Z,W)\\ &+(\alpha^2-\beta^2)[g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\\ &-\Phi(X,Z)\Phi(Y,W)+\Phi(Y,Z)\Phi(X,W)]\\ &+2\alpha\beta[g(Y,Z)\Phi(X,W)-g(X,Z)\Phi(Y,W)\\ &+g(X,W)\Phi(Y,Z)-g(Y,W)\Phi(X,Z)\\ &-\Phi(Y,Z)\eta(X)\eta(W)+\Phi(X,Z)\eta(Y)\eta(W)]\end{aligned} \tag{2.21}$$

for any X, Y, Z, W on M, where $\bar{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Proof. By virtue of Eq.(2.19), we can write

$$\begin{aligned}\bar{R}(X,Y,\phi Z,\phi W) &= g(\nabla_X\nabla_Y\phi Z,\phi W)-g(\nabla_Y\nabla_X\phi Z,\phi W)\\ &- g(\nabla_{[X,Y]}\phi Z,\phi W).\end{aligned} \tag{2.22}$$

By making use of Eq.(2.1) and Eq.(2.5), Eq.(2.22) takes the form

$$\begin{aligned}\bar{R}(X,Y,\phi Z,\phi W) &= \alpha g[g(Y,Z)\nabla_X\xi-(\nabla_X\eta)(Z)Y-\eta(\nabla_X Z)Y\\ &-\eta(Z)\nabla_X Y,\phi W]+\beta g[g(\phi Y,Z)\nabla_X\xi-(\nabla_X\eta)(Z)\phi Y\\ &-\eta(\nabla_X Z)\phi Y-\eta(Z)(\nabla_X\phi)Y-\eta(Z)\phi(\nabla_X Y),\phi W]\\ &-\alpha g[g(X,Z)\nabla_Y\xi-(\nabla_Y\eta)(Z)X-\eta(\nabla_Y Z)X\\ &-\eta(Z)\nabla_Y X,\phi W]-\beta g[g(\phi X,Z)\nabla_Y\xi-(\nabla_Y\eta)(Z)\phi X\end{aligned} \tag{2.23}$$

$$-\eta(\nabla_Y Z)\phi X - \eta(Z)(\nabla_Y \phi)X - \eta(Z)\phi(\nabla_Y X), \phi W]$$
$$+g[\alpha\eta(\nabla_X Z)Y + \beta\eta(\nabla_X Z)\phi Y - \alpha\eta(\nabla_Y Z)X$$
$$-\beta\eta(\nabla_Y Z)\phi X + \phi\nabla_X\nabla_Y Z - \phi\nabla_Y\nabla_X Z, \phi W]$$
$$-g[(\nabla_{[X,Y]}\phi)Z + \phi\nabla_{[X,Y]}Z, \phi W].$$

In view of Eq.(2.1)-Eq.(2.3) and Eq.(2.5)-Eq.(2.6), Eq.(2.23) turns to

$$\begin{aligned}
\bar{R}(X,Y,\phi Z,\phi W) &= g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)\\
&\quad -(\alpha^2-\beta^2)[g(\phi Y,Z)g(X,\phi W) - g(\phi X,Z)g(Y,\phi W)\\
&\quad -g(X,Z)g(Y,W) + g(Y,Z)g(X,W)\\
&\quad +g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)]\\
&\quad +2\alpha\beta[g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W)\\
&\quad -g(X,W)g(\phi Y,Z) + g(Y,W)g(\phi X,Z)\\
&\quad +g(\phi Y,Z)\eta(X)\eta(W) - g(\phi X,Z)\eta(Y)\eta(W)]\\
&\quad -\eta(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)\eta(W)
\end{aligned}$$

which by using Eq.(2.4) , Eq.(2.16) and Eq.(2.19), Eq.(2.21) follows. This completes the proof. □

Lemma 2.2. *In a 3-dimensional trans-Sasakian manifold the ∗-Ricci tensor is given by*

$$S^*(Y,Z) = S(Y,Z) - (\alpha^2-\beta^2)g(Y,Z) - (\alpha^2-\beta^2)\eta(Y)\eta(Z) \tag{2.24}$$

for any Y, Z on M.

Proof. Let $\{e_i\}, i = 1,2,3$ be an orthonormal basis of the tangent space at each point of the manifold. Thus from the equations Eq.(1.2) and Eq.(2.21), we have

$$\begin{aligned}
S^*(Y,Z) &= \sum_{i=1}^{3}\bar{R}(e_i,Y,\phi Z,\phi e_i)\\
&= \sum_{i=1}^{3}\bar{R}(e_i,Y,Z,e_i) + (\alpha^2-\beta^2)\sum_{i=1}^{3}[g(e_i,Z)g(Y,e_i) - g(Y,Z)g(e_i,e_i)
\end{aligned}$$

$$+g(\phi e_i, Z)g(Y, \phi e_i) - g(\phi Y, Z)g(e_i, \phi e_i)]$$
$$+2\alpha\beta\sum_{i=1}^{3}[g(Y,Z)g(e_i,\phi e_i) - g(e_i,Z)g(Y,\phi e_i) - g(e_i,e_i)g(\phi Y, Z)$$
$$+g(Y,e_i)g(\phi e_i, Z) + g(\phi Y, Z)\eta(e_i)\eta(e_i) - g(\phi e_i, Z)\eta(Y)\eta(e_i)].$$

By contracting the above equation we can easily find

$$\begin{aligned} S^*(Y,Z) &= S(Y,Z) - (\alpha^2-\beta^2)[g(Y,Z)+\eta(Y)\eta(Z)] \\ &\quad +2\alpha\beta g(e_i,\phi Z)g(e_i,\xi)\eta(Z) \end{aligned}$$

from which Eq.(2.24) follows. □

3. $*$-Ricci solitons on 3−dimensional trans-Sasakian manifolds

Let M be a 3−dimensional trans-Sasakian manifold admitting $*$−Ricci solitons. Then Eq.(1.3) holds and thus we have

$$(\pounds_\xi g)(Y,Z) + 2S^*(Y,Z) + 2\lambda g(Y,Z) = 0. \tag{3.1}$$

By using Eq.(2.18), we have

$$S^*(Y,Z) = -(\beta+\lambda)g(Y,Z) + \beta\eta(Y)\eta(Z). \tag{3.2}$$

Therefore, from Eq.(2.24) and Eq.(3.2), we obtain

$$S(Y,Z) = (\alpha^2-\beta^2-\beta-\lambda)g(Y,Z) + (\alpha^2-\beta^2+\beta)\eta(Y)\eta(Z). \tag{3.3}$$

It yields

$$QY = (\alpha^2-\beta^2-\beta-\lambda)Y + (\alpha^2-\beta^2+\beta)\eta(Y)\xi. \tag{3.4}$$

Taking $Z=\xi$ in Eq.(3.3), we find

$$S(Y,\xi) = [2(\alpha^2-\beta^2)-\lambda]\eta(Y). \tag{3.5}$$

From the equations Eq.(2.17) and Eq.(3.5), it follows that

$$\lambda = 0. \tag{3.6}$$

Thus we have the following:

Theorem 3.1. *Let M be a $3-$dimensional trans-Sasakian manifold. If the manifold admits $*$-Ricci soliton, then M is an η-Einstein manifold and the $*$-Ricci soliton is steady.*

Now, let (g, V, λ) be a $*$-Ricci soliton on a 3-dimensional trans-Sasakian manifold such that V is pointwise collinear with ξ, i.e., $V = h\xi$, where h is a function. Then Eq.(1.3) holds and thus we have

$$(\pounds_{h\xi}g)(X,Y) + 2S^*(X,Y) + 2\lambda g(X,Y) = 0. \tag{3.7}$$

Applying the property of the Lie derivative and Levi-Civita connection in Eq.(3.7), we have

$$hg(\nabla_X\xi, Y) + (Xh)\eta(Y) + hg(X, \nabla_Y\xi) + (Yh)\eta(X)$$
$$+2S^*(X,Y) + 2\lambda g(X,Y) = 0$$

which by using Eq.(2.24) takes the form

$$\begin{aligned} &hg(\nabla_X\xi, Y) + (Xh)\eta(Y) + hg(X, \nabla_Y\xi) + (Yh)\eta(X) \\ &+2S(X,Y) + 2[\lambda - (\alpha^2 - \beta^2)]g(X,Y) - 2(\alpha^2 - \beta^2)\eta(X)\eta(Y) = 0. \end{aligned} \tag{3.8}$$

By using Eq.(2.6), Eq.(3.8) turns to

$$\begin{aligned} &2h\beta[g(X,Y) - \eta(X)\eta(Y)] + (Xh)\eta(Y) + (Yh)\eta(X) \\ &+2S(X,Y) + 2[\lambda - (\alpha^2 - \beta^2)]g(X,Y) - 2(\alpha^2 - \beta^2)\eta(X)\eta(Y) = 0. \end{aligned} \tag{3.9}$$

Taking $Y = \xi$ and using Eq.(2.1), Eq.(2.3) and Eq.(3.5), Eq.(3.9) reduces to

$$(Xh) + (\xi h) = 0. \tag{3.10}$$

Putting $X = \xi$ in Eq.(3.10) and using Eq.(2.1), we get

$$(\xi h) = 0. \tag{3.11}$$

Combining the equations Eq.(3.10) and Eq.(3.11), we get $X(h) = 0$, that is, h is constant. Thus from (3.9) we obtain

$$\begin{aligned} S(X,Y) &= -[\lambda + h\beta - (\alpha^2 - \beta^2)]g(X,Y) \\ &\quad +(\alpha^2 - \beta^2 + h\beta)\eta(X)\eta(Y). \end{aligned} \tag{3.12}$$

Now putting $Y = \xi$ in Eq.(3.12) and using Eq.(2.17), we get $\lambda = 0$. Thus Eq.(3.12) reduces to

$$S(X,Y) \;=\; -[h\beta - (\alpha^2 - \beta^2)]g(X,Y) + (\alpha^2 - \beta^2 + h\beta)\eta(X)\eta(Y).$$

Therefore we have the following theorem:

Theorem 3.2. *If (g, V, λ) is a $*$-Ricci soliton in a 3-dimensional trans-Sasakian manifold such that V is pointwise collinear with ξ, then V is a constant multiple of ξ and the manifold is an η-Einstein manifold, Moreover, the Ricci soliton is steady.*

4. $*$-Ricci solitons on $3-$dimensional trans-Sasakian manifolds satisfying $R(\xi, X) \cdot S = 0$

Let M be a 3-dimensional trans-Sasakian manifold admitting $*$-Ricci solitons satisfies $R(\xi, X) \cdot S = 0$. Therefore we have

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0. \tag{4.1}$$

By using Eq.(2.13) in Eq.(4.1), we find

$$S(X,Z)\eta(Y) - S(\xi,Z)g(X,Y) + S(Y,X)\eta(Z) - S(Y,\xi)g(X,Z) = 0$$

where $\alpha^2 - \beta^2 \neq 0$, which in view of Eq.(3.5) takes the form

$$\begin{aligned} S(X,Z)\eta(Y) - [2(\alpha^2 - \beta^2) - \lambda]g(X,Y)\eta(Z) \\ -[2(\alpha^2 - \beta^2) - \lambda]g(X,Z)\eta(Y) + S(X,Y)\eta(Z) = 0. \end{aligned} \tag{4.2}$$

Putting $Z = \xi$ in Eq.(4.2) and using Eq.(2.1),Eq.(3.5), we get

$$S(X,Z) = [2(\alpha^2 - \beta^2) - \lambda]g(X,Z). \tag{4.3}$$

Now putting $Z = \xi$ in Eq.(4.3) and using Eq.(2.17), we get $\lambda = 0$. Thus Eq.(4.3) reduces to

$$S(X,Z) = 2(\alpha^2 - \beta^2)g(X,Z).$$

Therefore we have the following theorem:

Theorem 4.1. *If a 3-dimensional trans-Sasakian manifold admitting $*$-Ricci solitons satisfies $R(\xi, X) \cdot S = 0$, then the manifold is an Einstein manifold and the Ricci soliton is steady.*

5. $*$-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $S\cdot R=0$

Let M be a 3-dimensional trans-Sasakian manifold admitting $*$-Ricci solitons satisfies $(S(X,Y)\cdot R)(U,V)W=0$. This implies that

$$(X\wedge_S Y)R(U,V)W+R((X\wedge_S Y)U,V)W \tag{5.1}$$
$$+R(U,(X\wedge_S Y)V)W+R(U,V)(X\wedge_S Y)W=0.$$

By virtue of Eq.(2.20), Eq.(5.1) takes the form

$$S(Y,R(U,V)W)X-S(X,R(U,V)W)Y+S(Y,U)R(X,V)W \tag{5.2}$$
$$-S(X,U)R(Y,V)W+S(Y,V)R(U,X)W-S(X,V)R(U,Y)W$$
$$+S(Y,W)R(U,V)X-S(X,W)R(U,V)Y=0.$$

Taking the inner product of Eq.(5.2) with ξ, we have

$$S(Y,R(U,V)W)\eta(X)-S(X,R(U,V)W)\eta(Y)+S(Y,U)\eta(R(X,V)W)$$
$$-S(X,U)\eta(R(Y,V)W)+S(Y,V)\eta(R(U,X)W)-S(X,V)\eta(R(U,Y)W)$$
$$+S(Y,W)\eta(R(U,V)X)-S(X,W)\eta(R(U,V)Y)=0$$

which by putting $U=W=\xi$ and using Eq.(2.13)-Eq.(2.15) and Eq.(3.5) takes the form

$$S(Y,V)\eta(X)-S(X,V)\eta(Y)-[2(\alpha^2-\beta^2)-\lambda]g(X,V)\eta(Y) \tag{5.3}$$
$$+[2(\alpha^2-\beta^2)-\lambda]g(Y,V)\eta(X)=0.$$

Now replacing X by ξ and using Eq.(2.1), Eq.(2.3) and Eq.(3.5), Eq.(5.3) turns to

$$S(Y,V)=-[2(\alpha^2-\beta^2)-\lambda]g(Y,V)+2[2(\alpha^2-\beta^2)-\lambda]\eta(Y)\eta(V). \tag{5.4}$$

Now putting $V=\xi$ in Eq.(5.4) and using Eq.(2.17), we get $\lambda=0$. Thus Eq.(5.4) reduces to

$$S(Y,V)=-2(\alpha^2-\beta^2)g(Y,V)+4(\alpha^2-\beta^2)\eta(Y)\eta(V).$$

Therefore we have the following theorem:

Theorem 5.1. *If a 3-dimensional trans-Sasakian manifold admitting $*-$Ricci solitons satisfies $S \cdot R = 0$, then the manifold is an η-Einstein and the Ricci soliton is steady.*

6. $*$-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $Q \cdot R = 0$

Let M be a 3-dimensional trans-Sasakian manifold admitting $*$-Ricci solitons satisfies $Q \cdot R = 0$. Then we have

$$Q(R(X,Y)Z) - R(QX,Y)Z - R(X,QY)Z - R(X,Y)QZ = 0 \tag{6.1}$$

for all $X, Y, Z \in \chi(M)$. By virtue of Eq.(3.4), Eq.(6.1) takes the form

$$2[\lambda + \beta - (\alpha^2 - \beta^2)]R(X,Y)Z + (\alpha^2 - \beta^2 + \beta)[\eta(R(X,Y)Z)\xi - \eta(X)R(\xi,Y)Z$$

$$-\eta(Y)R(X,\xi)Z - \eta(Z)R(X,Y)\xi] = 0$$

which by taking the inner product with ξ and using Eq.(2.13), Eq.(2.14) and Eq.(2.16) reduces to

$$[\lambda + \beta - (\alpha^2 - \beta^2)](\alpha^2 - \beta^2)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) = 0. \tag{6.2}$$

Putting $X = \xi$ in Eq.(6.2), we get

$$[\lambda + \beta - (\alpha^2 - \beta^2)](\alpha^2 - \beta^2)g(\phi Y, \phi Z) = 0$$

from which it follows that

$$\lambda = (\alpha^2 - \beta^2) - \beta, \tag{6.3}$$

where $g(\phi Y, \phi Z) \neq 0$. Thus we can state the following:

Theorem 6.1. *If a 3-dimensional trans-Sasakian manifold admitting $*$-Ricci solitons satisfies $Q \cdot R = 0$, then either the the Ricci solitons is expanding or shrinking according as $(\alpha^2 - \beta^2) - \beta > 0$ or $(\alpha^2 - \beta^2) - \beta < 0$. Also for $\alpha = \beta = 0$ (or $\alpha = 0,\ \beta = -1$) the Ricci soliton is steady and in this case the manifold reduces to a cosymplectic (or $\beta-$Kenmotsu) manifold.*

7. $*$-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $R(X,\xi)\cdot P - P(X,\xi)\cdot R = 0$

Definition 7.1. The projective curvature tensor P in a 3-dimensional trans-Sasakian manifold is defined by [1]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2}[S(Y,Z)X - S(X,Z)Y], \tag{7.1}$$

where $X, Y, Z \in \chi(M)$.

From Eq.(7.1), we find the following results to use later

$$\begin{cases} P(X,\xi)Y = \frac{\lambda}{2}\eta(Y)X - (\alpha^2-\beta^2)g(X,Y)\xi + \frac{1}{2}S(X,Y)\xi, \\ P(X,Y)\xi = \frac{\lambda}{2}(\eta(Y)X - \eta(X)Y), \\ P(X,\xi)\xi = \frac{\lambda}{2}(X - \eta(X)\xi). \end{cases} \tag{7.2}$$

In this section, we consider 3-dimensional trans-Sasakian manifolds admitting $*$-Ricci solitons which satisfies the condition $R(X,\xi)\cdot P - P(X,\xi)\cdot R = 0$, then we have

$$\begin{aligned} &R(X,\xi)P(U,V)W - P(R(X,\xi)U,V)W \\ &-P(U,R(X,\xi)V)W - P(U,V)R(X,\xi)W \\ &-P(X,\xi)R(U,V)W + R(P(X,\xi)U,V)W \\ &+R(U,P(X,\xi)V)W + R(U,V)P(X,\xi)W = 0. \end{aligned} \tag{7.3}$$

Putting $U = W = \xi$ in Eq.(7.3), we have

$$\begin{aligned} &R(X,\xi)P(\xi,V)\xi - P(R(X,\xi)\xi,V)\xi \\ &-P(\xi,R(X,\xi)V)\xi - P(\xi,V)R(X,\xi)\xi. \\ &-P(X,\xi)R(\xi,V)\xi + R(P(X,\xi)\xi,V)\xi \\ &+R(\xi,P(X,\xi)V)\xi + R(\xi,V)P(X,\xi)\xi = 0. \end{aligned} \tag{7.4}$$

By using the equations Eq.(2.13)-Eq.(2.15), Eq.(3.5) and Eq.(7.2), Eq.(7.4) turns to

$$\lambda[2g(V,X)\xi - 2\eta(X)\eta(V)\xi + \eta(X)V + \eta(V)X]$$

$$-4(\alpha^2-\beta^2)g(V,X)\xi + 2S(V,X)\xi = 0, \ (\alpha^2-\beta^2) \neq 0$$

which by taking the inner product with ξ and using (2.1) gives

$$S(V,X)=[2(\alpha^2-\beta^2)-\lambda]g(V,X). \tag{7.5}$$

Now putting $X=\xi$ in Eq.(7.5) and using Eq.(2.17), we get $\lambda=0$. Thus Eq.(7.5) reduces to

$$S(V,X)=2(\alpha^2-\beta^2)g(V,X).$$

Therefore we have the following theorem:

Theorem 7.1. *If a 3-dimensional trans-Sasakian manifold admitting $*$-Ricci solitons satisfies $R(X,\xi)\cdot P-P(X,\xi)\cdot R=0$, then the manifold is an Einstein manifold and the Ricci soliton is steady.*

8. ϕ-projectively semisymmetric 3-dimensional trans-Sasakian manifolds admitting $*$-Ricci solitons

Definition 8.1. A 3-dimensional trans-Sasakian manifold admitting $*$-Ricci solitons is said to be ϕ-projectively semisymmetric if [3, 16]

$$P(X,Y)\cdot\phi=0$$

for all $X,Y\in\chi(M)$.

Let M be a 3-dimensional ϕ-projectively semisymmetric trans-Sasakian manifold admitting $*$-Ricci solitons. Therefore $P(X,Y)\cdot\phi=0$ turns into

$$(P(X,Y)\cdot\phi)Z=P(X,Y)\phi Z-\phi P(X,Y)Z=0. \tag{8.1}$$

Taking $X=\xi$ in Eq.(8.1), we have

$$(P(\xi,Y)\cdot\phi)Z=P(\xi,Y)\phi Z-\phi P(\xi,Y)Z=0. \tag{8.2}$$

From Eq.(7.1), we find

$$\begin{cases} P(\xi,Y)\phi Z=(\alpha^2-\beta^2)g(Y,\phi Z)\xi-\frac{1}{2}S(Y,\phi Z)\xi, \\ \phi P(\xi,Y)Z=-\frac{\lambda}{2}\eta(Z)\phi Y. \end{cases} \tag{8.3}$$

Therefore, by using Eq.(8.3) in Eq.(8.2), we obtain $S(Y,\phi Z)\xi = 2(\alpha^2-\beta^2)g(Y,\phi Z)\xi + \lambda\eta(Z)\phi Y$ which by taking the inner product with ξ reduces to

$$S(Y,\phi Z) = 2(\alpha^2-\beta^2)g(Y,\phi Z). \tag{8.4}$$

By replacing Z by ϕZ in Eq.(8.4) and using Eq.(2.1), we get

$$S(Y,Z) = 2(\alpha^2-\beta^2)g(Y,Z) - \lambda\eta(Y)\eta(Z). \tag{8.5}$$

Now putting $Z=\xi$ in Eq.(8.5) and using Eq.(2.16), we get $\lambda = 0$. Thus we have the following theorem:

Theorem 8.1. *A 3-dimensional ϕ-projectively semisymmetric trans-Sasakian manifold admitting $*$-Ricci solitons is an Einstein manifold and the Ricci soliton is steady.*

Example [4]: We consider the three dimensional manifold $M = [(x,y,z)\in R^3 \mid z\neq 0]$, where (x,y,z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = z\frac{\partial}{\partial x},\ e_2 = z\frac{\partial}{\partial y},\ e_3 = z\frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1,e_2) = g(e_2,e_3) = g(e_3,e_1) = 0,\ g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1,$$

Let η be the 1-form on M defined by $\eta(X) = g(X,e_3)$ for all $X\in\chi(M)$. Let ϕ be the $(1,1)$-tensor field on M defined by

$$\phi(e_1) = -e_2, \qquad \phi(e_2) = e_1, \qquad \phi(e_3) = 0.$$

Then using the linearity property of ϕ and g, we have

$$\phi^2X = -X + \eta(X)e_3,\ \eta(e_3) = 1 \text{ and } g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields $X,Y\in\chi(M)$. Thus for $e_3=\xi$, the structure (ϕ,ξ,η,g) defines an almost contact metric structure on M. Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1,e_2] = 0,\ [e_2,e_3] = -e_2,\ [e_1,e_3] = -e_1.$$

The Riemannian connection ∇ with respect to the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z)$$

$$-g([Y,Z],X) + g([Z,X],Y).$$

From above equation which is known as Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_3 = -e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

It can be easily shown that M is a trans-Sasakian manifold of type $(0,-1)$. It is known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

By using the above results, one can easily obtain the components of the curvature tensors as follows:

$$R(e_1,e_2)e_1 = e_2, \ R(e_1,e_2)e_2 = -e_1, \ R(e_1,e_2)e_3 = 0,$$

$$R(e_2,e_3)e_1 = 0, \ R(e_2,e_3)e_2 = e_3, \ R(e_2,e_3)e_3 = -e_2$$

$$R(e_1,e_3)e_1 = e_3, \ R(e_1,e_3)e_2 = 0, \ R(e_1,e_3)e_3 = -e_1.$$

From these curvature tensors, we calculate the components of Ricci tensor as follows:

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = -2. \tag{8.6}$$

From Eq.(3.3), we have $S(e_3,e_3) = 2(\alpha^2 - \beta^2) - \lambda$. By equating both the values of $S(e_3,e_3)$, we obtain $2(\alpha^2 - \beta^2) - \lambda = -2$, which for $\alpha = 0$ and $\beta = -1$ gives $\lambda = 0$. Thus a $*$-Ricci soliton (g, ξ, λ) on a 3-dimensional trans-Sasakian manifold is steady.

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