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CONVEX CONTRACTIONS OF ORDER n IN $CAT(0)$ SPACES

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Abstract. In this paper, we work on convex contraction of order n. Our first result in general metric spaces shows that each convex contraction of order n is a Bessaga mapping. We then turn our attention to $CAT(0)$ spaces. We prove a demiclosedness principle for such mappings in this setting. Next, we consider modified Mann iteration process and prove some convergence theorems for fixed points of such mappings in $CAT(0)$ spaces. Our results are new in $CAT(0)$ setting. Our results remain true in linear spaces like Hilbert and Banach spaces. Finally, we give an example in order to support our main results and to demonstrate the efficiency of modified Mann iteration process.

1. Introduction

Istrătescu [10], [11] introduced some classes of convex contractions such as convex contractions of order 2, two-sided convex contractions and convex contractions of order n. He proved that convex contraction of order 2 and two-sided convex contractions have a single fixed point. These classes of mappings have begun to attract attention in recent years and Alghamdi et al. [12] carried Banach contraction principle to the class of convex contractions in cone metric spaces. In 2013, Miandaragh et al. [13] proved that generalized convex contractions have approximate fixed points in metric spaces. In 2015, Ramezani [14] gave theorems about the existence and uniqueness of the fixed points of such mappings in orthogonal metric spaces. Afterwards, Khan et al. [15] proved some results about existence and uniqueness of fixed points of two-sided convex contractions in b-metric spaces and 2-metric spaces.

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In this paper, motivated by [10] and [11], we first show that each convex conraction of order n is a Bessaga mapping. Then we prove a demiclosedness principle for such mappings. Next, we consider modified Mann iteration process and prove some convergence theorems for fixed points of such mappings in $CAT(0)$ spaces. Our results are new in $CAT(0)$ setting. Finally, we give an example in order to support our main results and to demonstrate the efficiency of the this iteration process.

2. Preliminaries

In this section, we give some definitions and known results from the existing literature. From here on, we denote the set of all fixed points of a mapping f by $F(f)$.

Definition 1. [10] *Let (X,d) be a metric space. A continuous mapping $f : X \to X$ is said to be a convex contraction of order n if there exist positive constants $a_0, a_1, a_2, ..., a_{n-1} \in (0,1)$ such that the following conditions hold:*
(i) $a_0 + a_1 + a_2 + ... + a_{n-1} < 1$
(ii) for all $x, y \in X$,

$$d\left(f^n(x), f^n(y)\right) \leq a_0 d(x,y) + a_1 d\left(f(x), f(y)\right) + a_2 d\left(f^2(x), f^2(y)\right) + ... + a_{n-1} d\left(f^{n-1}(x), f^{n-1}(y)\right). \tag{2.1}$$

If we take $n = 2$ in the above definition, we obtain convex contraction of order 2.

Theorem 1. [10] *Let X be a complete metric space with the metric d. If $f : X \to X$ is a convex contraction of order n, then f has a unique fixed point p, i.e. $F(f) = \{p\}$.*

Definition 2. *Let X be a nonempty set. A map $f : X \to X$ is said to be a Bessaga mapping if there exists $p \in X$ such that $F(f^n) = \{p\}$ for all $n \in \mathbb{N}$.*

Let (X,d) be a metric space. A geodesic from x to y in X is a map c from a closed interval $[0,l] \subset R$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t\prime)) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image of c is called a geodesic segment joining x and y.

A geodesic triangle (x_1, x_2, x_3) in a geodesic metric space (X,d) consists of three points x_1, x_2, x_3 in X and a geodesic segment between each pair of vertices. A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(x_i, x_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists [16].

A metric space X is a $CAT\,(0)$ space if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane [1]. Fixed point theory in $CAT(0)$ spaces has been first studied by Kirk [2, 3]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point.

Lemma 1. [8] *Let X be a $CAT\,(0)$ space. Then*

$$d\left((1-\alpha)\, x \oplus \alpha y, z\right) \leq (1-\alpha)\, d\,(x, z) + \alpha d\,(y, z)$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$.

Lemma 2. [8] *Let X be a $CAT\,(0)$ space. Then*

$$d^2\left((1-\alpha)\, x \oplus \alpha y, z\right) \leq (1-\alpha)\, d^2\,(x, z) + \alpha d^2\,(y, z) - \alpha\,(1-\alpha)\, d^2\,(x, y)$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$.

Let $\{x_n\}$ be a bounded sequence in a $CAT\,(0)$ space X. For $x \in X$, we set

$$r\,(x, \{x_n\}) = \limsup_{n\to\infty} d\,(x, x_n)\,.$$

The asymptotic radius $r\,(\{x_n\})$ of $\{x_n\}$ is given by

$$r\,(\{x_n\}) = \inf\{r\,(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A\,(\{x_n\})$ of $\{x_n\}$ is the set

$$A\,(\{x_n\}) = \{x \in X : r\,(x, \{x_n\}) = r\,(\{x_n\})\}\,.$$

It is known from Proposition 7 of [5] that $A\,(\{x_n\})$ consist of exactly one point in $CAT\,(0)$ spaces.

Lemma 3. [9] *If K is a closed convex subset of a complete $CAT\,(0)$ space and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Definition 3. [6] *A sequence* $\{x_n\}$ *in* X *is said to* Δ*-converge to* $x \in X$ *if* x *is the unique asymptotic center of* $\{u_n\}$ *for every subsequence* $\{u_n\}$ *of* $\{x_n\}$*. In this case, we write* $\Delta - \lim_{n\to\infty} x_n = x$ *and call* x *the* Δ*-limit of* $\{x_n\}$.

Lemma 4. [7] *Let* $\{x_n\}$ *be sequence in* X *such that* $\{x_n\}$ Δ*-converges to* x *and let* $y \in X$ *with* $y \neq x$*. Then* $\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y)$.

This condition is known as the Opial's property in Banach spaces.

Lemma 5. [16] *Every bounded sequence in a complete* $CAT(0)$ *space always has a* Δ*-convergent subsequence.*

Lemma 6. *Let* p, x, y *be points of a* $CAT(0)$ *space* X *and let* $\alpha \in [0, 1]$*. Then*

$$d((1-\alpha)p \oplus \alpha x, (1-\alpha)p \oplus \alpha y) \leq \alpha d(x, y)$$

If x, y_1, y_2 *are points in a* $CAT(0)$ *space and if* $y_0 = \frac{1}{2}y_1 + \frac{1}{2}y_2$*; then the* $CAT(0)$ *inequality implies*

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

This is the (CN) inequality of Bruhat and Tits [8].

In the next section of this paper, we use the following modified Mann iteration process for convergence results.

$$\{.x_1 \in K x_{n+1} = (1-\alpha_n) x_n \oplus \alpha_n f^n(x_n), n \geq 1 \tag{2.2}$$

where f is a convex contraction of order n on a closed convex subset K of a $CAT(0)$ space X. This iteration process is a $CAT(0)$ version of the one given by Schu [4].

3. Main results

Our first result in this section shows a relation between convex contractions of order n and Bessaga mappings.

Theorem 2. *Let* (X, d) *be a metric space. If* $f : X \to X$ *is a convex contraction of order* n*, then* f *is a Bessaga mapping.*

Proof. From Theorem 1, f has a unique fixed point p. Since $F(f) \subset F(f^n)$, p is the fixed point of the mapping f^n, too. We then need to show that f^n has a unique fixed point. Suppose that q is the another fixed point of the mapping f^n. From (2.1),

$$\begin{aligned} d(p,q) &= d\left(f^n(p), f^n(q)\right) \leq a_0 d(p,q) + a_1 d\left(f(p), f(q)\right) \\ &\quad + a_2 d\left(f^2(p), f^2(q)\right) + \cdots + a_{n-1} d\left(f^{n-1}(p), f^{n-1}(q)\right) \\ &\leq a_0 d(p,q) + a_1 a_0 d(p,q) + a_2\left(a_0 d(p,q) + a_1 a_0 d(p,q)\right) \\ &\quad + a_3 \begin{pmatrix} a_0 d(p,q) + a_1 a_0 d(p,q) \\ + a_2\left(a_0 d(p,q) + a_1 a_0 d(p,q)\right) \end{pmatrix} + \ldots \end{aligned}$$

$$\begin{aligned} &\quad + a_{n-1} \begin{pmatrix} a_0 d(p,q) + a_1 a_0 d(p,q) \\ + a_2\left(a_0 d(p,q) + a_1 a_0 d(p,q)\right) \\ + a_3 \begin{pmatrix} a_0 d(p,q) + a_1 a_0 d(p,q) + \\ a_2\left(a_0 d(p,q) + a_1 a_0 d(p,q)\right) \end{pmatrix} + \ldots \end{pmatrix} \\ &= d(p,q) \begin{pmatrix} a_0 + a_1 a_0 + a_2\left(a_0 + a_1 a_0\right) + \\ a_3\left(a_0 + a_1 a_0 + a_2\left(a_0 + a_1 a_0\right)\right) + \ldots \\ + a_{n-1} \begin{pmatrix} a_0 + a_1 a_0 + a_2\left(a_0 + a_1 a_0\right) + \\ a_3\left(a_0 + a_1 a_0 + a_2\left(a_0 + a_1 a_0\right)\right) + \ldots \end{pmatrix} \end{pmatrix} \\ &= a_0 d(p,q) \begin{pmatrix} 1 + a_1 + a_2(1 + a_1) + + a_3\left(1 + a_1 + a_2(1 + a_1)\right) \\ + \cdots + a_{n-1} \begin{pmatrix} 1 + a_1 + a_2(1 + a_1) \\ + a_3\left(1 + a_1 + a_2(1 + a_1)\right) + \ldots \end{pmatrix} \end{pmatrix} \end{aligned}$$

That is,

$$\begin{aligned} d(p,q) &\leq a_0(1 + a_1) d(p,q) \begin{pmatrix} 1 + a_2 + a_3(1 + a_2) + \ldots \\ + a_{n-1}\left(1 + a_2 + a_3(1 + a_2) + \ldots\right) \end{pmatrix} \\ &= a_0(1 + a_1)(1 + a_2) d(p,q)\left(1 + a_3 + \cdots + a_{n-1}(1 + a_3 + \ldots)\right) \\ &= a_0(1 + a_1)(1 + a_2)(1 + a_3)\ldots(1 + a_{n-1}) d(p,q). \end{aligned}$$

Since $a_0, a_1, a_2, ..., a_{n-1} \in (0,1)$ and $a_0 + a_1 + a_2 + a_3 + \cdots + a_{n-1} < 1$, therefore

$$A = a_0 (1 + a_1)(1 + a_2)(1 + a_3) \ldots (1 + a_{n-1}) < 1. \tag{3.1}$$

Hence

$$d(p,q) \leq Ad(p,q) \Longrightarrow (1 - A) d(p,q) \leq 0 \Longrightarrow d(p,q) = 0 \Longrightarrow p = q.$$

Thus, the mapping f^n has a unique fixed point. That is, f is a Bessaga mapping. □

We next prove a demiclosedness principle for convex contraction of order n in $CAT(0)$ spaces as follows. It is well-known that demiclosedness principle plays a key role in studying the asymptotic and ergodic behavior of mappings.

Lemma 7. *Let K be a closed convex subset of a complete $CAT(0)$ space X and let $T : K \to K$ be a convex contraction of order n. If a sequence $\{x_n\}$ Δ-converges to x and $d(x_n, f(x_n)) \to 0$, then $x \in K$ and $f(x) = x$.*

Proof. From the definition of convex contraction of order n and the inequality (3.1), we get that

$$\begin{aligned}
\limsup_{n\to\infty} d(f^n(x), x_n) &\leq \limsup_{n\to\infty} d(f^n(x), f^n(x_n)) + \limsup_{n\to\infty} d(f^n(x_n), x_n) \\
&\leq \limsup_{n\to\infty} a_0 (1 + a_1)(1 + a_2)(1 + a_3) \ldots \\
&\qquad (1 + a_{n-1}) d(x, x_n) + \limsup_{n\to\infty} d(f^n(x_n), x_n) \qquad (3.2) \\
&\leq A \limsup_{n\to\infty} d(x, x_n) + \limsup_{n\to\infty} d(f^n(x_n), x_n) \\
&\leq \limsup_{n\to\infty} d(x, x_n) \qquad (3.3) \\
&= r(x, x_n).
\end{aligned}$$

In here, since $\lim_{n\to\infty} d(x_n, f(x_n)) = 0$, therefore $\lim_{n\to\infty} d(f^n(x_n), x_n) = 0$. By induction we prove that

$$\lim_{n\to\infty} d(f^n(x_n), x_n) = 0 \text{ for each } n \geq 1. \tag{3.4}$$

It is clear that this limit is true for $n = 1$. We suppose that this limit holds for $n = k \geq 1$, that is $\lim_{n\to\infty} d\left(f^k(x_n), x_n\right) = 0$. Now we prove that it is also true for

$n = k+1$. Since f is a convex contraction of order n, we have

$$\begin{aligned} d\left(x_n, f^{k+1}(x_n)\right) &\leq d\left(x_n, f^k(x_n)\right) + d\left(f^k(x_n), f^{k+1}(x_n)\right) \\ &= d\left(x_n, f^k(x_n)\right) + d\left(f^k(x_n), f^k(f(x_n))\right) \\ &\leq d\left(x_n, f^k(x_n)\right) + a_0(1+a_1)(1+a_2)(1+a_3) \\ &\quad \ldots (1+a_{k-1})\, d(x_n, f(x_n)) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, the existence of the limit (3.4) is proved. From unique of the asymptotic center and (3.2), we obtain that $x \in K$ and $f^n(x) = x$. From Theorem 2, the mapping f^n has a unique fixed point. Therefore $f(x) = x$. □

We now exploit the iteration process (2.2) to prove our convergence results. But before that, we have a couple of results as follows.

Lemma 8. *Let X be a complete $CAT(0)$ space and let K be a nonempty closed convex subset of X. Let $f : K \to K$ be a convex contraction of order n and let $\{x_n\}$ defined by the iteration process (2.2). Then $\lim_{n\to\infty} d(x_n, p)$ exists for $p \in F(f)$.*

Proof. Let $p \in F(f)$. From (2.1) and (2.2), we can write

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)x_n \oplus \alpha_n f^n(x_n), p) \\ &= (1-\alpha_n)\, d(x_n, p) + \alpha_n d(f^n(x_n), p) \\ &= (1-\alpha_n)\, d(x_n, p) + \alpha_n d(f^n(x_n), f^n(p)) \\ &\leq (1-\alpha_n)\, d(x_n, p) \\ &\quad + \alpha_n \begin{pmatrix} a_0 d(x_n, p) + a_1 d(f(x_n), f(p)) \\ + a_2 d(f^2(x_n), f^2(p)) + \ldots \\ + a_{n-1} d(f^{n-1}(x_n), f^{n-1}(p)) \end{pmatrix}. \end{aligned} \tag{3.5}$$

Following the steps as in the above Theorem 2, we have

$$d(x_{n+1}, p) \leq d(x_n, p)$$

which obviously implies that $\{d(x_n, p)\}$ is a decreasing sequence and hence $\lim_{n\to\infty} d(x_n, p)$ exists for $p \in F(f)$. □

Lemma 9. *Let X be a complete $CAT(0)$ space, K be a nonempty convex subset of X and $f : K \to K$ be a convex contraction of order n. Let the sequence $\{x_n\}$ be defined by (2.2) with $0 < b \leq \alpha_n \leq c < 1$. Then $\lim_{n\to\infty} d(f(x_n), x_n) = 0$.*

Proof. Let f be a convex contraction of order n and $p \in F(f)$. Using Lemma 3, we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2((1-\alpha_n)x_n \oplus \alpha_n f^n(x_n), p) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(f^n(x_n), p) \\ &\quad -\alpha_n(1-\alpha_n)d^2(f^n(x_n), x_n) \\ &= (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(f^n(x_n), f^n(p)) \\ &\quad -\alpha_n(1-\alpha_n)d^2(f^n(x_n), x_n) \\ &\leq (1-\alpha_n)d^2(x_n, p) \\ &\quad +\alpha_n[a_0(1+a_1)(1+a_2)(1+a_3)\ldots(1+a_{n-1})]^2 d^2(x_n, p) \\ &\quad -\alpha_n(1-\alpha_n)d^2(f^n(x_n), x_n) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(x_n, p) \\ &\quad -\alpha_n(1-\alpha_n)d^2(f^n(x_n), x_n) \\ &= d^2(x_n, p) - \alpha_n(1-\alpha_n)d^2(f^n(x_n), x_n). \end{aligned}$$

That is

$$d^2(x_{n+1}, p) \leq d^2(x_n, p) - \alpha_n(1-\alpha_n)d^2(f^n(x_n), x_n). \tag{3.6}$$

We know that $\lim_{n\to\infty} d(x_n, p)$ exists for $p \in F(f)$ from Lemma 8. Hence using (3.6) we have

$$\lim_{n\to\infty} d(f^n(x_n), x_n) = 0.$$

From (2.2), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, (1-\alpha_n)x_n \oplus \alpha_n f^n(x_n)) \\ &\leq (1-\alpha_n)d(x_n, x_n) + \alpha_n d(f^n(x_n), x_n). \end{aligned}$$

Hence $d\left(x_n, x_{n+1}\right) \to 0$ as $n \to \infty$. Now using triangle inequality and continuity of f, we have

$$\begin{aligned} d\left(x_n, f\left(x_n\right)\right) &\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, f^{n+1}\left(x_{n+1}\right)\right) \\ &\quad + d\left(f^{n+1}\left(x_{n+1}\right), f^{n+1}\left(x_n\right)\right) + d\left(f^{n+1}\left(x_n\right), f\left(x_n\right)\right) \\ &\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, f^{n+1}\left(x_{n+1}\right)\right) \\ &\quad + a_0\left(1+a_1\right)\left(1+a_2\right)\left(1+a_3\right)\ldots\left(1+a_n\right) d\left(x_n, x_{n+1}\right) \\ &\quad + d\left(f^{n+1}\left(x_n\right), f\left(x_n\right)\right) \\ &= \left[1 + a_0\left(1+a_1\right)\left(1+a_2\right)\left(1+a_3\right)\ldots\left(1+a_n\right)\right] d\left(x_n, x_{n+1}\right) \\ &\quad + d\left(x_{n+1}, f^{n+1}\left(x_{n+1}\right)\right) + d\left(f^{n+1}\left(x_n\right), f\left(x_n\right)\right) \end{aligned}$$

which implies that

$$\lim_{n\to\infty} d\left(f\left(x_n\right), x_n\right) = 0.$$

This completes the proof. □

We are now in a position to give our convergence results. Our first result is on Δ-convergence.

Theorem 3. *Let X be a $CAT\left(0\right)$ space, K be a nonempty convex subset of X and $f : K \to K$ be a convex contraction of order n. Let the sequence $\{x_n\}$ be defined by (2.2) with $0 < b \leq \alpha_n \leq c < 1$. Then the sequence $\{x_n\}$, Δ-converges to a fixed point of f.*

Proof. By Lemmas 8 and 9, $\lim_{n\to\infty} d\left(f\left(x_n\right), x_n\right) = 0$ and $\lim_{n\to\infty} d\left(x_n, p\right)$ exists for $p \in F\left(f\right)$. Thus the sequence $\{x_n\}$ is bounded. First, we show that $w_\triangle(x_n) \subseteq F(f)$. Let $u \in w_\triangle(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle$-$\lim_{n\to\infty} v_n = v$ for some $v \in K$. From Lemma 7, $v \in F(T)$. Also by Lemma 8, $\lim_{n\to\infty} d(x_n, v)$ exists. Now, we will show that $u = v$. Suppose that $u \neq v$. Then, by the uniqueness of asymptotic centers, we get that

$$\begin{aligned}
\limsup_{n\to\infty} d\left(v_n, v\right) &< \limsup_{n\to\infty} d\left(v_n, u\right) \\
&\leq \limsup_{n\to\infty} d\left(u_n, u\right) \\
&< \limsup_{n\to\infty} d\left(u_n, v\right) \\
&= \limsup_{n\to\infty} d\left(x_n, v\right) \\
&= \limsup_{n\to\infty} d\left(v_n, v\right)
\end{aligned}$$

which is a contradiction. Thus, $u = v \in F(f)$ and hence $w_{\triangle}(x_n) \subseteq F(f)$. To show that the sequence $\{x_n\}$ $\triangle$-converges to a fixed point of f, we show that $w_{\triangle}(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle$-$\lim_{n\to\infty} v_n = v$ for some $v \in K$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. Therefore $u = v$ and $v \in F(f)$. Finally, we claim that $x = v$. If not, then existence of $\lim_{n\to\infty} d(x_n, v)$ and uniqueness of asymptotic centers imply that

$$\begin{aligned}
\limsup_{n\to\infty} d\left(v_n, v\right) &< \limsup_{n\to\infty} d\left(v_n, x\right) \\
&\leq \limsup_{n\to\infty} d\left(x_n, x\right) \\
&< \limsup_{n\to\infty} d\left(x_n, v\right) \\
&= \limsup_{n\to\infty} d\left(v_n, v\right)
\end{aligned}$$

which is a contradiction and hence $x = v \in F(f)$. Therefore, $w_{\triangle}(x_n) = \{x\}$. □

We now prove a strong convergence theorem.

Theorem 4. *Let X be a $CAT\,(0)$ space, K be a nonempty convex subset of X and $f : K \to K$ be a convex contraction of order n. Let the sequence $\{x_n\}$ be defined by (2.2). Then $\{x_n\}$ converges strongly to a fixed point of f if and only if $\liminf_{n\to\infty} d(x_n, F(f)) = 0$ where $d(x, F(f)) = \inf\{d(x, p) : p \in F(f)\}$.*

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(f) = 0$. As proved in Lemma 8, we have $d\left(x_{n+1}, p\right) \leq\ d\left(x_n, p\right)$for $p \in F(f)$. This implies that $d\left(x_{n+1}, F\left(f\right)\right) \leq\ d\left(x_n, F\left(f\right)\right)$ so that $\lim_{n\to\infty} d(x_n, F(f))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(f) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in K. Let $\varepsilon >$

0 be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(f)) = 0$, there exists a positive integer n_0 such that $d(x_n, F(f)) < \frac{\varepsilon}{4}$, $\forall n \geq n_0$. In particular, $\inf\{d(x_{n_0}, p) : p \in F(f)\} < \frac{\varepsilon}{4}$. Thus there must exist $p^* \in F(f)$ such that $d(x_{n_0}, p^*) < \frac{\varepsilon}{2}$. Now, for all $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p) + d(p, x_n) \\ &\leq 2d(x_{n_0}, p) \\ &= \varepsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in a closed subset K of a complete $CAT(0)$ space and so it must converge to a point q in K. Now, $\lim_{n\to\infty} d(x_n, F(f)) = 0$ gives that $d(q, F(f)) = 0$ and closedness of $F(f)$ forces q to be in $F(f)$. □

In order to support our main results and to demonstrate the efficiency of the iteration process (2.2), we give the following example.

Example 1. *Let $f : [0,1] \to [0,1]$ be defined by $f(x) = \frac{2x^2+1}{4}$. Then f is a convex contraction of order n. Define $\{x_n\}$ as*

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n f^m(x_n), \; n = 1, 2, 3, \ldots$$

and let $\alpha_n = \frac{2n}{3n+1}$ be a sequence in $(0,1)$. The following tables demonstrate convergence of our iteration process for different choices of the initial guess x_1 and different values of m.

Table 3.8 Convergence test for the iteration process (2.2) with initial value $x_1 = 1$ and $m = 1, 2, 3$.

$x_1 = 1$ and $m = 1$	$x_1 = 1$ and $m = 2$	$x_1 = 1$ and $m = 3$
$x_2 = 0.8750000$	$x_2 = 0.7656250$	$x_2 = 0.6955566$
$x_3 = 0.7366071$	$x_3 = 0.5552529$	$x_3 = 0.4802752$
$x_4 = 0.6074199$	$x_4 = 0.4211030$	$x_4 = 0.3722061$
$x_5 = 0.5009950$	$x_5 = 0.3510991$	$x_5 = 0.3248733$
$x_{10} = 0.3066557$	$x_{10} = 0.2936697$	$x_{10} = 0.2931504$
$x_{15} = 0.2935451$	$x_{15} = 0.2929019$	$x_{15} = 0.2928950$
$x_{19} = 0.2929479$	$x_{19} = 0.2928934$	$x_{19} = 0.2928932$

$x_{20} = 0.2929225$	$x_{20} = 0.2928933$	
$x_{21} = 0.2929089$	$x_{21} = 0.2928933$	
$x_{22} = 0.2929016$	$x_{22} = 0.2928932$	
$x_{23} = 0.2928977$		
$x_{27} = 0.2928936$		
$x_{28} = 0.2928934$		
$x_{29} = 0.2928933$		

Table 3.9 Convergence test for the iteration process (2.2) with initial value $x_1 = 1$ and $m = 10, 50, 100$.

$x_1 = 1$ and $m = 10$	$x_1 = 1$ and $m = 50$	$x_1 = 1$ and $m = 100$
$x_2 = 0.6464581$	$x_2 = 0.6464466$	$x_2 = 0.6464466$
$x_3 = 0.4444233$	$x_3 = 0.4444161$	$x_3 = 0.4444161$
$x_4 = 0.3.535058$	$x_4 = 0.3535024$	$x_4 = 0.3535024$
$x_5 = 0.3162060$	$x_5 = 0.3162044$	$x_5 = 0.3162044$
$x_{10} = 0.2930438$	$x_{10} = 0.2930438$	$x_{10} = 0.2930438$
$x_{15} = 0.2928940$	$x_{15} = 0.2928940$	$x_{15} = 0.2928940$
$x_{16} = 0.2928935$	$x_{16} = 0.2928935$	$x_{16} = 0.2928935$
$x_{17} = 0.2928933$	$x_{17} = 0.2928933$	$x_{17} = 0.2928933$
$x_{18} = 0.2928933$	$x_{18} = 0.2928933$	$x_{18} = 0.2928933$
$x_{19} = 0.2928932$	$x_{19} = 0.2928932$	$x_{19} = 0.2928932$

Table 3.10 Convergence test for the iteration process (2.2) with initial value $x_1 = 0$ and $m = 1, 2, 3$.

$x_1 = 0$ and $m = 1$	$x_1 = 0$ and $m = 2$	$x_1 = 0$ and $m = 3$
$x_2 = 0.1250000$	$x_2 = 0.1406250$	$x_2 = 0.1447754$
$x_3 = 0.2008929$	$x_3 = 0.2224226$	$x_3 = 0.2279361$
$x_4 = 0.2424645$	$x_4 = 0.2616130$	$x_4 = 0.2660718$
$x_5 = 0.265190$	$x_5 = 0.2793223$	$x_5 = 0.2821881$
$x_{10} = 0.2915342$	$x_{10} = 0.2927185$	$x_{10} = 0.2928077$
$x_{15} = 0.2928300$	$x_{15} = 0.2928913$	$x_{15} = 0.2928926$

$x_{18} = 0.2928834$	$x_{18} = 0.2928931$	$x_{18} = 0.2928932$
$x_{19} = 0.2928879$	$x_{19} = 0.2928932$	
$x_{20} = 0.2928904$		
$x_{23} = 0.2928928$		

Table 3.11 Convergence test for the iteration process (2.2) with initial value $x_1 = 0$ and $m = 10, 50, 100$.

$x_1 = 0$ and $m = 10$	$x_1 = 0$ and $m = 50$	$x_1 = 0$ and $m = 100$
$x_2 = 0.1464463$	$x_2 = 0.1464466$	$x_2 = 0.1464466$
$x_3 = 0.2301300$	$x_3 = 0.2301304$	$x_3 = 0.2301304$
$x_4 = 0.2677878$	$x_4 = 0.2677881$	$x_4 = 0.2677881$
$x_5 = 0.2832372$	$x_5 = 0.2832374$	$x_5 = 0.2832374$
$x_{10} = 0.2928308$	$x_{10} = 0.2928308$	$x_{10} = 0.2928308$
$x_{15} = 0.2928929$	$x_{15} = 0.2928929$	$x_{15} = 0.2928929$
$x_{16} = 0.2928931$	$x_{16} = 0.2928931$	$x_{16} = 0.2928931$
$x_{17} = 0.2928932$	$x_{17} = 0.2928932$	$x_{17} = 0.2928932$

Table 3.12 Convergence test for the iteration process (2.2) with initial value $x_1 = 0.5$ and $m = 1, 2, 3$.

$x_1 = 0.5$ and $m = 1$	$x_1 = 0.5$ and $m = 2$	$x_1 = 0.5$ and $m = 3$
$x_2 = 0.4375000$	$x_2 = 0.4101563$	$x_2 = 0.4006500$
$x_3 = 0.3850446$	$x_3 = 0.3505333$	$x_3 = 0.3410620$
$x_4 = 0.3484957$	$x_4 = 0.3193112$	$x_4 = 0.3129734$
$x_5 = 0.3252519$	$x_5 = 0.3045316$	$x_5 = 0.3009418$
$x_{10} = 0.2946006$	$x_{10} = 0.2930449$	$x_{10} = 0.2929577$
$x_{15} = 0.2929729$	$x_{15} = 0.2928949$	$x_{15} = 0.2928937$
$x_{18} = 0.2929056$	$x_{18} = 0.2928933$	$x_{18} = 0.2928932$
$x_{19} = 0.2928999$	$x_{19} = 0.2928933$	
$x_{20} = 0.2928968$	$x_{20} = 0.2928932$	
$x_{21} = 0.2928951$		
$x_{28} = 0.2928932$		

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