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APPROXIMATION OF GENERALIZED SZÁSZ-MIRAKJAN OPERATORS DEPENDING ON CERTAIN PARAMETERS

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Abstract. Motivated by certain generalizations, in this paper we consider a new analogue of generalized Szász-Mirakjan operators whose construction depends on τ, with extra parameters μ and λ. Depending on the selection of μ and λ, these operators are more flexible than the generalized Szász-Mirakjan operators. We investigate approximation properties. Also, we study local and global approximation, Voronovskaya type theorem. Finally, quantitative estimates for the local approximation are discussed.

1. Introduction

The well-known Weierstrass Approximation Theorem, proved by Karl Weierstrass in 1885, states that for any continuous function g defined in interval $[a, b]$ and $\epsilon > 0$, there exists a polynomial P such that $|g(y) - P(y)| < \epsilon$. Since the proof of the theorem is lengthy and complicated, many researchers studied to find simple and effective proof. In 1912, S.N. Bernstein [4] proposed the famous polynomial, which is constructed by probabilistic method to give the simple, short and most elegant proof of Weierstrass theorem [22] as follows:

$$\mathcal{B}_m(g; y) = \sum_{j=0}^{m} b_{m,j}(y) g\left(\frac{j}{m}\right), \tag{1.1}$$

where $y \in [0, 1], m = 1, 2, 3, ...$, and the basis of Bernstein functions $b_{m,j}$ are defined as follows:

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$$b_{m,j}(y) = \binom{m}{j} y^j (1-y)^{m-j}. \tag{1.2}$$

In order to obtain more flexibility, Stancu [20] applied another technique for choosing nodes. He observed that the distance between two successive nodes and between 0 and first node and similarly between last and 1 goes to zero when $m \to \infty$. After these observation Stancu introduced the following positive linear operators

$$S_m^{\mu,\lambda}(g; y) = \sum_{k=0}^{m} \binom{m}{k} y^k (1-y)^{m-k} f\left(\frac{k+\mu}{m+\lambda}\right) \tag{1.3}$$

converge to continuous function $g(y)$ uniformly in [0,1] for each real μ, λ such that $0 \leq \mu \leq \lambda$. For various generalization of stancu type operators one can see [3, 10, 12, 13, 14, 15, 16, 17, 18, 19].

To presents a better degree of approximation, a new generalization of Bernstein type operators was given by Cárdenas et al. [5] which depends on τ.

For $m \geq 1$, $y \geq 0$, and suitable functions g defined on $[0, \infty)$. A similar modification of Szász-Mirakyan type operators was introduced by Aral et al. [2] which depends on τ as follows:

$$\mathcal{S}_m^{\tau}(g; y) = e^{-m\tau(y)} \sum_{j=0}^{\infty} \frac{(m\tau(y))^j}{j!} \left(go\tau^{-1}\right) \left(\frac{j}{m}\right). \tag{1.4}$$

Where, τ having following properties

(τ_1) τ be a continuously differentiable function on $[0, \infty)$,

(τ_2) $\tau(0) = 0$ and $\inf\limits_{y \in [0,\infty)} \tau^{'}(y) \geq 1$.

If we put $\tau(y) = y$ then (1.4) reduces to the Szász-Mirakyan operators defined in [21] as

$$\mathcal{S}_m(g; y) = e^{-my} \sum_{j=0}^{\infty} \frac{(my)^j}{j!} g\left(\frac{j}{m}\right), \tag{1.5}$$

Motivated by various Stancu type generalizations and by the above mentioned work, we introduce Stancu variant of operators (1.4) which depend on a suitable function τ as follows:

Definition 1.1. For $m \geq 1$, $y \geq 0$, and suitable functions g defined on $[0, \infty)$ with $0 \leq \mu \leq \lambda$. We define Stancu variant of generalized Szász-Mirakjan operators as

$$\mathcal{S}^{*\mu,\lambda}_{m,\tau}(g; y) = e^{-m\tau(y)} \sum_{j=0}^{\infty} \frac{(m\tau(y))^j}{j!} \left(go\tau^{-1}\right) \left(\frac{j+\mu}{m+\lambda}\right), \ y \geq 0 \tag{1.6}$$

The new constructed operators (1.6) are positive and linear. For $\mu = \lambda = 0$, the operators (1.6) turn out to be generalized Szász-Mirakyan operators defined in (1.4). Next, we prove some Lemma's for (1.6) which play an important role to prove our main results.

Lemma 1.1. *For the operators $\mathcal{S}^{*\mu,\lambda}_{m,\tau}$ be given by (1.6), we have*

(i) $\mathcal{S}^{*\mu,\lambda}_{m,\tau}(1; y) = 1$,

(ii) $\mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau; y) = \frac{m}{m+\lambda}\tau(y) + \frac{\mu}{m+\lambda}$,

(iii) $\mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau^2; y) = \frac{m^2}{(m+\lambda)^2}\tau^2(y) + \frac{m+2\mu m}{(m+\lambda)^2}\tau(y) + \frac{\mu^2}{(m+\lambda)^2}$,

(iv) $\mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau^3; y) = \frac{m^3}{(m+\lambda)^3}\tau^3(y) + \frac{3m^2+6\mu m^2}{(m+\lambda)^3}\tau^2(y) + \frac{m+6\mu m+3\mu^2}{(m+\lambda)^3}\tau(y) + \frac{\mu^3}{(m+\lambda)^3}$,

(v) $\mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau^4; y) = \frac{m^4}{(m+\lambda)^4}\tau^4(y) + \frac{6m^3+4\mu m^3}{(m+\lambda)^4}\tau^3(y) + \frac{7m^2+6\mu^2 m^2+8\mu m^2}{(m+\lambda)^4}\tau^2(y)$
$+ \frac{m+6\mu^2 m+4\mu m+4m^2\mu}{(m+\lambda)^4}\tau(y) + \frac{\mu^4}{(m+\lambda)^4}$.

Proof.

(i)

$$\mathcal{S}^{*\mu,\lambda}_{m,\tau}(1; y) = e^{-m\tau(y)} \sum_{j=0}^{\infty} \frac{(m\tau(y))^j}{j!} \left(go\tau^{-1}\right) = 1$$

(ii)

$$\begin{aligned}
\mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau; y) &= e^{-m\tau(y)} \sum_{j=0}^{\infty} \frac{(m\tau(y))^j}{j!} \left(go\tau^{-1}\right) \left(\frac{j+\mu}{m+\lambda}\right) \\
&= e^{-m\tau(y)} \sum_{j=0}^{\infty} \frac{(m\tau(y))^j}{j!} \left(go\tau^{-1}\right) \left(\frac{j}{m+\lambda}\right) \\
&+ e^{-m\tau(y)} \sum_{j=0}^{\infty} \frac{(m\tau(y))^j}{j!} \left(go\tau^{-1}\right) \left(\frac{\mu}{m+\lambda}\right) \\
&= \frac{m}{m+\lambda}\tau(y) + \frac{\mu}{m+\lambda}.
\end{aligned}$$

(iii)

$$\begin{aligned}
\mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau^2;y) &= e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\left(\frac{j+\mu}{m+\lambda}\right)^2 \\
&= e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\left(\frac{j}{m+\lambda}\right)^2 \\
&+ e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\left(\frac{\mu}{m+\lambda}\right)^2 \\
&+ e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\frac{2\mu j}{(m+\lambda)^2} \\
&= \frac{m^2}{(m+\lambda)^2}\tau^2(y)+\frac{m+2\mu m}{(m+\lambda)^2}\tau(y)+\frac{\mu^2}{(m+\lambda)^2}.
\end{aligned}$$

(iv)

$$\begin{aligned}
\mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau^3;y) &= e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\left(\frac{j+\mu}{m+\lambda}\right)^3 \\
&= e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\left(\frac{j}{m+\lambda}\right)^3 \\
&+ e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\left(\frac{\mu}{m+\lambda}\right)^3 \\
&+ e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\frac{2j^2\mu}{(m+\lambda)^3} \\
&+ e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\frac{2\mu^2 j}{(m+\lambda)^3} \\
&= \frac{m^3}{(m+\lambda)^3}\tau^3(y)+\frac{3m^2+6\mu m^2}{(m+\lambda)^3}\tau^2(y) \\
&+ \frac{m+6\mu m+3\mu^2}{(m+\lambda)^3}\tau(y)+\frac{\mu^3}{(m+\lambda)^3}.
\end{aligned}$$

(v) Finally,

$$\begin{aligned}
\mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau^4;y) &= e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\left(\frac{j+\mu}{m+\lambda}\right)^4 \\
&= e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\left(\frac{j}{m+\lambda}\right)^4
\end{aligned}$$

$$
\begin{aligned}
&+ \quad e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\left(\frac{\mu}{m+\lambda}\right)^4 \\
&+ \quad e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\frac{6j^2\mu^2}{(m+\lambda)^4} \\
&+ \quad e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\frac{4\mu^2 j^3}{(m+\lambda)^4} \\
&+ \quad e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\left(go\tau^{-1}\right)\frac{4\mu^3 j}{(m+\lambda)^4} \\
&= \quad \frac{m^4}{(m+\lambda)^4}\tau^4(y)+\frac{6m^3+4\mu m^3}{(m+\lambda)^4}\tau^3(y)+\frac{7m^2+6\mu^2m^2+8\mu m^2}{(m+\lambda)^4}\tau^2(y) \\
&+ \quad \frac{m+6\mu^2 m+4\mu m+4m^2\mu}{(m+\lambda)^4}\tau(y)+\frac{\mu^4}{(m+\lambda)^4}.
\end{aligned}
$$

□

Corollary 1.1. *By using the linearity of operators $\mathcal{S}^{*\mu,\lambda}_{m,\tau}$ and by Lemma 1.1,we can acquire the central moments as*

(i) $\mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau(\xi)-\tau(y);y) = \left(\frac{m}{m+\lambda}-1\right)\tau(y)+\frac{\mu}{m+\lambda}$,

(ii) $\mathcal{S}^{*\mu,\lambda}_{m,\tau}((\tau(\xi)-\tau(y))^2;y) = \left(\frac{m^2}{(m+\lambda)^2}-\frac{2m}{m+\lambda}+1\right)\tau^2(y)+\left(\frac{m+2\mu m}{(m+\lambda)^2}-\frac{2\mu}{m+\lambda}\right)\tau(y)$
$+\frac{\mu^2}{(m+\lambda)^2}$,

(iii) $\mathcal{S}^{*\mu,\lambda}_{m,\tau}((\tau(\xi)-\tau(y))^3;y) = \left(\frac{m^3}{(m+\lambda)^3}-\frac{3m^2}{(m+\lambda)^2}+\frac{3m}{m+\lambda}-1\right)\tau^3(y)$
$+\left(\frac{3m^2+6\mu m^2}{(m+\lambda)^3}-\frac{6\mu m+3m}{(m+\lambda)^2}+\frac{3\mu}{(m+\lambda)}\right)\tau^2(y)+\left(\frac{m+6\mu m+3\mu^2}{(m+\lambda)^3}-\frac{3\mu^2}{(m+\lambda)^2}\right)\tau(y)+\frac{\mu^3}{(m+\lambda)^3}$,

(iv) $\mathcal{S}^{*\mu,\lambda}_{m,\tau}((\tau(\xi)-\tau(y))^4;y) = \left(\frac{m^4}{(m+\lambda)^4}-\frac{4m^3}{(m+\lambda)^3}+\frac{6m^2}{(m+\lambda)^2}-\frac{4m}{(m+\lambda)}+1\right)\tau^4(y)$
$+\left(\frac{6m^3+6m^3\mu}{(m+\lambda)^4}-\frac{24m^2\mu+12m^2}{(m+\lambda)^3}+\frac{12m\mu+6m}{(m+\lambda)^2}-\frac{4\mu}{(m+\lambda)}\right)\tau^3(y)$
$+\left(\frac{7m^2+6m^2\mu^2+8m^2\mu}{(m+\lambda)^4}-\frac{24m\mu m+4m+12\mu^2}{(m+\lambda)^3}+\frac{6\mu^2}{(m+\lambda)^2}\right)\tau^2(y)$
$+\left(\frac{m+6m\mu^2+4m^2\mu+4m\mu}{(m+\lambda)^4}+\frac{\mu^3}{(m+\lambda)^3}\right)\tau(y)+\frac{\mu^4}{(m+\lambda)^4}$.

2. Weighted approximation

In this section, by using weighted space we discuss some convergence properties of new constructed operators $\mathcal{S}^{*\mu,\lambda}_{m,\tau}$.

Let $\Psi(y) = 1+\tau^2(y)$ be a weight function and $\mathcal{B}_\Psi[0,\infty)$ be the weighted spaces defined as:

$$\mathcal{B}_\Psi[0,\infty) = \{g : [0,\infty) \to \mathbb{R} \big| |g(y)| \leq \mathcal{M}_g\Psi(y), y \geq 0\},$$

where $\mathcal{M}_g$ is a constant. $\mathcal{B}_\Psi[0,\infty)$ is a normed linear space equipped with the norm

$$\| g \|_\Psi = \sup_{y\in[0,\infty)} \frac{|g(y)|}{\Psi(y)}.$$

Also, the subspaces $\mathcal{C}_\Psi[0,\infty), U_\Psi[0,\infty)$ and $U_\Psi[0,\infty)$ of $\mathcal{B}_\Psi[0,\infty)$ are defined as

$$\mathcal{C}_\Psi[0,\infty) = \{g \in \mathcal{B}_\Psi[0,\infty) : \ g \ is \ continuous \ on \ [0,\infty)\},$$
$$\mathcal{C}^*_\Psi[0,\infty) = \left\{ g \in \mathcal{C}_\Psi[0,\infty) : \ \lim_{y\to\infty} \frac{g(y)}{\Psi(y)} = \mathcal{M}_g = Constant \right\},$$
$$U_\Psi[0,\infty) = \{g \in \mathcal{C}_\Psi[0,\infty) : \ \frac{g(y)}{\Psi(y)} \text{ is uniformly continuous on } [0,\infty)\}.$$

It is Obvious that $\mathcal{C}^*_\Psi[0,\infty) \subset U_\Psi[0,\infty) \subset \mathcal{C}_\Psi[0,\infty) \subset \mathcal{B}_\Psi[0,\infty)$.

In [7], the weighted Korovkin type theorems are proved by Gadjiev.

Lemma 2.1. [7] *For $m \geq 1$, $\mathcal{Q}_m : \mathcal{B}_\Psi[0,\infty) \to \mathcal{B}_\Psi[0,\infty)$ if and only if the inequality*

$$|\mathcal{Q}_m(\Psi; y)| \leq \mathcal{M}_m \Psi(y), \ y \geq 0,$$

holds, where $\mathcal{M}_m > 0$ is a constant.

Theorem 2.1. [7] *For $m \geq 1$, $\mathcal{Q}_m : \mathcal{B}_\Psi[0,\infty) \to \mathcal{B}_\Psi[0,\infty)$ and satisfying*

$$\lim_{m\to\infty} \| \mathcal{Q}_m \tau^i - \tau^i \|_\Psi = 0, \ i = 0, 1, 2.$$

Then for any function $g \in C^*_\Psi[0,\infty)$ we have

$$\lim_{m\to\infty} \| \mathcal{Q}_m(g) - g \|_\Psi = 0.$$

Therefore, we can prove the following results.

Theorem 2.2. *For each function $g \in C^*_\Psi[0,\infty)$ with $0 \leq \mu \leq \lambda$. We have*

$$\lim_{m\to\infty} \| \mathcal{S}^{*\mu,\lambda}_{m,\tau}(g) - g \|_\Psi = 0.$$

Proof. It is clear from Lemma 1.1 that

$$\| \mathcal{S}^{*\mu,\lambda}_{m,\tau}(1; y) - 1 \|_\Psi = 0.$$

$$\| \mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau;y) - \tau \|_\Psi = \left(\frac{m}{m+\lambda} - 1\right) \sup_{y\in[0,\infty)} \frac{\tau(y)}{1+\tau^2(y)} + \frac{\mu}{m+\lambda} \leq \frac{\mu-\lambda}{m+\lambda}.$$

Again by Lemma 1.1 (iii), we have

$$\begin{aligned} \| \mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau^2;y) - \tau^2 \|_\Psi &= \left(\frac{m^2}{(m+\lambda)^2} - 1\right) \sup_{y\in[0,\infty)} \frac{\tau^2(y)}{1+\tau^2(y)} \\ &+ \frac{2\mu m + 2m}{(m+\lambda)^2} \sup_{y\in[0,\infty)} \frac{\tau(y)}{1+\tau^2(y)} + \frac{\mu^2}{(m+\lambda)^2} \\ &\leq \frac{\mu^2 - \lambda^2 - 2m\lambda + 2m\mu + m}{(m+\lambda)^2}. \end{aligned} \tag{2.1}$$

Then from Lemma 1.1 and (2.1) we get $\lim_{m\to\infty} \| \mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau^i) - \tau^i \|_\Psi = 0,\ i = 0,1,2.$ □

3. Rate of convergence

In this section, by using weighted modulus of continuity $\omega_\tau(f;\delta)$ we determine the rate of convergence for $\mathcal{S}^{*\mu,\lambda}_{m,\tau}$ which was recently considered by Holhoş [9] as follows:

$$\omega_\tau(g;\delta) = \sup_{y,\xi\in[0,\infty),|\tau(\xi)-\tau(y)|\leq\lambda} \frac{|g(\xi)-g(y)|}{\Psi(\xi)+\Psi(y)},\ \lambda > 0, \tag{3.1}$$

where $g \in \mathcal{C}_\Psi[0,\infty)$, having following properties:

(i) $\omega_\tau(g;0) = 0$,

(ii) $\omega_\tau(g;\lambda) \geq 0$, $\lambda \geq 0$ for $g \in \mathcal{C}_\Psi[0,\infty)$,

(ii) $\lim_{\lambda\to 0} \omega_\tau(g;\lambda) = 0$, for each $g \in U_\Psi[0,\infty)$.

Theorem 3.1. [9] *Let $\mathcal{Q}_m : \mathcal{C}_\Psi[0,\infty) \to \mathcal{B}_\Psi[0,\infty)$ be a sequence of positive linear operators with*

$$\parallel \mathcal{Q}_m(\tau^0) - \tau^0 \parallel_{\Psi^0} = a_m, \tag{3.2}$$

$$\parallel \mathcal{Q}_m(\tau) - \tau \parallel_{\Psi^{\frac{1}{2}}} = b_m, \tag{3.3}$$

$$\parallel \mathcal{Q}_m(\tau^2) - \tau^2 \parallel_{\Psi} = c_m, \tag{3.4}$$

$$\parallel \mathcal{Q}_m(\tau^3) - \tau^3 \parallel_{\Psi^{\frac{3}{2}}} = d_m, \tag{3.5}$$

where the sequences a_m, b_m, c_m and d_m converge to zero as $m \to \infty$. Then

$$\parallel \mathcal{Q}_m(g) - g \parallel_{\Psi^{\frac{3}{2}}} \leq (7 + 4a_m + 2c_m)\omega_\tau(g; \lambda_m) + \parallel g \parallel_\Psi a_m, \tag{3.6}$$

for all $g \in \mathcal{C}_\Psi[0,\infty)$, where

$$\lambda_m = 2\sqrt{(a_m + 2b_m + c_m)(1 + a_m)} + a_m + 3b_m + 3c_m + d_m.$$

Theorem 3.2. *Let for each $g \in \mathcal{C}_\Psi[0,\infty)$ with $0 \leq \mu \leq \lambda$. Then we have*

$$\parallel \mathcal{S}^{*\mu,\lambda}_{m,\tau}(g) - g \parallel_{\Psi^{\frac{3}{2}}} \leq \left(7 + \frac{2\mu^2 - 2\lambda^2 - 4m\lambda + 4m\mu + 2m}{(m+\lambda)^2}\right)\omega_\tau(g; \delta_m),$$

where

$$\begin{aligned}
\delta_m &= 2\sqrt{\frac{2\mu - 2\lambda}{m+\lambda} + \frac{\mu^2 - \lambda^2 - 2m\lambda + 2m\mu + 2m}{(m+\lambda)^2}} \\
&+ \frac{3\mu - 3\lambda}{m+\lambda} + \frac{3\mu^2 - 3\lambda^2 - 6m\lambda + 6m\mu + 3m}{(m+\lambda)^2} \\
&+ \frac{3m^2 + 6\mu m^2 + m + 6\mu m + 3\mu^2 m + \mu^3 - \lambda^3 - 3m^2\lambda - 3m\lambda^2}{(m+\lambda)^3}.
\end{aligned}$$

Proof. If we calculate the sequences (a_m), (b_m), (c_m) and (d_m), then by using Lemma 1.1, clearly we have

$$\parallel \mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau^0) - \tau^0 \parallel_{\Psi^0} = 0 = a_m,$$

$$\parallel \mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau) - \tau \parallel_{\Psi^{\frac{1}{2}}} \leq \frac{\mu - \lambda}{m+\lambda} = b_m,$$

and

$$\parallel \mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau^2) - \tau^2 \parallel_{\Psi} \leq \frac{\mu^2 - \lambda^2 - 2m\lambda + 2m\mu + m}{(m+\lambda)^2} = c_m.$$

Finally,

$$\begin{aligned}\| \mathcal{S}^{*\mu,\lambda}_{m,\tau}(\tau^3) - \tau^3 \|_{\Psi^{\frac{3}{2}}} & \\ &\leq \frac{3m^2+6\mu m^2+m+6\mu m+3\mu^2 m+\mu^3-\lambda^3-3m^2\lambda-3m\lambda^2}{(m+\lambda)^3} \\ &= d_m.\end{aligned}$$

Thus the conditions (3.1)-(3.5) are satisfied. Now by Theorem 3.1, the proof is competed. □

Remark 1. For $\lim_{\lambda\to 0} \omega_\tau(g;\lambda) = 0$ in Theorem 3.2, we get

$$\lim_{m\to\infty} \| \mathcal{S}^{*\mu,\lambda}_{m,\tau}(g) - g \|_{\Psi^{\frac{3}{2}}} = 0, \text{ for } g \in U_\Psi[0,\infty).$$

4. Voronovskaya type theorem

In this section, we establish Voronovskaya-type result for $\mathcal{S}^{*\mu,\lambda}_{m,\tau}$.

Theorem 4.1. *Let $g \in \mathcal{C}_\Psi[0,\infty)$, $y \in [0,\infty)$ with $0 \leq \mu \leq \lambda$. and suppose that $(go\tau^{-1})'$ and $(go\tau^{-1})''$ exist at $\tau(y)$. If $(go\tau^{-1})''$ is bounded on $[0,\infty)$, then we have*

$$\lim_{m\to\infty} m\left[\mathcal{S}^{*\mu,\lambda}_{m,\tau}(g;y) - g(y)\right] = \tau(y)\left(go\tau^{-1}\right)'\mu + \tau(y)\left(go\tau^{-1}\right)''\tau(y)$$

Proof. Let $g \in \mathcal{C}_\Psi[0,\infty)$ and $\tau(y) \in [0,\infty)$. By Taylor expansion of $(go\tau^{-1})$ we may write

$$\begin{aligned}g(\xi) = \left(go\tau^{-1}\right)(\tau(\xi)) &= \left(go\tau^{-1}\right)(\tau(y)) + \left(go\tau^{-1}\right)'(\tau(y))(\tau(\xi)-\tau(y)) \\ &+ \frac{\left(go\tau^{-1}\right)''(\tau(y))(\tau(\xi)-\tau(y))^2}{2} + \lambda_y(\xi)(\tau(\xi)-\tau(y))^2,\end{aligned} \tag{4.1}$$

where

$$\lambda_y(\xi) = \frac{\left(go\tau^{-1}\right)''(\tau(\xi)) - \left(go\tau^{-1}\right)''(\tau(y))}{2}. \tag{4.2}$$

Therefore, $\lim_{\xi\to y} \lambda_y(\xi) = 0$. Applying $\mathcal{S}^{*\mu,\lambda}_{m,\tau}$ to (4.1), we obtain

$$\begin{aligned}\left[\mathcal{S}^{*\mu,\lambda}_{m,\tau}(g;y) - g(y)\right] &= \left(go\tau^{-1}\right)'(\tau(y))\mathcal{S}^{*\mu,\lambda}_{m,\tau}((\tau(\xi)-\tau(y));y) \\ &+ \frac{\left(go\tau^{-1}\right)''(\tau(y))\mathcal{S}^{*\mu,\lambda}_{m,\tau}((\tau(\xi)-\tau(y))^2;y)}{2} \\ &+ \mathcal{S}^{*\mu,\lambda}_{m,\tau}\left(\lambda^y(\xi)\left((\tau(\xi)-\tau(y))^2;y\right)\right).\end{aligned} \tag{4.3}$$

From Lemma 1.1 and Corollary 1.1, we obtain

$$\lim_{m\to\infty} m\mathcal{S}^{*\mu,\lambda}_{m,\tau}\left((\tau(\xi)-\tau(y));y\right)\leq\mu, \tag{4.4}$$

and

$$\lim_{m\to\infty} m\mathcal{S}^{*\mu,\lambda}_{m,\tau}\left((\tau(\xi)-\tau(y))^2;y\right)\leq 2\tau(y) \tag{4.5}$$

Since from (4.2), for every $\epsilon>0$, $\lim_{\xi\to y}\lambda_y(\xi)=0$. Let $\delta>0$ such that $|\lambda_y(\xi)|<\epsilon$ for every $\xi\geq 0$. From Cauchy-Schwartz inequality, we get immediately

$$\begin{aligned}\lim_{m\to\infty} m\mathcal{S}^{*\mu,\lambda}_{m,\tau}\left(|\lambda_y(\xi)|\,(\tau(\xi)-\tau(y))^2;y\right) &\leq \epsilon\lim_{m\to\infty} m\mathcal{S}^{*\mu,\lambda}_{m,\tau}\left((\tau(\xi)-\tau(y))^2;y\right)\\ &+ \frac{\mathcal{K}}{\delta^2}\lim_{m\to\infty}\mathcal{S}^{*\mu,\lambda}_{m,\tau}\left((\tau(\xi)-\tau(y))^4;y\right).\end{aligned}$$

Since

$$\lim_{m\to\infty} m\mathcal{S}^{*\mu,\lambda}_{m,\tau}\left((\tau(\xi)-\tau(y))^4;y\right)=0, \tag{4.6}$$

we obtain

$$\lim_{m\to\infty} m\mathcal{S}^{*\mu,\lambda}_{m,\tau}\left(|\lambda_y(\xi)|\,(\tau(\xi)-\tau(y))^2;y\right)=0. \tag{4.7}$$

Thus, by taking into account the equations (4.4),(4.5) and (4.7) to equation (4.3) the proof is completed. □

5. Local Approximation

Let $\mathcal{C}_B[0,\infty)$, be the space of real-valued continuous and bounded functions g with the norm $\|\cdot\|$ is given by

$$\| g \| = \sup_{0\leq y<\infty} |\, g(y)\,| \,.$$

We begin by considering the $\mathcal{K}$-functional as:

$$\mathcal{K}_2(g,\delta)=\inf_{s\in W^2}\{\| g-s \| +\delta \| g^{''} \|\},$$

where $\delta>0$ and $W^2=\{s\in\mathcal{C}_B[0,\infty): s^{'}, s^{''}\in\mathcal{C}_B[0,\infty)\}$.
Then, in view of known result [6], there exists an absolute constant $\mathcal{C}>0$ such that

$$\mathcal{K}(g,\delta)\leq\mathcal{C}\omega_2(g,\sqrt{\delta}), \tag{5.1}$$

where

$$\omega_2(g,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{u\in[0,\infty)} \mid g(y+2h)-2g(y+h)+g(y) \mid$$

is second order modulus of smoothness of $g \in C_B[0,\infty)$.

Also,

$$\omega(g,\delta) = \sup_{0<h\leq\delta} \sup_{y\in[0,\infty)} \mid g(y+h)-g(y) \mid$$

is the usual modulus of continuity of $g \in C_B[0,\infty)$

Theorem 5.1. *There exists an absolute constant* $\mathcal{C} > 0$ *such that*

$$\left|\mathcal{S}^{*\mu,\lambda}_{m,\tau}(g;y) - g(y)\right| \leq \mathcal{C}\mathcal{K}\left(g,\delta_m(y)\right),$$

where $g \in \mathcal{C}_B[0,\infty)$, $0 \leq \mu \leq \lambda$ *and*

$$\delta_m(y) = \left\{\left(\frac{m^2}{(m+\lambda)^2} - \frac{2m}{m+\lambda}+1\right)\tau^2(y) + \left(\frac{2\mu m+m}{(m+\lambda)^2} - \frac{2\mu}{m+\lambda}\right)\tau(y) + \frac{\mu^2}{(m+\lambda)^2}\right\}$$

Proof. By using Taylor's formula and for $s \in W^2$ also $y,\xi \in [0,\infty)$. We have

$$s(\xi) = s(y) + \left(so\tau^{-1}\right)'(\tau(y))(\tau(\xi)-\tau(y)) + \int_{\tau(y)}^{\tau(\xi)} (\tau(\xi)-v)\left(so\tau^{-1}\right)''(v)dv. \tag{5.2}$$

By using the equality

$$\left(so\tau^{-1}\right)''(\tau(y)) = \frac{s''(y)}{\left(\tau'(y)\right)^2} - s''(y)\frac{\tau''(y)}{\left(\tau'(y)\right)^3}. \tag{5.3}$$

Now, in the last term of equality (5.2) put $v = \tau(y)$, we obtain

$$\begin{aligned}
\int_{\tau(y)}^{\tau(\xi)} (\tau(\xi)-v)\left(so\tau^{-1}\right)''(v)dv &= \int_{y}^{\xi} (\tau(\xi)-\tau(y))\left[\frac{s''(y)\tau'(y)-s'(y)\tau''(v)}{(\tau'(y))^2}\right]dy \\
&= \int_{\tau(y)}^{\tau(\xi)} (\tau(\xi)-v)\frac{s''(\tau^{-1}(v))}{(\tau'(\tau^{-1}(v))^2}dv \qquad (5.4)\\
&- \int_{\tau(y)}^{\tau(\xi)} (\tau(\xi)-v)\frac{s'(\tau^{-1}(v))\tau''(\tau^{-1}(v))}{(\tau'(\tau^{-1}(v))^3}dv.
\end{aligned}$$

By applying ${\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}$ to (5.2) and also by using Lemma 1.1 and (5.4) and we deduce

$$\begin{aligned}{\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}(s;y) &= s(y)+{\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}\bigg(\int_{\tau(y)}^{\tau(\xi)}(\tau(\xi)-v)\frac{s''(\tau^{-1}(v))}{(\tau'(\tau^{-1}(v))^2}dv;u\bigg)\\ &- {\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}\bigg(\int_{\tau(y)}^{\tau(\xi)}(\tau(\xi)-v)\frac{s'(\tau^{-1}(v))\tau''(\tau^{-1}(v))}{(\tau'(\tau^{-1}(v))^3}dv;y\bigg).\end{aligned}$$

By using the conditions (τ_1) and (τ_2) given above we get

$$\big|{\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}(s;y)-s(y)\big|\leq \mathcal{M}_{m,2}^{\tau}(y)\big(\|s''\|+\|s'\|\|\tau''\|\big),$$

where

$$\mathcal{M}_{m,2}^{\tau}(y)={\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}((\tau(\xi)-\tau(y))^2;y).$$

For $g\in\mathcal{C}_B[0,\infty)$, we have

$$\begin{aligned}\big|{\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}(s;y)\big| &\leq \|go\tau^{-1}\|e^{-m\tau(y)}\sum_{j=0}^{\infty}\frac{(m\tau(y))^j}{j!}\\ &\leq \|g\|{\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}(1;y)=\|g\|. \qquad (5.5)\end{aligned}$$

Hence we have

$$\begin{aligned}\big|{\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}(g;y)-g(y)\big| &\leq \big|{\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}(g-s;y)\big|+\big|{\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}(s;y)-s(y)\big|+\big|s(y)-g(y)\big|\\ &\leq 2\|g-s\|+\bigg\{\bigg(\frac{m^2}{(m+\lambda)^2}-\frac{2m}{m+\lambda}+1\bigg)\tau^2(y)\\ &+ \bigg(\frac{2\mu m+m}{(m+\lambda)^2}-\frac{2\mu}{m+\lambda}\bigg)\tau(y)+\frac{\mu^2}{(m+\lambda)^2}\bigg\}\big(\|s''\|+\|s'\|\|\tau''\|\big),\end{aligned}$$

if we choose $\mathcal{C}=\max\{2,\|\tau''\|\}$,then

$$\begin{aligned}\big|{\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}(g;y)-g(y)\big| &\leq \mathcal{C}\bigg(2\|g-s\|+\bigg\{\bigg(\frac{m^2}{(m+\lambda)^2}-\frac{2m}{m+\lambda}+1\bigg)\tau^2(y)\\ &+ \bigg(\frac{2\mu m+m}{(m+\lambda)^2}-\frac{2\mu}{m+\lambda}\bigg)\tau(u)+\frac{\mu^2}{(m+\lambda)^2}\bigg\}\|s''\|_{W^2}\bigg).\end{aligned}$$

Taking the infimum on right hand side over all $s\in W^2$, we obtain

$$\big|{\mathcal{S}^*}_{m,\tau}^{\mu,\lambda}(g;y)-g(y)\big|\leq \mathcal{C}\mathcal{K}\left(g,\delta_m(y)\right).$$

□

Let $0 < \alpha \leq 1$, τ be a function with conditions (τ_1), (τ_2) and $Lip_{\mathcal{M}}(\tau(y);\alpha)$, $\mathcal{M} \geq 0$ satisfying

$$\left|g(\xi) - g(y)\right| \leq \mathcal{M}\left|\tau(\xi) - \tau(y)\right|^{\alpha}, y, \xi \geq 0.$$

Moreover, $\mathcal{E} \subset [0, \infty)$ be a bounded subset and the function $g \in Lip_{\mathcal{M}}(\tau(y);\alpha)$, $0 < \alpha \leq 1$ on $\mathcal{E}$ if

$$\left|g(\xi) - g(y)\right| \leq \mathcal{M}_{\alpha,g}\left|\tau(\xi) - \tau(y)\right|^{\alpha}, u \in \mathcal{E} \text{ and } \xi \geq 0,$$

where $\mathcal{M}_{\alpha,g}$ is a constant depending on α and g.

Theorem 5.2. *Let $0 < \alpha \leq 1$ and for every $g \in Lip_{\mathcal{M}}(\rho(y);\alpha)$, with $0 \leq \mu \leq \lambda$. Then for every $y \in (0, \infty), m \in \mathbb{N}$, we have*

$$\left|\mathcal{S}^{*\mu,\lambda}_{m,\rho}(g; y) - g(y)\right| \leq \mathcal{M}\left(\delta_m(y)\right)^{\frac{\alpha}{2}}, \tag{5.6}$$

where

$$\delta_m(y) = \left\{\left(\frac{m^2}{(m+\lambda)^2} - \frac{2m}{m+\lambda} + 1\right)\rho^2(y) + \left(\frac{2\mu m + m}{(m+\lambda)^2} - \frac{2\mu}{m+\lambda}\right)\rho(y) + \frac{\mu^2}{(m+\lambda)^2}\right\}$$

Proof. Assume that $\alpha = 1$. Then, for $g \in Lip_{\mathcal{M}}(\alpha; 1)$ and $y \in (0, \infty)$, we have

$$\begin{aligned} |\mathcal{S}^{*\mu,\lambda}_{m,\rho}(g; y) - g(y)| &\leq \mathcal{S}^{*\mu,\lambda}_{m,\rho}(|g(\xi) - g(y)|; y) \\ &\leq \mathcal{M}\mathcal{S}^{*\mu,\lambda}_{m,\rho}(|\rho(\xi) - g(y)|; y). \end{aligned}$$

By Cauchy Schwartz inequality, we obtain

$$\begin{aligned} |\mathcal{S}^{*\mu,\lambda}_{m,\rho}(g; y) - g(y)| &\leq \mathcal{M}\left[\mathcal{S}^{*\mu,\lambda}_{m,\rho}((\rho(\xi) - \rho(y))^2; y)\right]^{\frac{1}{2}} \\ &\leq \mathcal{M}\sqrt{\delta_m(y)}. \end{aligned}$$

Let us assume that $\alpha \in (0, 1)$. Then, for $g \in Lip_{\mathcal{M}}(\alpha; 1)$ and $y \in (0, \infty)$, we have

$$\begin{aligned} |\mathcal{S}^{*\mu,\lambda}_{m,\rho}(g; y) - g(y)| &\leq \mathcal{S}^{*\mu,\lambda}_{m,\rho}(|g(\xi) - g(y)|; y) \\ &\leq \mathcal{M}\mathcal{S}^{*\mu,\lambda}_{m,\rho}(|\rho(\xi) - g(y)|^{\alpha}; y). \end{aligned}$$

For $g \in Lip_{\mathcal{M}}(\rho(y);\alpha)$ and by Hölder's inequality with $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, we have

$$|\mathcal{S}^{*\mu,\lambda}_{m,\rho}(g; y) - g(y)| \leq \mathcal{M}\left[\mathcal{S}^{*\mu,\lambda}_{m,\rho}(|(\rho(\xi) - \rho(y)|; y)\right]^{\alpha}.$$

Finally from Cauchy-Schwartz inequality, we get

$$\left|\mathcal{S}^{*\mu,\lambda}_{m,\rho}(g;y)-g(y)\right|\leq \mathcal{M}\left(\delta_m(y)\right)^{\frac{\alpha}{2}}.$$

□

Theorem 5.3. *Let $\mathcal{E}$ be a bounded subset of $[0,\infty)$ and τ be a function satisfying the conditions (τ_1), (τ_2). Then for any $g\in Lip_{\mathcal{M}}(\tau(y);\alpha)$, $0<\alpha\leq 1$ on $\mathcal{E}$ $\alpha\in(0,1]$, we have*

$$\left|\mathcal{S}^{*\mu,\lambda}_{m,\tau}(g;y)-g(y)\right|\leq \mathcal{M}_{\alpha,g}\left\{(\delta_m(y))^{\frac{\alpha}{2}}+2[\tau'(y)]^{\alpha}d^{\alpha}(y,\mathcal{E})\right\}, y\in[0,\infty), m\in\mathbb{N},$$

where $d(y,\mathcal{E})=inf\{\|y-x\|:x\in\mathcal{E}\}$ and $\mathcal{M}_{\alpha,g}$ is a constant depending on α and g.

where

$$\delta_m(y)=\left\{\left(\frac{m^2}{(m+\lambda)^2}-\frac{2m}{m+\lambda}+1\right)\tau^2(y)+\left(\frac{2\mu m+m}{(m+\lambda)^2}-\frac{2\mu}{m+\lambda}\right)\tau(y)+\frac{\mu^2}{(m+\lambda)^2}\right\}.$$

Proof. Let $\mathcal{E}$ be a bounded subset of $[0,\infty)$ and $\overline{\mathcal{E}}$ be its closure. Then, there exists a point $y_0\in\overline{\mathcal{E}}$ such that $d(y,\mathcal{E})=|y-y_0|$.
Using the monotonicity of $\mathcal{S}^{*\mu,\lambda}_{m,\tau}$ and the hypothesis of g, we obtain

$$\begin{aligned}|\mathcal{S}^{*\mu,\lambda}_{m,\tau}(g;y)-g(y)| &\leq \mathcal{S}^{*\mu,\lambda}_{m,\tau}\left(|g(\xi)-g(y_0)|;y\right)+\mathcal{S}^{*\mu,\lambda}_{m,\tau}\left(|g(y)-g(y_0)|;y\right)\\ &\leq \mathcal{M}_{\alpha,g}\left\{\mathcal{S}^{*\mu,\lambda}_{m,\tau}\left(|\tau(\xi)-\tau(y_0)|^{\alpha};y\right)+|\tau(y)-\tau(y_0)|^{\alpha}\right\}\\ &\leq \mathcal{M}_{\alpha,g}\left\{\mathcal{S}^{*\mu,\lambda}_{m,\tau}\left(|\tau(\xi)-\tau(y)|^{\alpha};y\right)+2|\tau(y)-\tau(y_0)|^{\alpha}\right\}.\end{aligned}$$

Let $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$ and by using the fact $|\tau(y)-\tau(y_0)|=\tau'(y)|\tau(y)-\tau(y_0)|$ in the last inequality along with Hölder's inequality we immediately have

$$\left|\mathcal{S}^{*\mu,\lambda}_{m,\tau}(g;y)-g(y)\right|\leq \mathcal{M}_{\alpha,g}\left\{\left[\mathcal{S}^{*\mu,\lambda}_{m,\tau}((\tau(\xi)-\tau(y))^2;y)\right]^{\frac{1}{2}}+2[\tau'(y)|\tau(y)-\tau(y_0)|]^{\alpha}\right\}.$$

Hence, by Corollary1.1 we get the proof. □

Now, we recall local approximation in terms of α order generalized Lipschitz-type maximal function given by Lenze [11] for $g\in\mathcal{C}_B[0,\infty)$ as

$$\widetilde{\omega}^{\tau}_{\alpha}(g;y)=\sup_{\xi\neq y,\xi\in(0,\infty)}\frac{|g(\xi)-g(y)|}{|\xi-y|^{\alpha}},\ y\in[0,\infty)\text{ and }\alpha\in(0,1]. \tag{5.7}$$

Then we get the next result

Theorem 5.4. *Let* $g \in \mathcal{C}_B[0,\infty)$ *and* $\alpha \in (0,1]$ *with* $0 \leq \mu \leq \lambda$*. Then, for all* $y \in [0,\infty)$*, we have*

$$\left|\mathcal{S}^{*\mu,\lambda}_{m,\tau}(g;y) - g(y)\right| \leq \widetilde{\omega}^{\tau}_{\alpha}(g;y)\,(\delta_m(y))^{\frac{\alpha}{2}}.$$

where

$$\delta_m(y) = \left\{\left(\frac{m^2}{(m+\lambda)^2} - \frac{2m}{m+\lambda} + 1\right)\tau^2(y) + \left(\frac{2\mu m + m}{(m+\lambda)^2} - \frac{2\mu}{m+\lambda}\right)\tau(y) + \frac{\mu^2}{(m+\lambda)^2}\right\}.$$

Proof. We know that

$$|\mathcal{S}^{*\mu,\lambda}_{m,\tau}(g;y) - g(y)| \leq \mathcal{S}^{*\mu,\lambda}_{m,\tau}(|g(t) - g(y)|;y).$$

From equation (5.7), we have

$$|\mathcal{S}^{*\mu,\lambda}_{m,\tau}(g;y) - g(y)| \leq \widetilde{\omega}^{\tau}_{\alpha}(g;y)\mathcal{S}^{*\mu,\lambda}_{m,\tau}(|\tau(\xi) - \tau(y)|^{\alpha};y).$$

From Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$\begin{aligned}|\mathcal{S}^{*\mu,\lambda}_{m,\tau}(g;y) - g(y)| &\leq \widetilde{\omega}^{\tau}_{\alpha}(g;y)\left[\mathcal{S}^{*\mu,\lambda}_{m,\tau}((\tau(\xi) - \tau(y))^2;y)\right]^{\frac{\alpha}{2}} \\ &\leq \widetilde{\omega}^{\tau}_{\alpha}(g;y)\,(\delta_m(y))^{\frac{\alpha}{2}}.\end{aligned}$$

which proves the desired result ☐

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